Determinants:

for <u>square</u> matrices only

Solve a general system (2×2)

$$\begin{aligned} ax + by &= f \\ cx + dy &= g \end{aligned} \xrightarrow{\boxed{2} - \frac{c}{a}\boxed{1}} \begin{aligned} ax + by &= f \\ \left(d - \frac{bc}{a}\right) y &= g - \frac{fc}{a} \\ \left(\frac{ad - bc}{a}\right) y &= \frac{ag - fc}{a} \end{aligned}$$

$$\xrightarrow{\boxed{2} \times \frac{a}{ad-bc}} \begin{aligned} ax + by &= f \\ y &= \frac{ag - fc}{ad - bc} \end{aligned}$$

Solve a general system (2×2, ctnd.)

$$\xrightarrow{\boxed{1} - b\boxed{2}} \qquad \begin{aligned} ax &= f - b\frac{ag - fc}{ad - bc} \\ &= \frac{fad - fbc - bag + bfc}{ad - bc} = a\frac{fd - bg}{ad - bc} \\ y &= \frac{ag - fc}{ad - bc} \end{aligned}$$

$$\xrightarrow{\boxed{1} \times \frac{1}{a}} \qquad \begin{aligned} x &= \frac{fd - bg}{ad - bc} \\ y &= \frac{ag - fc}{ad - bc} \end{aligned}$$

The value $ad - bc$ came up in the denominator — and we saw earlier that the system has a solution exactly when $ad - bc$. It <u>determines</u> when a solution exists.

Definition

The *determinant* of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$, denoted either $\det A$ or $|A|$.

Note: From the general system above:

$$x = \frac{fd - bg}{ad - bc} = \frac{\det \begin{bmatrix} f & b \\ g & d \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} \qquad y = \frac{ag - fc}{ad - bc} = \frac{\det \begin{bmatrix} a & f \\ c & g \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$

Solve a general system (3×3)

$$
\begin{aligned}
a_1 x + b_1 y + c_1 z &= p \\
a_2 x + b_2 y + c_2 z &= q \\
a_3 x + b_3 y + c_3 z &= r
\end{aligned}
\quad
\xrightarrow{\substack{\boxed{2} - \frac{a_2}{a_1}\boxed{1} \\ \boxed{3} - \frac{a_3}{a_1}\boxed{1}}}
$$

$$
\begin{aligned}
a_1 x + b_1 y + c_1 z &= p \\
\frac{a_1 b_2 - a_2 b_1}{a_1} y + \frac{a_1 c_2 - a_2 c_1}{a_1} z &= \frac{a_1 q - a_2 p}{a_1} \\
\frac{a_1 b_3 - a_3 b_1}{a_1} y + \frac{a_1 c_3 - a_3 c_1}{a_1} z &= \frac{a_1 r - a_3 p}{a_1}
\end{aligned}
$$

$$
\xrightarrow{\boxed{3} - \frac{a_1 b_3 - a_3 b_1}{a_1 b_2 - a_2 b_1}\boxed{2}}
$$

$\rightarrow$

$$a_1 x + b_1 y + c_1 z = p$$

$$\frac{a_1 b_2 - a_2 b_1}{a_1} y + \frac{a_1 c_2 - a_2 c_1}{a_1} z = \frac{a_1 q - a_2 p}{a_1}$$

$$\frac{a_1 b_2 c_3 - a_2 b_1 c_3 - a_3 b_2 c_1 - a_1 b_3 c_2 + a_2 b_3 c_1 + a_3 b_1 c_2}{a_1 b_2 - a_2 b_1} z$$
$$= \frac{a_1 b_2 r - a_2 b_1 r - a_3 b_2 p - a_1 b_3 q + a_2 b_3 p + a_3 b_1 q}{a_1 b_2 - a_2 p}$$

ETC.

So:

Definition

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, then

$$\det A = a_1 b_2 c_3 - a_2 b_1 c_3 - a_3 b_2 c_1 - a_1 b_3 c_2 + a_2 b_3 c_1 + a_3 b_1 c_2 \ .$$

And then we get from the general system above:

$$z = \frac{\det \begin{bmatrix} a_1 & b_1 & p \\ a_2 & b_2 & q \\ a_3 & b_3 & r \end{bmatrix}}{\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}} \ .$$

A a square matrix $\to \det A$ or $|A|$, a number built out of the entries of A

Basic Properties

1. $\det(I_n) = 1$ (for any n)
2. Reversing two rows changes sign.
3. Det is "multilinear", i.e., linear in each row separately. [So, for example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2(2)+3(4) & 2(-1)+3(7) & 2(3)+3(2) \end{vmatrix}$$
$$= 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & -1 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 7 & 2 \end{vmatrix}.]$$

Warning: By 3., $\det(cA) = c^n \det A$ (not $c \det A$), if A is $n \times n$.

Secondary Properties

4. If two rows are equal, $\det = 0$.
   - Pf: Reversing two rows changes the sign but, if the rows are equal, the matrix is the same. Only $d = 0$ satisfies $-d = d$.
5. Subtracting a scalar multiple of one row from another doesn't change det.
   - Pf:

$$\begin{vmatrix} \vdots \\ \mathbf{a}_k \\ \vdots \\ \mathbf{a}_j + c\mathbf{a}_k \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ \mathbf{a}_k \\ \vdots \\ \mathbf{a}_j \\ \vdots \end{vmatrix} + c \begin{vmatrix} \vdots \\ \mathbf{a}_k \\ \vdots \\ \mathbf{a}_k \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ \mathbf{a}_k \\ \vdots \\ \mathbf{a}_j \\ \vdots \end{vmatrix} + c(0) = \begin{vmatrix} \vdots \\ \mathbf{a}_k \\ \vdots \\ \mathbf{a}_j + c\mathbf{a}_k \\ \vdots \end{vmatrix}.$$

6. If A has a row of 0's, $\det A = 0$.
   - Pf: If $\mathbf{a}_k = \mathbf{0} = 0\mathbf{a}_k$, then $\det A = 0(\det A) = 0$.

Secondary Properties (ctnd.)

7. If A is triangular, $\det A$ is the product of its main diagonal entries.
   - Pf: If a main diagonal entry is 0, then by row ops we can make a row of zeros, so det is 0. If not, we can make it diagonal and use multilinearity to turn it into I.
8. $\det A = 0$ iff A is singular ($=$ not invertible).
   - Pf: A is singular iff its rref R has 0 on its diagonal; but $\det A = k \det R$ where $k \neq 0$.

9*. $\det(AB) = (\det A)(\det B)$.
   - Pf: If A or B is singular, both sides are 0. If neither, A is a product of elementary matrices, and we can check $\det(EB) = (\det E)(\det B)$ for each of the 3 kinds of E.

$9\frac{1}{2}$. $\det(A^{-1}) = 1/(\det A)$.
   - Pf: $(\det A)(\det A^{-1}) = \det I$.

Secondary Properties (ctnd. again)

10. $\det(A^T) = \det A$.
    - Pf: If A is singular, so is A^T. If not, A is a product of elementaries E, and $\det E = \det E^T$ for each kind of E. (For two of them, $E = E^T$. For the third, $\det E = 1 = \det E^T$.) So if $A = E_1 E_2 E_3$, then

$$\begin{aligned}
\det A^T &= (\det E_3^T)(\det E_2^T)(\det E_1^T) \\
&= (\det E_3)(\det E_2)(\det E_1) \\
&= (\det E_1)(\det E_2)(\det E_3) \quad \text{(mult of scalars is comm)} \\
&= \det A \ .
\end{aligned}$$

Claims:

- The Basic Properties imply the "Big Formula" for determinants that we'll do soon.
- Because the Big Formula gives the determinant, it follows that any rule that has the Basic Properties <u>is</u> the determinant.

Check that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ has the Basic Properties.

Evaluating determinants I: Pivots

Do row ops to get a diagonal matrix, keeping track of how determinant changes.

Example:

$$\begin{vmatrix} 0 & 3 & 5 \\ 2 & 0 & 4 \\ 1 & 1 & 1 \end{vmatrix} \overset{\boxed{1}\leftrightarrow\boxed{2}}{=} - \begin{vmatrix} 2 & 0 & 4 \\ 0 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix} \overset{\boxed{1}\times(1/2)}{=} -2 \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\overset{\boxed{3}-\boxed{1}}{=} -2 \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 0 & 1 & -1 \end{vmatrix} \overset{\boxed{3}-(1/3)\boxed{2}}{=} -2 \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & -8/3 \end{vmatrix}$$

$$= -2(1)(3)(-8/3) = 16$$

This is the most efficient method, i.e., fewest arithmetic operations.

Note also, from $PA = LU$ and $|P| = \pm 1$ and $|L| = 1$, we get that $|A|$ is plus or minus the product of the pivots of A.

Evaluating determinants II: The Big Formula

Recall:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Note in each term the first subscript goes from 1 to n, and the second varies over a rearrangement of 1 to n, with all the rearrangements in different terms (i.e., each term is a product of entries, one from each row and one from each column). We need the signs of the terms.

In the same way, from solving bigger systems:

$$\det A = \sum_{\sigma} (\operatorname{sgn} \sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where σ varies over all the permutations (rearrangements) of 1,2,..., n and $\operatorname{sgn} \sigma$ is -1 to the power the number of backward pairs in σ.

A "backward pair" is a pair from 1 through n that is out of its usual order in σ.

Example: If σ is 41325, then of the ten pairs

$$1,2 \quad 1,3 \quad 1,4 \quad 1,5 \quad 2,3 \quad 2,4 \quad 2,5 \quad 3,4 \quad 3,5 \quad 4,5$$

1,4 and 2,3 and 2,4 and 3,4 are backwards in σ, so $\operatorname{sgn} \sigma = (-1)^4 = 1$.

Because there are $n(n-1)(n-2)\dots(2)(1) = n!$ permutations of 1 through n, using the Big Formula, a determinant has $n!$ terms of products of n entries.

So a 10×10 matrix requires

$$10!(9) = 32,659,200$$

arithmetic operations, even without trying to figure out the sgn for each term.

Example

1234	1
1243	−1
1324	−1
1342	1
1423	1
1432	−1
2134	−1
2143	1
2314	1
2341	−1
2413	−1
2431	1
3124	1
3142	−1
3214	−1
3241	1
3412	1
3421	−1
4123	−1
4132	1
4213	1
4231	−1
4312	−1
4321	1

$$
\begin{aligned}
&\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 0 & 5 \\ 0 & 1 & 4 & 3 \end{vmatrix} \\
&= 0 - 0 - 0 + 0 + 0 - 0 \\
&- 1(1)(0)(3) + 1(1)(5)(4) + 1(2)(2)(3) \\
&\quad - 1(2)(5)(0) - 1(0)(2)(4) + 1(0)(0)(0) \\
&+ 2(1)(1)(3) - 2(1)(5)(1) - 2(-1)(2)(3) \\
&\quad + 2(-1)(5)(0) + 2(0)(2)(1) - 2(0)(1)(0) \\
&- 3(1)(1)(4) + 3(1)(0)(1) + 3(-1)(2)(4) \\
&\quad - 3(-1)(0)(0) - 3(2)(2)(1) + 3(2)(1)(0) \\
&= 20 + 12 + 6 - 10 - (-12) - 12 + (-24) - 12 \\
&= -8
\end{aligned}
$$

Evaluating determinants III: Cofactors

The only way to keep the Big Formula straight:

Definition

The (i, j)-th *cofactor* of the matrix A is $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting from A the i-th row and the j-th column.

$$\begin{aligned} \det A &= \sum_i a_{i,j} C_{i,j} \text{ for any } i \\ &= \sum_j a_{i,j} C_{i,j} \text{ for any } j \end{aligned}$$

Example:

$$
\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 0 & 5 \\ 0 & 1 & 4 & 3 \end{vmatrix}
$$

$$
\begin{aligned}
&= 0 + 1(-1)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 5 \\ 1 & 4 & 3 \end{vmatrix} + 2(1)\begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 1 & 4 & 3 \end{vmatrix} + 0 \\
&= -\left[1(-1)\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} + 0 + 5(-1)\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}\right] \\
&\qquad + 2\left[-1(-1)\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} + 2(1)\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + 0\right] \\
&= -[-(-6) - 5(2)] + 2[1(-6) + 2(0)] = -8
\end{aligned}
$$

Cramer's Rule

As we saw earlier, we can solve $A\mathbf{x} = \mathbf{b}$ (for A nonsingular) by letting B_i be the result of replacing the i-th column of A (the coefficients of x_i) with $\mathbf{b}$ and setting

$$x_i = \frac{\det B_i}{\det A} \text{ for each } i \ .$$

Why does this work? Well, by the multilinearity <u>on columns</u> and the fact that a matrix with two equal columns has determinant 0:

$$\begin{aligned}
|B_i| &= \begin{vmatrix} \mathbf{a}_1 & \dots & \mathbf{b} & \dots & \mathbf{a}_n \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{a}_1 & \dots & (x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n) & \dots & \mathbf{a}_n \end{vmatrix} \\
&= x_1 \begin{vmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_1 & \dots & \mathbf{a}_n \end{vmatrix} \\
&\qquad + \cdots + x_i \begin{vmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_i & \dots & \mathbf{a}_n \end{vmatrix} \\
&\qquad + \cdots + x_n \begin{vmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n & \dots & \mathbf{a}_n \end{vmatrix} \\
&= 0 + \cdots + x_i|A| + \cdots + 0 = x_i|A| \ .
\end{aligned}$$

Example

$$\begin{array}{rcrcrcr} & & 3y & + & 2z & = & 7 \\ x & - & 2y & + & z & = & 12 \\ 3x & + & 4y & & & = & -6 \end{array}$$

$$\begin{vmatrix} 0 & 3 & 2 \\ 1 & -2 & 1 \\ 3 & 4 & 0 \end{vmatrix} = 0 - 3(0-3) + 2(4-(-6)) = 29 \ :$$

$$x = \frac{\begin{vmatrix} 7 & 3 & 2 \\ 12 & -2 & 1 \\ 6 & 4 & 0 \end{vmatrix}}{29} = \frac{110}{29}\ , \qquad y = \frac{\begin{vmatrix} 0 & 7 & 2 \\ 1 & 12 & 1 \\ 3 & 6 & 0 \end{vmatrix}}{29} = \frac{-39}{29}\ ,$$

$$z = \frac{\begin{vmatrix} 0 & 3 & 7 \\ 1 & -2 & 12 \\ 3 & 4 & 6 \end{vmatrix}}{29} = \frac{160}{29}\ .$$

Inverses via the "classical adjoint"

To find the columns of A^{-1}, we need to solve the n systems

$$A\mathbf{x}_1 = \mathbf{e}_1 \ , \quad \ldots \ , \quad A\mathbf{x}_n = \mathbf{e}_n$$

where the $\mathbf{e}$'s are the columns of the identity matrix. Suppose we solve them each by Cramer's rule, evaluating each $|B|$ by cofactors down the $\mathbf{e}$ column. Then, for example, to find the $(1, n)$-th entry in A^{-1}, we replace the first column in A with $\mathbf{e}_n$ and evaluate the determinant of the result down the first column — which gives the cofactor C_{n1} of A — and divide by $|A|$. In general,

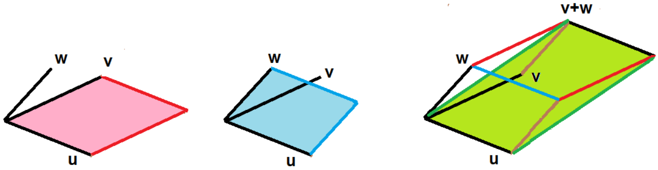
$$A^{-1} = (1/|A|) \begin{bmatrix} C_{11} & \ldots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \ldots & C_{nn} \end{bmatrix} .$$

Note this is the <u>transpose</u> of the obvious matrix of cofactors.

Determinants are signed areas

1. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is the area of the parallelogram with vertices (0,0), (a, c), (b, d) and $(a + b, c + d)$, with sign determined by whether (b, d) is counterclockwise or clockwise from (a, c).
2. $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is the volume of the parallelopiped from (0,0,0) determined by the columns, with sign determined by whether the columns give a right-handed system.

Why? Well, signed area has the three Basic Properties (the third is the hard one to see), so it is the determinant function.



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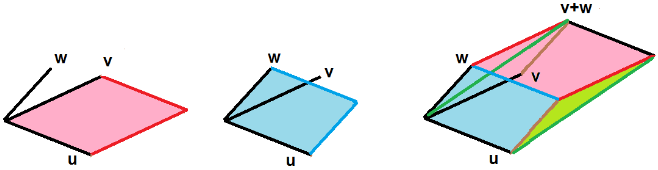
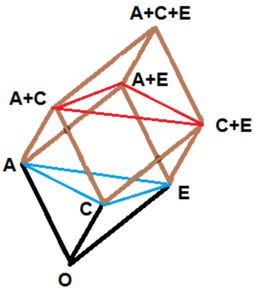
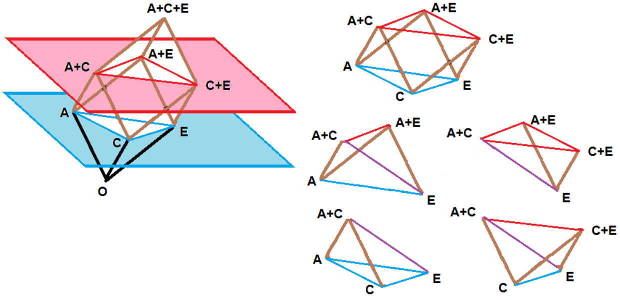


Area of a triangle not at (0,0)

Area $\mathfrak{A}$ of triangle with vertices $(a, b), (c, d), (e, f)$ is half the absolute value of $\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$.

Why? Volume of parallelopiped with these vertices in plane $x_3 = 1$ (call those points A, C, E) is that determinant.





But it is also sum of areas of:

- the upside-down pyramid with vertex (0,0,0) and base triangle A, C, E,
- four tetrahedra that assemble to give the piece between $x_3 = 1$ and $x_3 = 2$; and
- the pyramid from triangle $A + C, A + E, C + E$ in $x_3 = 2$ to vertex $A + C + E$ in $x_3 = 3$.

All 6 have same volume (4 clear, 2 harder).

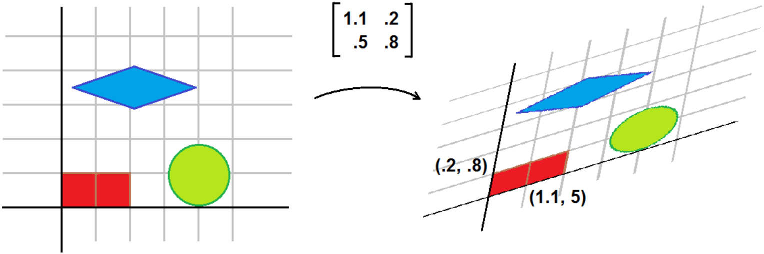
So

$$\det = \text{volume} = 6(\frac{1}{3}\mathfrak{A}) \quad \Longrightarrow \quad \mathfrak{A} = \frac{1}{2}\det \ .$$

Example: The area of the triangle with vertices (1,2), (2,5), $(-2,3)$ is

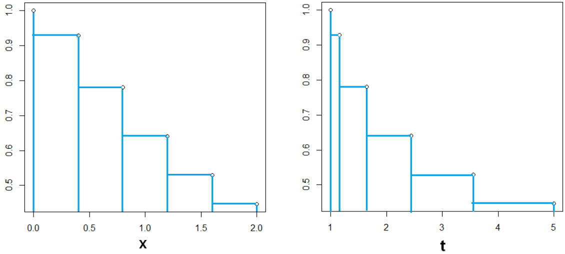
$$\frac{1}{2}\left|\det\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ -2 & 3 & 1 \end{bmatrix}\right| = 5 \ .$$

If a region $\mathcal{R}$ in the plane has area $\mathfrak{A}$ and the plane is transformed by multiplication by a matrix A, then each one of the unit squares in the plane is converted to a parallelogram with area $|A|$, so the area of the the transformed region $A(\mathcal{R})$ is $\mathfrak{A}|A|$.



Change of variable, one variable

If we set $t = 1 + x^2$, then the values of the functions $1/\sqrt{t}$ and $1/\sqrt{1+x^2}$ agree at corresponding values:

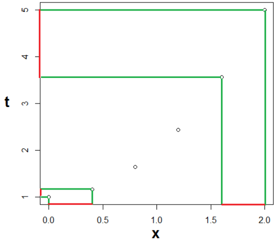


but clearly

$$\int_0^2 \frac{1}{\sqrt{1+x^2}}\,dx \neq \int_1^5 \frac{1}{\sqrt{t}}\,dt\ .$$

Change of variable, one variable, ctnd.

To get the correct equality for integrals, we need to adjust the — ultimately infinitesimal – widths of the rectangles, and that adjustment has to vary with the x and t values.



The equation $dt = 2x\,dx$ gives the (varying) horizontal adjustment in the widths from the x-line to the t-line:

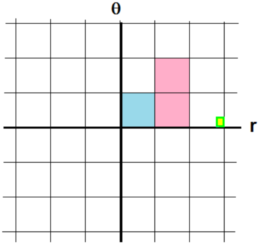
$$\int_0^2 \frac{2x}{\sqrt{1+x^2}}\,dx = \int_1^5 \frac{1}{\sqrt{t}}\,dt\ .$$

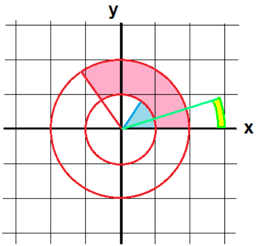
Change of variable, two variables

So how do we make a similar change of variable in multivariate integral?

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx = ?$$

The change of variable $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$ simplifies the integrand to $1/r$. But what about the varying change of (infinitesimal) area (in place of length) from $dr\, d\theta$ in the $r\theta$-plane to $dy\, dx$ in the xy-plane:





Change of variable, two variables, ctnd.

That change is done with the determinant of the matrix of partial derivatives, the *Jacobian*, which is the factor that the area of an infinitesimal square in the $r\theta$ plane changes by as it becomes an infinitesimal parallelogram in the xy plane:

$$\left| \begin{array}{cc} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{array} \right| = \left| \begin{array}{cc} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{array} \right| = r\cos^2\theta + r\sin^2\theta = r \; .$$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{x^2+y^2}} \, dy \, dx = \int_{-\pi}^{\pi} \int_0^1 \frac{1}{r} \, r \, dr \, d\theta = 2\pi \; .$$

In general, if the change of variable is $(x, y) = \mathbf{f}(u, v)$,

$$\begin{array}{ccc} \text{rectangle} & & \text{parallelogram} \\ du \left[\begin{array}{c} 1 \\ 0 \end{array} \right], \; dv \left[\begin{array}{c} 0 \\ 1 \end{array} \right] & \rightarrow & du \left[\begin{array}{c} \partial x/\partial u \\ \partial y/\partial u \end{array} \right], \; dv \left[\begin{array}{c} \partial x/\partial v \\ \partial y/\partial v \end{array} \right] \end{array}$$

and the area changes from $du\,dv$ to $J(\mathbf{f}) du\,dv$.