The problem:

Two systems differing only in constant terms:

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 2 \\ 2x & + & 3y & + & z & = & 11 \\ x & - & y & - & 2z & = & -2 \end{array} \qquad \begin{array}{rcrcrcr} x & + & y & - & z & = & 1 \\ 2x & + & 3y & + & z & = & 6 \\ x & - & y & - & 2z & = & -2 \end{array}$$

The solution:

Solve them both at once — the operations depend only on the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 2 & 3 & 1 & 11 & 6 \\ 1 & -1 & -2 & -2 & -2 \end{bmatrix}$$

$$\xrightarrow[\boxed{3} - \boxed{1}]{\boxed{2} - 2\boxed{1}} \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 3 & 7 & 4 \\ 0 & -2 & -1 & -4 & -3 \end{bmatrix}$$

$$\xrightarrow{\boxed{3} + 2\boxed{2}} \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 3 & 7 & 4 \\ 0 & 0 & 5 & 10 & 5 \end{bmatrix}$$

$$\xrightarrow{\boxed{3} \times 1/5} \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 3 & 7 & 4 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

(continued)

$$\xrightarrow{\substack{\boxed{2}-3\boxed{3}\\ \boxed{1}+\boxed{3}}} \begin{bmatrix} 1 & 1 & 0 & 4 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{\boxed{1}-\boxed{2}} \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}.$$

We've continued the elimination process until the coefficient matrix is the identity matrix, so we can just read off the solutions to the two systems:

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{respectively.}$$

Definition

Gauss-Jordan reduction: The use of elementary row operations on a matrix to get it into "reduced row echelon form": The first nonzero entry in any row is a 1 (called a "leading 1"); each leading 1 is to the right of the one above it; the entries above the leading 1's are 0's; and any rows of all 0's come last.

Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 5 & 7 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The problem:

To find the inverse of

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix} :$$

The solution:

Solve <u>three</u> systems at once, with columns of constants the columns of I_3:

$$\left[\begin{array}{rrrrrr} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & -1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\boxed{2}-2\boxed{1} \\ \boxed{3}-\boxed{1}}} \left[\begin{array}{rrrrrr} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\boxed{3}+2\boxed{2}} \left[\begin{array}{rrrrrr} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 5 & -5 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{\boxed{3}\times 1/5} \left[\begin{array}{rrrrrr} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2/5 & 1/5 \end{array}\right]$$

(continued)

$$\xrightarrow[]{\substack{\boxed{2}-3\boxed{3}\\ \boxed{1}+\boxed{3}}} \begin{bmatrix} 1 & 1 & 0 & 0 & 2/5 & 1/5 \\ 0 & 1 & 0 & 1 & -1/5 & -3/5 \\ 0 & 0 & 1 & -1 & 2/5 & 1/5 \end{bmatrix}$$

$$\xrightarrow[]{\boxed{1}-\boxed{2}} \begin{bmatrix} 1 & 0 & 0 & -1 & 3/5 & 4/5 \\ 0 & 1 & 0 & 1 & -1/5 & -3/5 \\ 0 & 0 & 1 & -1 & 2/5 & 1/5 \end{bmatrix}.$$

So

$$A^{-1} = \begin{bmatrix} -1 & 3/5 & 4/5 \\ 1 & -1/5 & -3/5 \\ -1 & 2/5 & 1/5 \end{bmatrix}.$$

Proposition

For a square matrix A, the following statements are equivalent:

(1) *A is singular, i.e., not invertible.*

(2) *A is not a product of elementary matrices.*

(3) *For a system $A\mathbf{x} = \mathbf{b}$, "elimination breaks down permanently," i.e., the system does not have a unique solution. (It has no solution or infinitely many, depending on the choice of $\mathbf{b}$.)*

(4) *The rows of A are dependent.*

(5) *The columns of A are dependent.*

(6) $\det(A) = 0$ — *but we haven't defined determinant except for 2×2 matrices, so we'll set this one aside for now.*

(1) $\implies$ (2) (or its equivalent, not (2) $\implies$ not (1)): Each elementary matrix has an inverse, so the product of elementary matrices has an inverse, namely, the product of their inverses in reverse order.

(2) $\implies$ (1) (or its equivalent, not (1) $\implies$ not (2)): If A has an inverse, then Gauss-Jordan reduction will find it as a product of elementary matrices; so A is the product of the inverses of those elementary matrices in reverse order.

(2) $\implies$ (3): If A is not the product of elementary matrices, then row operations on it will never reach the identity. So it must reach a row in which it is impossible to find a nonzero pivot, even by reversing rows from below. So there is a column without a nonzero pivot; and because the matrix is square, that means there is a row of zeros and hence no unique solution to a system $A\mathbf{x} = \mathbf{0}$.

(3) $\implies$ (1) (or its equivalent, not (1) $\implies$ not (3)): If A is invertible, then the unique solution to any $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$.

(3) $\implies$ (4): The row operations of elimination lead to a row of 0's, so some row is a linear combination of the rows above it.

(4) $\implies$ (3): There are row operations that lead to a row of 0's, and that new system $E_1 E_2 \dots E_n A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. But row operations don't change solutions, so $A\mathbf{x} = \mathbf{0}$ also has infinitely many solutions.

(3) $\implies$ (5): $(0, 0, \ldots, 0)$ is always a solution to $A\mathbf{x} = \mathbf{0}$, so in saying that the solution isn't unique, we are assuming that we can find a nonzero solution $\mathbf{x} = \mathbf{c}$ to it. Let $\mathbf{a}_j$ denote the j-th column of A, and suppose $c_n \neq 0$. Then we have

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0} \quad \implies \quad -\frac{c_1}{c_n}\mathbf{a}_1 - \cdots - \frac{c_{n-1}}{c_n}\mathbf{a}_{n-1} = \mathbf{a}_n \ ,$$

so the columns of A are dependent.

(5) $\implies$ (3): If the columns of A are dependent, say

$$c_1\mathbf{a}_1 + \cdots + c_{n-1}\mathbf{a}_{n-1} = \mathbf{a}_n \ ,$$

then $\mathbf{x} = (c_1, \ldots, c_{n-1}, -1)$ is a nonzero solution to $A\mathbf{x} = \mathbf{0}$. But $\mathbf{x} = (0, 0, \ldots, 0)$ is always a solution to it, so the solution isn't unique.