Definition

The *transpose* A^T of the matrix A is the result of turning the rows into columns and vice versa:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Notes:

- $(A + B)^T = A^T + B^T$.
- $(cA)^T = cA^T$.
- $(AB)^T = B^T A^T$. (So $(A^{-1})^T = (A^T)^{-1}$.)
- If $A^T = A$, then A is *symmetric* (and square).
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$ (the last a product as matrices, so that the resulting 1×1 matrix is just a number).

Example

If A and B are symmetric (of the same size), must $AB - BA$ be symmetric? Well,

$$(AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB \ .$$

This is the negative of the original difference matrix, so unless $AB = BA$ (i.e., the original matrix is all 0's), $AB - BA$ cannot be symmetric.

Example

If A and B are symmetric (of the same size), must $ABAB$ be symmetric? Well,

$$(ABAB)^T = B^T A^T B^T A^T = BABA \ .$$

This doesn't look the same as $ABAB$, but it may be; for example, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$:

$$AB = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

have $ABAB = -I_2 = BABA$. But the same A and $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ doesn't work:

$$ABAB = \begin{bmatrix} 8 & 16 \\ 4 & 8 \end{bmatrix} \qquad \text{(not symmetric).}$$

Uniqueness of LDU decomposition

The text says that if A has two LDU decompositions, they must be the same. Let's show that's true if A has all nonzero pivots: Suppose $L_1 D_1 U_1 = L_2 D_2 U_2$, where the L's, D's and U's have the usual properties — and all the entries on the main diagonal of D_1 are nonzero. Then

$$L_2^{-1} L_1 D_1 = D_2 U_2 U_1^{-1} \ .$$

The left side is all 0's above the main diagonal, the right side is all 0's below it; so both sides are diagonal; and the main diagonal entries on both sides agree, so we have $D_1 = D_2$. Now suppose $L_2^{-1} L_1$ has a nonzero entry off the main diagonal; then the left side has a nonzero entry below the main diagonal, so the same is true of the right side, a contradiction. So $L_2^{-1} L_1$ is diagonal, and because both L's have 1's on the main diagonal, the product is the identity, i.e., $L_1 = L_2$. Similarly for the other side, so $U_1 = U_2$ also.

Nonuniqueness of LDU decomposition

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

no matter what x is.

LU Decomposition in Matlab

```
>> A = [1 3 5 ; 2 4 6 ; 5 5 2]
A =
     1     3     5
     2     4     6
     5     5     2
>> lu(A)
ans =
     5.0000    5.0000     2.0000
     0.4000    2.0000     5.2000
     0.2000    1.0000    -0.6000
>> [L,U,P] = lu(A)
L =
     1.0000         0          0
     0.4000    1.0000          0
     0.2000    1.0000     1.0000
```

```
U =
     5.0000    5.0000     2.0000
          0    2.0000     5.2000
          0         0    -0.6000
P =
     0     0     1
     0     1     0
     1     0     0
>> P*A-L*U
ans =
     0     0     0
     0     0     0
     0     0     0
```

LU Decomposition in R

```
> library(Matrix)
> A <-matrix(c(1,3,5,2,4,6,5,5,2), nrow=3, byrow=TRUE)
> lu(A)
'MatrixFactorization' of Formal class 'denseLU' [package "Matrix"]
with 3 slots
  ..@ x    : num [1:9] 5 0.4 0.2 5 2 ...
  ..@ perm : int [1:3] 3 2 3
  ..@ Dim  : int [1:2] 3 3
```

(continued)

```
> F <- expand(lu(A))
> F$L
3 x 3 Matrix of class "dtrMatrix" (unitriangular)
     [,1] [,2] [,3]
[1,]  1.0    .    .
[2,]  0.4  1.0    .
[3,]  0.2  1.0  1.0
> F$U
3 x 3 Matrix of class "dtrMatrix"
     [,1] [,2] [,3]
[1,]  5.0  5.0  2.0
[2,]    .  2.0  5.2
[3,]    .    . -0.6
```

(continued again)

> F$P

3 x 3 sparse Matrix of class "pMatrix"

[1;]	.	.	|
[2;]	.	|	.
[3;]	|	.	.

> F$P%*%A−F$L%*%F$U

3 x 3 Matrix of class "dgeMatrix"

	[; 1]	[; 2]	[; 3]
[1;]	0	0	0
[2;]	0	0	0
[3;]	0	0	0