Example (from last time)

$$A = \begin{bmatrix} -7 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -7 \end{bmatrix} :$$

$$\begin{aligned}
\begin{vmatrix} -7-\lambda & 1 & 0 \\ 1 & -3-\lambda & 2 \\ 0 & 2 & -7-\lambda \end{vmatrix}
&= (-7-\lambda)^2(-3-\lambda) + 0 + 0 - 0 - 4(-7-\lambda) - 1(-7-\lambda) \\
&= (-7-\lambda)[21 + 10\lambda + \lambda^2 - 5] = (-7-\lambda)(\lambda^2 + 10\lambda + 16) \\
&= (-7-\lambda)(\lambda+2)(\lambda+8) \\
&\qquad = 0 \text{ when } \lambda = -7, -2, -8
\end{aligned}$$

Example (continued)

$$A = \begin{bmatrix} -7 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -7 \end{bmatrix} :$$

$$\lambda = -7 \; : \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = -2 \; : \quad \begin{bmatrix} -5 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} 1/2 \\ 5/2 \\ 1 \end{bmatrix}$$

$$\lambda = -8 \; : \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Proposition

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigvecs of A corresponding to <u>different</u> eigvals $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are lin ind.

Proof.

As an eigvec, $\mathbf{v}_1 \neq \mathbf{0}$. WLOG, $\mathbf{v}_k$ is the first one that is in span of earlier ones, say $\mathbf{v}_k = c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1}$. Multiply by A, multiply by λ_k and take difference:

$$\mathbf{0} = \sum_{i=1}^{k-1} c_i(\lambda_i - \lambda_k)\mathbf{v}_i \ .$$

All $(\lambda_i - \lambda_k)$'s are nonzero and some c_i is nonzero, say c_{k-1}, so

$$\mathbf{v}_{k-1} = \sum_{i=1}^{k-2} -\frac{c_i(\lambda_i - \lambda_k)}{c_{k-1}(\lambda_{k-1} - \lambda_k)}\mathbf{v}_i \ ,$$

contradicting our choice of $\mathbf{v}_k$. $\square$

Example (resumed)

So as eigvecs corresponding to different eigvals,

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 5/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

are lin ind, and $S = \begin{bmatrix} -2 & 1/2 & 1/2 \\ 0 & 5/2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix}$ is invertible.

Trace of a square matrix

The sum of the main diagonal entries of a square matrix is the *trace*. We have $\operatorname{trace}(A+B) = \operatorname{trace} A + \operatorname{trace} B$, but NOT $\operatorname{trace}(AB) = (\operatorname{trace} A)(\operatorname{trace} B)$.

Who cares? Well, it provides a check on the eigenvalues:

$$\begin{aligned} \det A &= \text{ product of the eigvals of } A \\ \operatorname{trace} A &= \text{ sum of the eigvals of } A \end{aligned}$$

Verifying the last claim

Consider characteristic polynomial

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = (\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda)$$

where λ_i's are the eigvals.

Constant term, when $\lambda = 0$, is $|A|$, but it is also product of λ_i's.

Coefficient of λ^{n-1} is $(-1)^{n-1}$ times the sum of the λ_i's, but also:

The only term in the Big Formula that has λ^{n-1} in it is from the main diagonal — the rest have at least 2 factors of constants. And the coefficient of λ^{n-1} in $(a_{11}-\lambda)(a_{22}-\lambda)\dots(a_{nn}-\lambda)$ is $(-1)^{n-1}$ times the sum of the a_{ii}'s, i.e., the trace.

Example again

$$A = \begin{bmatrix} -7 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -7 \end{bmatrix}, \text{ eigvals } -7, -2, -8:$$

$$\operatorname{trace} A = -7 + (-3) + -7 = -7 + -2 + -8$$

$$\begin{aligned} \det A &= (-7)^2(-3) + 0 + 0 - 0 - 4(-7) - 1(-7) \\ &= -112 = (-7)(-2)(-8) \end{aligned}$$

Surprise!

Because now we are trying to solve some polynomials (in λ) that are <u>not</u> linear, some roots may not be real:

Example

$C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix}$:

$$\begin{aligned} \begin{vmatrix} 5-\lambda & 7 \\ -3 & -4-\lambda \end{vmatrix} &= (5-\lambda)(-4-\lambda)+21 \\ &= 1-\lambda+\lambda^2 \\ &= 0 \text{ when } \lambda = \frac{1 \pm \sqrt{1-4}}{2} \\ &= \frac{1 \pm i\sqrt{3}}{2} \end{aligned}$$

The set $\mathbb{C}$ of complex numbers is also a field, so we can start taking "scalars" from there.

The best part is, a polynomial with coefficients from $\mathbb{C}$ has all its roots in $\mathbb{C}$, and we can always factor it into linear factors.

Continuing example

$$\lambda = \frac{1+i\sqrt{3}}{2} \quad : \quad \begin{bmatrix} \frac{9-i\sqrt{3}}{2} & 7 \\ -3 & \frac{-9-i\sqrt{3}}{2} \end{bmatrix} \to \begin{bmatrix} 1 & \frac{14}{9-i\sqrt{3}} \\ 0 & 0 \end{bmatrix} .$$

Note

$$\frac{14}{9-i\sqrt{3}} = \frac{14(9+i\sqrt{3})}{81+3} = \frac{9+i\sqrt{3}}{6} ,$$

so a corr eigvec is $\begin{bmatrix} -\frac{9+i\sqrt{3}}{6} \\ 1 \end{bmatrix}$.

An eigvec corr to $\lambda = \frac{1-i\sqrt{3}}{2}$ must be $\begin{bmatrix} -\frac{9-i\sqrt{3}}{6} \\ 1 \end{bmatrix}$ — take complex conjugates everywhere.

New World Order

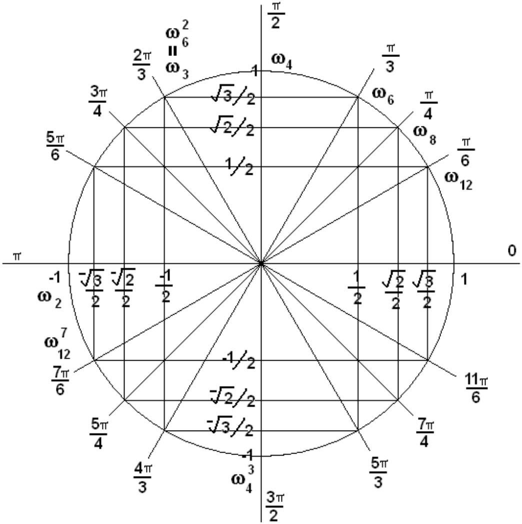
Just as we can picture real scalars as points on a line, we can think of complex scalars as points on a plane: $x + iy$ corresponds to the point (x, y). Complex numbers add like vectors, as usual. But multiplication is handier if we write complex numbers in "polar form": $r(\cos\theta + i\sin\theta)$, where r is the distance from the origin and θ is the angle with the positive real axis. (By a coincidence of Maclaurin series, we can abbreviate this to $re^{i\theta}$.)

$$
\begin{aligned}
r(\cos\theta + i\sin\theta)s(\cos\varphi + i\sin\varphi) \\
&= rs[(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(cos\theta\sin\varphi + \sin\theta\cos\varphi)] \\
&= rs[\cos(\theta+\varphi) + i\sin(\theta+\varphi)]
\end{aligned}
$$

Thus, to multiply two complex numbers, multiply their distances from the origin and <u>add</u> their angles from the positive real axis. (This looks even easier in exponential notation: $(re^{i\theta})(se^{i\varphi}) = (rs)e^{i(\theta+\varphi)}$.)

Roots of Unity

Thus, in $\mathbb{C}$, there are *roots of unity* spaced evenly around the unit circle: $\omega_n^n = 1$



Any point on the unit circle in the complex plane is $\cos\theta + i\sin\theta$, where θ is the angle to the positive real axis. So:

$$\omega_2 = \cos\pi + i\sin\pi = -1$$

$$\omega_4 = \cos(\pi/2) + i\sin(\pi/2) = i$$

$$\omega_4^2 = \cos\pi + i\sin\pi = \omega_2 = -1 \qquad\qquad \omega_4^3 = -i$$

$$\omega_3 = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\omega_3^2 = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\omega_6 = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\omega_6^2 = \omega_3 \qquad\qquad \omega_6^3 = \omega_2 = -1$$

$$\omega_8 = \cos(\pi/4) + i\sin(\pi/4) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

There are corresponding ω_n^k for all n, k, but the sines and cosines aren't nice. (Well, ω_5 isn't <u>too</u> bad.)

Eigstuff $\rightarrow$ Diagonalization

Theorem

Suppose the $n \times n$ matrix A has n lin ind eigvecs $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, with corr eigvals $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Form the matrix S with cols the $\mathbf{v}_i$'s in order, and the diagonal matrix Λ with main diagonal entries the λ_i's in order. Then $A = S\Lambda S^{-1}$.

Proof.

First, S is invertible. We show $S^{-1}AS = \Lambda$, one column at a time: First column of S is $\mathbf{v}_1$, and $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, and

$$S^{-1}\lambda_1\mathbf{v}_1 = \lambda_1(\text{first column of I}) = \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \text{first column of } \Lambda \; .$$

Similarly for the other columns. $\square$

Therefore,

$$A = \begin{bmatrix} -7 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1/2 & 1/2 \\ 0 & 5/2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} -2 & 1/2 & 1/2 \\ 0 & 5/2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{9+i\sqrt{3}}{6} & -\frac{9-i\sqrt{3}}{6} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{1-i\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -\frac{9+i\sqrt{3}}{6} & -\frac{9-i\sqrt{3}}{6} \\ 1 & 1 \end{bmatrix}^{-1}$$

First Application of Eigstuff

Powers of diagonal matrices are easy:

$$\begin{bmatrix} -7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -8 \end{bmatrix}^4 = \begin{bmatrix} (-7)^4 & 0 & 0 \\ 0 & (-2)^4 & 0 \\ 0 & 0 & (-8)^4 \end{bmatrix}$$

So if we can diagonalize a matrix, then we can find its powers easily:

$$A = S\Lambda S^{-1}, \text{ so}$$

$$A^4 = (S\Lambda S^{-1})(S\Lambda S^{-1})(S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^4 S^{-1} \ .$$

Example 1

$$\begin{bmatrix} -7 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -7 \end{bmatrix}^4 = \begin{bmatrix} -2 & 1/2 & 1/2 \\ 0 & 5/2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (-7)^4 & 0 & 0 \\ 0 & (-2)^4 & 0 \\ 0 & 0 & (-8)^4 \end{bmatrix} \begin{bmatrix} -2 & 1/2 & 1/2 \\ 0 & 5/2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

Example 2

More interesting: Because $\frac{1+i\sqrt{3}}{2} = \omega_6$ and $\frac{1-i\sqrt{3}}{2} = \omega_6^5$,

$$C = S \begin{bmatrix} \omega_6 & 0 \\ 0 & \omega_6^5 \end{bmatrix} S^{-1}, \text{ so}$$

$$\begin{aligned} C^6 &= S \begin{bmatrix} \omega_6^6 & 0 \\ 0 & \omega_6^{30} \end{bmatrix} S^{-1} = S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S^{-1} = I \\ C^3 &= -I \end{aligned}$$

Because $1024 = 6(170) + 4$ by long division, we have

$$C^{1024} = (C^6)^{170} C^3 C = I^{170}(-I)C = -C \ .$$

A Discrete Linear Process

Example

A gene can be in one of two forms, dominant A or recessive b; and an animal in which this gene appears can be one of three "genotypes": pure dominant AA, pure recessive bb, or hybrid Ab. Suppose we always mate females with hybrid males and keep the female offspring. In the long run, what will the distribution of genotypes in the population we keep?

Suppose the distribution was initially d_0 dominant, h_0 hybrid and r_0 recessive (d_0, h_0, r_0 fractions adding to 1). After one breeding season, the distribution is

$$\begin{aligned} d_1 &= \frac{1}{2}d_0 + \frac{1}{4}h_0 \\ h_1 &= \frac{1}{2}d_0 + \frac{1}{2}h_0 + \frac{1}{2}r_0 \\ r_1 &= \frac{1}{4}h_0 + \frac{1}{2}r_0 \end{aligned} \quad \text{or} \quad \begin{bmatrix} d_1 \\ h_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} d_0 \\ h_0 \\ r_0 \end{bmatrix}$$

Write $\mathbf{f}_n = \begin{bmatrix} d_n \\ h_n \\ r_n \end{bmatrix}$ and $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$.

Then $\mathbf{f}_{n+1} = A\mathbf{f}_n$ for each n, and we are asking what $\mathbf{f}_n$ approaches as $n \to \infty$. Using `R`, I can check

$$A^{1024} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \text{ so } \mathbf{f}_{1024} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$$

(with some error so small that `R` won't show it to me), no matter what $\mathbf{f}_0$ was, because $d_0 + h_0 + r_0 = 1$.

But why did all the columns of A^{1024} come out the same?

$$\begin{vmatrix} \frac{1}{2}-\lambda & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2}-\lambda & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2}-\lambda \end{vmatrix} = (\frac{1}{2}-\lambda)(-\lambda+\lambda^2)$$

so eigvals of A are $\frac{1}{2}, 0, 1$. Suppose corresponding eigvecs are $\mathbf{u}, \mathbf{v}, \mathbf{w}$ respectively, a basis for $\mathbb{R}^3$. So if

$$\mathbf{f}_0 = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \ , \quad \text{then}$$

$$\mathbf{f}_{1024} = A^{1024}\mathbf{f}_0 = a(\frac{1}{2})^{1024}\mathbf{u} + b(0)^{1024}\mathbf{v} + c(1)^{1024}\mathbf{w} \approx c\mathbf{w} \ .$$

Find $\mathbf{w}$: When $\lambda = 1$,

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$$

Conclusion: If A is diagonalizable and 1 is the largest eigval of A in absolute value, then $A^n\mathbf{f}$ is going to approach an eigvec $\mathbf{w}$ of A corresponding to 1, i.e., a *steady state:* $A\mathbf{w} = \mathbf{w}$. Because the columns of A^n are just A^n times the columns of I, each column of A^n will also go to such an eigvec.

Suppose the columns of A all add up to 1. Then the columns of $A - I$ add up to 0, so $A - I$ is singular; i.e., 1 <u>is</u> an eigval of A. If it's the largest in absolute value and the eigenspace corr to it has dim 1, then all the columns of A^n will go to the same vector, the eigvec with sum 1.

And that was true of $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$.

One Last Pitfall

If an $n \times n$ matrix A has n different eigvals, then each has an eigvec; these form a basis for $\mathbb{R}^n$ or $\mathbb{C}^n$, and A is diagonalizable.

But if A has a <u>repeated</u> eigval, there may or may not be enough eigvecs to give a basis.

Example

$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is diagonalizable: $S = I$, $\Lambda = A$. Char poly is $(3-\lambda)^2$, with repeated root 3:

$$\begin{bmatrix} 3-3 & 0 \\ 0 & 3-3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} : \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigspace corr to 3 is all of $\mathbb{R}^2$, so we can pick a basis for $\mathbb{R}^2$ consisting of eigvecs of A.

Example

$B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable: Char poly is again $(3-\lambda)^2$, with repeated root 3, and

$$\begin{bmatrix} 3-3 & 1 \\ 0 & 3-3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} : \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Eigval 3 has *algebraic multiplicity* 2, but *geometric multiplicity* only 1. We can't make a basis of $\mathbb{R}^2$ consisting of eigvecs of B, so B isn't diagonalizable.