Symmetric matrices and dot products

Proposition

An $n \times n$ matrix A is symmetric iff, for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$.

Proof.

If A is symmetric, then $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = \mathbf{x}^T A\mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$.

If equality holds for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, let $\mathbf{x}, \mathbf{y}$ vary over the standard basis of $\mathbb{R}^n$. □

Corollary

If A is symmetric and $\mathbf{x}, \mathbf{y}$ are eigvecs corresponding to <u>different</u> eigvals λ, μ, then $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof.

$\lambda\mathbf{x} \cdot \mathbf{y} = (A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y}) = \mathbf{x} \cdot (\mu\mathbf{y}) = \mu\mathbf{x} \cdot \mathbf{y}$, so $(\lambda - \mu)(\mathbf{x} \cdot \mathbf{y}) = 0$, but $\lambda - \mu \neq 0$. □

Proposition

Every eigenvalue of a symmetric matrix (with real entries) is real, and we can pick the corresponding eigenvector to have real entries.

Proof.

Suppose A is symmetric and $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$ (maybe with complex entries). Then $A\overline{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$, so

$$\lambda(\mathbf{x} \cdot \overline{\mathbf{x}}) = (A\mathbf{x}) \cdot \overline{\mathbf{x}} = \mathbf{x} \cdot (A\overline{\mathbf{x}}) = \overline{\lambda}(\mathbf{x} \cdot \overline{\mathbf{x}}) \ ;$$

but $\mathbf{x} \cdot \overline{\mathbf{x}}$ is real and positive, so $\overline{\lambda} = \lambda$.

And if $\mathbf{x}$ is an eigvec with complex entries corr to λ, then we can multiply by some complex number and get at least one real nonzero entry, and then $\mathbf{x} + \overline{\mathbf{x}}$ is an eigvec corr to λ with real entries. □

Lemma

If A is a symmetric matrix with (real) eigvec $\mathbf{x}$, then if $\mathbf{y}$ is perpendicular to $\mathbf{x}$, then so is $A\mathbf{y}$ (i.e., A multiplies the orthogonal complement $(\mathbb{R}\mathbf{x})^\perp$ into itself).

Proof.

Let λ be the corr eigval. Because $\mathbf{y} \cdot \mathbf{x} = 0$, we have

$$(A\mathbf{y}) \cdot \mathbf{x} = \mathbf{y} \cdot (A\mathbf{x}) = \mathbf{y} \cdot (\lambda \mathbf{x}) = \lambda(\mathbf{y} \cdot \mathbf{x}) = 0 \ .$$

□

Symmetrics have orthogonal diagonalization

Theorem (Spectral Theorem)
If A is symmetric, then there is an orthogonal matrix Q and a diagonal matrix Λ for which $A = Q\Lambda Q^T$.

Proof
Let $\lambda_1, \mathbf{x}_1$ be a real eigval and eigvec of A, with $||\mathbf{x}|| = 1$. Pick an orthonormal basis $\mathcal{B}$ for $(\mathbb{R}\mathbf{x})^\perp$; then $\mathbf{x}, \mathcal{B}$ form an orthonormal basis for $\mathbb{R}^n$. Use them as columns of an orthogonal matrix Q_1, with $\mathbf{x}_1$ first; then because A multiplies $(\mathbb{R}\mathbf{x})^\perp$ into itself, we have

$$A = Q_1 \begin{bmatrix} \lambda_1 & 0 \\ 0 & A_2 \end{bmatrix} Q_1^T \ .$$

Because

$$Q_1 \begin{bmatrix} \lambda_1 & 0 \\ 0 & A_2 \end{bmatrix} Q_1^T = A = A^T = Q_1 \begin{bmatrix} \lambda_1 & 0 \\ 0 & A_2^T \end{bmatrix} Q_1^T \ ,$$

A_2 is still symmetric; so we can repeat the process with A_2; and if

$$A_2 = Q_2 \begin{bmatrix} \lambda_2 & 0 \\ 0 & A_3 \end{bmatrix} Q_2^T$$

then

$$A = Q_1 \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q_2^T \end{bmatrix} Q_1^T \ .$$

Continuing, we get to the desired form. □

Example

```
> A
     [,1] [,2] [,3]
[1,]   -2    4    4
[2,]    4    1    1
[3,]    4    1    1
> eigen(A)
$values
[1]  6.000000e+00 -1.933132e-16 -6.000000e+00

$vectors
          [,1]       [,2]       [,3]
[1,] 0.5773503  0.0000000  0.8164966
[2,] 0.5773503 -0.7071068 -0.4082483
[3,] 0.5773503  0.7071068 -0.4082483
```

For a symmetic A, R gives orthonormal eigenvectors (and eigenvalues in decreasing order).

Suppose we have A symmetric, $Q = \left[\begin{array}{ccc} \mathbf{x}_1 & \dots & \mathbf{x}_n \end{array} \right]$ and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ with $A = Q\Lambda Q^T$. Write

$$\Lambda = \left[\begin{array}{cccc} \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \dots & 0 \end{array} \right] + \cdots + \left[\begin{array}{cccc} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{array} \right].$$

Then

$$
\begin{aligned}
& Q \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \dots & 0 \end{bmatrix} Q^T \\
&= \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots \mathbf{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_n^T \end{bmatrix} \\
&= \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T
\end{aligned}
$$

And so on, so we get

$$A = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \cdots + \lambda_n \mathbf{x}_n \mathbf{x}_n^T \ .$$

The $\mathbf{x}_i \mathbf{x}_i^T$'s are the projection matrices P_i's on the subspaces $\mathbb{R}\mathbf{x}_i$'s. (And they are rank 1 matrices.)

Because they are the orthogonal projections onto an orthonormal basis, they add up to I — if we apply them all to the same vector and add up the results, we'll get back the original vector.

Example:

$$A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}:$$

$$\lambda_1 = 5,\ \mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \qquad \lambda_2 = 10,\ \mathbf{x}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\begin{aligned} A &= 5 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \\ &\quad + 10 \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \\ &= 5 \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} + 10 \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} \end{aligned}$$

and $\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} + \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} = I.$

Definition

A *quadratic form* on $\mathbb{R}^n$ is a function $q : \mathbb{R}^n \to \mathbb{R}$ given by a polynomial in which every term has degree 2: $q(\mathbf{x}) = \sum_{i,j} a_{i,j} x_i x_j$. Because $x_i x_j = x_j x_i$, we can assume $a_{i,j} = a_{j,i}$, and rewrite $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ where A is symmetric.

- If $q(\mathbf{x}) > 0$ for every nonzero $\mathbf{x}$, then q and A are *positive definite.*
- If $q(\mathbf{x}) \geq 0$ for every $\mathbf{x}$, then they are *positive semidefinite.*

Example 1

$$x_1^2 + 2x_2^2 - 3x_3^2 - 4x_1x_2 + x_1x_3 + 2x_2x_3 = \mathbf{x}^T \begin{bmatrix} 1 & -2 & 1/2 \\ -2 & 2 & 1 \\ 1/2 & 1 & -3 \end{bmatrix} \mathbf{x}$$

Because of the -3 on the main diagonal, this quadratic form is not even pos semidef (nor is its associated matrix):

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1/2 \\ -2 & 2 & 1 \\ 1/2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -3$$

Moral: If any main diagonal entry is ≤ 0, the matrix is not pos def.

Example 2

Positive definite:

$$
\begin{aligned}
(x+3y)^2 + 4(2x-y)^2 &= 17x^2 - 10xy + 13y^2 \\
&= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 17 & -5 \\ -5 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{aligned}
$$

Example 3

$$
\begin{aligned}
&(x+3y)^2 + 4(2y-z)^2 + 2(2x+3z)^2 \\
&\qquad = 9x^2 + 25y^2 + 22z^2 + 6xy - 16yz + 24xz \\
&\qquad = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 9 & 3 & 12 \\ 3 & 25 & -8 \\ 12 & -8 & 22 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{aligned}
$$

Certainly always nonnegative, but if $x = -3y = -3z/2$, all 3 squares are 0; so pos semidef.

Example 4

What about $f(x,y,z) = 23x^2 + 14y^2 + 8z^2 - 28xy + 4xz + 32yz$?

$$\begin{aligned} f(x,y) &= 23(x^2 + \frac{4}{23}xz + \frac{4}{529}z^2) + 14(y^2 - \frac{16}{7}yz + \frac{64}{49}z^2) \\ &\quad + \left(8 - \frac{4}{23} - \frac{128}{7}\right) z^2 - 28xy \\ &= 23(x + \frac{2}{23}z)^2 + 14(y - \frac{8}{7}z)^2 - \frac{1684}{161}z^2 - 28xy \ . \end{aligned}$$

Can we make this negative for some x, y, z? Well, try setting $x = -(2/23)z$ and $y = (8/7)z$: That will make f into $-(1652/161)z^2$, which is negative for any $z \neq 0$. So f is not even pos semidef.

Application: Second Derivative Test

Suppose $z = f(x, y)$ is a function of two variables, and at a point (a, b) the first derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are both 0. Then the Taylor series for f at that point is

$$\begin{aligned} f(a,b) &+ \frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + \frac{\partial^2 f}{\partial x\,\partial y}(a,b)(x-a)(y-b) \\ &+ \frac{\partial^2 f}{\partial y\,\partial x}(a,b)(y-b)(x-a) + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2 + \dots \end{aligned}$$

where the rest is higher-degree terms. So near (a, b), f behaves like the quadratic part, which is a quadratic form. When this form is positive definite, the smallest value f takes on in a small neighborhood of (a, b) <u>at</u> (a, b). In other words, (a, b) is a local minimum of f.

So it would be useful to have a quick way to decide whether the matrix is positive definite:

$$\begin{bmatrix} \frac{\partial^2 f}{\partial^2 x}(a,b) & \frac{\partial^2 f}{\partial x\,\partial y}(a,b) \\ \frac{\partial^2 f}{\partial y\,\partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{bmatrix}$$

Theorem

Let A be a symmetric matrix. TFAE:

(a) *All pivots of A are positive.*

(b) *All upper left subdeterminants are positive. (This is* **Sylvester's criterion***.)*

(c) *All eigenvalues of A are positive.*

(d) *The quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite. (This is the* **energy-based** *definition.)*

(e) $A = R^T R$ *where R has independent columns.*

Proof

(a) $\implies$ (b): $a_{1,1}$ is first pivot, hence pos, and first subdet. By row ops, upper left subdets of A are equal to subdets of a matrix, say B, with the rest of the first column 0's. Second entry on B's main diagonal is A's 2nd pivot; because it and $a_{1,1}$ are pos, B's 2nd upper left subdet is also pos. Etc.

(b) $\implies$ (a): First subdet is $a_{1,1} > 0$, first pivot. Repeat the rest of (a) $\implies$ (b), except reversing causality: k-th pivot is pos <u>because</u> $k \times k$ subdet is pos.

(c) $\iff$ (d): Write $A = Q\Lambda Q^T$; then

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T Q\Lambda Q^T \mathbf{x} = (Q^T\mathbf{x})^T \Lambda (Q^T\mathbf{x}) = \sum_i \lambda_i y_i^2 \ ,$$

where the y_i's are the entries of $Q^T\mathbf{x}$. So if all the λ_i's are positive, $\mathbf{x}^T A\mathbf{x}$ is positive definite; while if one is negative, then there are $\mathbf{x}$'s for which $\mathbf{x}^T A\mathbf{x}$ is negative.

(c) $\implies$ (e): Write $A = Q\Lambda Q^T$. Then by (c), $\sqrt{\Lambda}$ makes sense and is invertible. Set $R = Q\sqrt{\Lambda}Q^T$. Then $A = R^T R$, and the columns of R are independent.

(e) $\implies$ (d): Given $A = R^T R$, $\mathbf{x}^T A\mathbf{x} = (R\mathbf{x}) \cdot (R\mathbf{x}) = ||R\mathbf{x}||^2 \geq 0$. And if R has ind cols, then the only $\mathbf{x}$ with $||R\mathbf{x}||^2 = 0$ is $\mathbf{x} = \mathbf{0}$.

(a) $\implies$ (e): Because A is symmetric, (a) says it has an LDU-decomposition $A = LDL^T$ where L is lower triangular with 1's on the main diagonal, and D is diagonal with the pivots of A on its diagonal. Set $R = \sqrt{D}L^T$.

(d) $\implies$ (b): Let $b_k =$ the upper left $k \times k$ subdet of A, and $c_{kj} =$ the cofactor of a_{kj} if we evaluate b_k along its last row. Then for $i < k$, $\sum_{j=1}^{k} a_{ij}c_{kj}$ is the value of the same det as b_k except that its last row is equal to its i-th row, so it is 0; but $\sum_{j=1}^{k} a_{kj}c_{kj} = b_k$. Note $c_{kk} = b_{k-1}$. We prove by induction (and pos defness) that all b_k's are pos:

Because A is pos def, $b_1 = a_{11} > 0$. Assume $b_{k-1} > 0$. Set

$$\mathbf{x} = (c_{k1}, c_{k2}, \ldots, c_{kk}, 0, \ldots, 0) .$$

Then

$$0 < \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(0, \ldots, 0, b_k, ?, \ldots, ?) = b_{k-1}b_k \ ,$$

so $b_k > 0$. $\square$

Notes on proof:

(e): The proof of (c) $\implies$ (e) gives one square, in fact symmetric, R that works (the **Cholesky decomposition**), but any R with independent columns, even if not square, works.

Example:

$A = R^T R = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix}$, and R has independent columns, so A is positive definite.

Therefore, the eigenvalues of $-A$ are all negative, so the solutions of

$$\frac{d\mathbf{u}}{dt} = -A\mathbf{u}$$

approach 0 as $t \to \infty$.

(c): The (c) $\implies$ (d) part of the proof shows us the right way to complete squares in quadratic forms. (Example, next page.)

Example 4, revisited:

For $f(x,y,z) = 23x^2 + 14y^2 + 8z^2 - 28xy + 4xz + 32yz$, the matrix is

$$A = \begin{bmatrix} 23 & -14 & 2 \\ -14 & 14 & 16 \\ 2 & 16 & 8 \end{bmatrix} = Q\Lambda Q^T \text{ where}$$

$$Q = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} -9 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{bmatrix}, \quad \text{so}$$

$$\begin{aligned} \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T Q\Lambda Q^T\mathbf{x} &= (Q^T\mathbf{x})^T\Lambda(Q^T\mathbf{x}) \\ &= -9(\frac{1}{3}x + \frac{2}{3}y - \frac{2}{3}z)^2 + 18(\frac{2}{3}x + \frac{1}{3}y + \frac{2}{3}z)^2 \\ &\quad + 36(-\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z)^2 ; \end{aligned}$$

again, not even pos semidef.

Proposition

A symmetric matrix is positive semidefinite iff its eigenvalues are nonnegative, or equivalently iff it has the form $R^T R$.

Proof.

Same as the positive definite case, except that, if the cols of R are not ind, there are nonzero $\mathbf{x}$'s for which $R\mathbf{x} = \mathbf{0}$. $\square$

Example 1 revisited:

$$\mathbf{x}^T \begin{bmatrix} 1 & -2 & 1/2 \\ -2 & 2 & 1 \\ 1/2 & 1 & -3 \end{bmatrix} \mathbf{x}$$

$$\det[1] = 1, \ \det \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} = -2, \ \det \begin{bmatrix} 1 & -2 & 1/2 \\ -2 & 2 & 1 \\ 1/2 & 1 & -3 \end{bmatrix} = 5/2$$

Not even positive semidefinite. Eigenvalues from `R`: 3.5978953, -0.2047827, -3.3931126

Example 2 revisited:

Positive definite:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 17 & -5 \\ -5 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det[17] = 17, \ \det \begin{bmatrix} 17 & -5 \\ -5 & 13 \end{bmatrix} = 196$$

Eigenvalues from `R`: 20.385165, 9.614835

Example 3 revisited:

$$
\begin{aligned}
(x+3y)^2 + 4(2y-z)^2 + 2(2x+3z)^2 \\
&= 9x^2 + 25y^2 + 22z^2 + 6xy - 16yz + 24xz \\
&= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 9 & 3 & 12 \\ 3 & 25 & -8 \\ 12 & -8 & 22 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{aligned}
$$

$$
\det[9] = 9 \ , \quad \det \begin{bmatrix} 9 & 3 \\ 3 & 25 \end{bmatrix} = 216 \ , \quad \det \begin{bmatrix} 9 & 3 & 12 \\ 3 & 25 & -8 \\ 12 & -8 & 22 \end{bmatrix} = 0 \ ;
$$

pos semidef. (Eigvals from R: 33.29150, 22.70850, 0.)

Example 4 re-revisited:

$$23x^2 + 14y^2 + 8z^2 - 28xy + 4xz + 32yz = \mathbf{x}^T A\mathbf{x} \text{ where}$$

$$A = \begin{bmatrix} 23 & -14 & 2 \\ -14 & 14 & 16 \\ 2 & 16 & 8 \end{bmatrix}$$

$$\det[23] = 23 \ , \quad \det \begin{bmatrix} 23 & -14 \\ -14 & 14 \end{bmatrix} = 126 \ , \quad \det A = -5832 \ ,$$

so not pos def. (We saw eigvals were -9, 18, 36.)

Application: Tilted Ellipses

The equation of an ellipse in standard position is $x^2/a^2 + y^2/b^2 = 1$, i.e.,

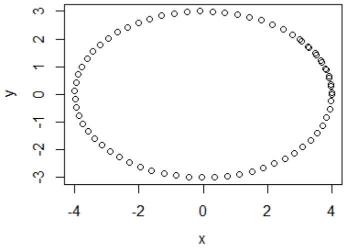
$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \, .$$

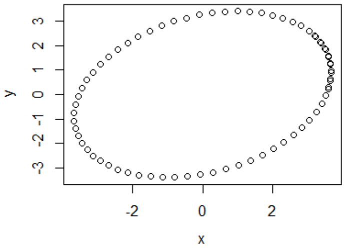
The semiaxes are a and b (the positive square roots of the reciprocals of the eigvals). But a rotation matrix Q gives a tilted ellipse. The axes are in the directions of the columns of Q.

Example 5

$$\Lambda = \begin{bmatrix} 1/16 & 0 \\ 0 & 1/9 \end{bmatrix}, \quad Q = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}:$$

$$A = Q\Lambda Q^T = \begin{bmatrix} 288/3600 & -84/3600 \\ -84/3600 & 337/3600 \end{bmatrix}$$





$$\frac{1}{16}x^2 + \frac{1}{9}y^2 = 1 \qquad \frac{288}{3600}x^2 - \frac{168}{3600}xy + \frac{337}{3600}y^2 = 1$$

Example 6

Describe the ellipse $13x^2 - 6\sqrt{3}xy + 7y^2 = 4$.

Divide by 4:

$$\frac{13}{4}x^2 - \frac{3\sqrt{3}}{2}xy + \frac{7}{4}y^2 = 1$$

The quadratic form has matrix

$$A = \begin{bmatrix} 13/4 & -3\sqrt{3}/4 \\ -3\sqrt{3}/4 & 7/4 \end{bmatrix} = Q\Lambda Q^T$$

where

$$Q = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

So the ellipse has axes tilted 60° from the xy-axes, with half-lengths 1 and $1/2$.

Example 7

Describe the ellipse $x^2 + 6xy + 4y^2 = 16$.

The matrix of the quadratic form is

$$A = \begin{bmatrix} 1/16 & 3/16 \\ 3/16 & 1/4 \end{bmatrix},$$

and R shows that its eigvals are 0.35688137 and -0.05338137. The negative eigenvalue means that it is not an ellipse. The graph of the quadratic form $z = (x^2 + 6xy + 4y^2)/16$ is a saddle, and the cross section $z = 1$ is a hyperbola.

Or,

$$\det[1/16] = 1/16, \ \det \begin{bmatrix} 1/16 & 3/16 \\ 3/16 & 1/4 \end{bmatrix} = -5/256 < 0$$