Test 1
Math 310
October 1, 2003

NAME__________________________________
Please, print!

- No calculators needed. I have kept the computations simple.
- If you have any questions, please raise your hand and ask. The worst that will happen is that I will say, "I can't tell you."
- Do the problems that you find easiest first.
- On proofs, state your game plan even if you are unable to complete it. Coherent thoughts that relate directly to the proof can be worth points even without a complete proof.
- There are 150 points on this exam. It is written to be taken in 3 hours, and you have unlimited time.
- I hope you all do well. Good luck!

1. (Total 15 pts.) Consider the traffic situation at the intersection, depicted in Figure 1, involving the following 10 traffic subjects:

   - four pedestrians: $P1, P2, P3, P4$,
   - two cars going straight: $S1, S2$,
   - two cars turning right: $R1, R2$ and
   - two cars turning left: $L1, L2$.

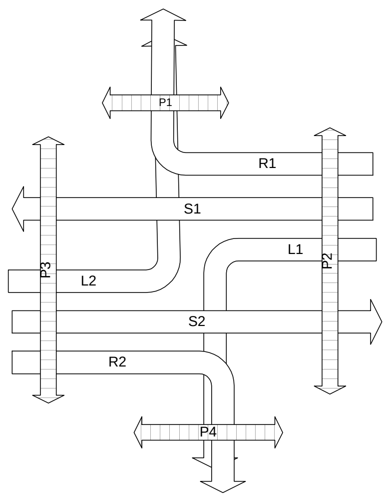

Figure 1: Intersection with traffic lights.

   (a) (5 pts.) Draw the graph G, representing the model of this intersection.

(b) (2 pts.) Determine the number of vertices v and edges e of G.

(c) (3 pts.) Determine whether G is planar or not.

(d) (3 pts.) Determine the chromatic number of G and give its interpretation in terms of traffic lights.

(e) (2 pts.) Find a Hamilton circuit in G or prove that there is none.

2. (Total 15 pts.) Let G be a graph on n vertices. The sequence $[d_1, d_2, \ldots, d_n]$ is called the **degree sequence** of G if $d_1 \geq d_2 \geq \cdots \geq d_n$ and there is a labelling of vertices $x_1, x_2, ..., x_n$ of G such that the vertex x_i has degree d_i.

   (a) (2 pts.) Determine the degree sequence of $K_{2,3,7}$.

   (b) (2 pts.) Determine the degree sequence of $K_{r,s,t}$.

   (c) (2 pts.) A sequence $[d_1, d_2, \ldots, d_n]$ consisting of non-negative integers: $d_1 \geq d_2 \geq ... \geq d_n$ is called **graphical** if it is a degree sequence of some graph. Show that $[1, 1, 0]$ is a graphical sequence and show that $[1, 1, 1]$ is not a graphical sequence.

   (d) (3 pts.) Prove the following statement. If $[d_1, d_2, \ldots, d_n]$ is a graphical sequence, then $[d_1, d_2, \ldots, d_n, 0]$ is also a graphical sequence.

(e) (3 pts.) Prove the following statement. Let $[d_1, d_2, \ldots, d_n]$ be a graphical sequence and k be an integer, such that $d_1 \leq k \leq n$ then $[k, d_1 + 1, d_2 + 1, ...d_k + 1, d_{k+1}, d_{k+2}, ..., d_n]$ is also a graphical sequence.

(f) (3 pts.) Prove the following statement. Sequence $[k, d_2, \ldots, d_n]$ is graphical if and only if an appropriate permutation of $[d_2 - 1, d_3 - 1, ...d_{k+1} - 1, d_{k+2}, d_{k+3}, ..., d_n]$ is a graphical sequence.

3. (Total 15 pts.) In the following consider the graphs determined by moves of chess pieces (chessmen): Queen (Q), King (K), Rook (R) and Bishop (B) on an $n \times n$ chessboard. The corresponding graphs are denoted by $Q(n), K(n), R(n), B(n)$. So $Q(n)$ has vertices squares of the chessboard and two squares s_1 and s_2 are adjacent if and only if a queen can move from s_1 to s_2 in a single move.

   (a) (4 pts.) Draw the following graphs: $Q(3), K(3), R(3), B(3)$.

   (b) (4 pts.) Prove that for no value of $n, n > 1$, the graphs $K(n)$ contains an Euler cycle.

   (c) (3 pts.) Prove that for no value of $n, n > 1$, the graphs $B(n)$ contains an Euler cycle.

   (d) (4 pts.) Determine all values of $n, n > 1$, for which the graph $R(n)$ contains an Euler cycle.

4. (Total 15 pts.) A **path** P_n with endvertices u and v is a graph with vertices $u = x_1, x_2, ..., x_n = v$ and the following andjacencies: $x_1 \sim x_2, x_2 \sim x_3, \ldots, x_{n-1} \sim x_n$.

   A **sun** with k **rays** and **range** n_1 is a rooted tree $S(k; n_1, n_2, ..., n_k), n_1 \geq n_2 \geq ... \geq n_k$ formed out of k disjoint paths $P_{n_1+1}, P_{n_2+1}, ..., P_{n_k+1}$ by identifying a common endvertex, the root.

   (a) (5 pts.) Determine the number of suns on $1, 2, 3, 4, 5$ and 6 vertices and fill in this table.

n	1	2	3	4	5	6
# suns						

   (b) (5 pts.) Determine all suns that are isomorphic (as unrooted graphs) to a path.

   (c) (5 pts.) Determine which suns $S(k;, n_1, n_2, ..., n_k)$ have two centers and none of them is the root.

5. (Total 15 pts.)

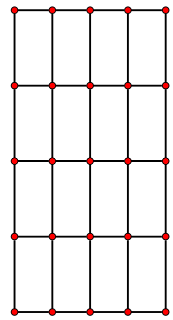

Figure 2: The grid graph $G(5, 5)$.

A **grid graph** $G(m, n)$ is a graph on $m \times n$ vertices, obtained by interlocking m horizontal copies of P_n and n vertical copies of P_m. For instance, a circuit graph C_4 can be regarded as $G(2, 2)$.

(a) (2 pts.) Prove that no sun with 5 rays is a spanning tree of a grid graph $G(m, n)$.

(b) (3 pts.) For each $m \geq n \geq 1$ determine the minimal number of colors needed to color properly the edges of $G(m, n)$.

(c) (5 pts.) Prove that each grid graph $G(m, n)$ contains a Hamilton path.

(d) (5 pts.) Determine all values of m and n for which $G(m, n)$ contains a Hamilton circuit.

6. (Total 15 pts.)

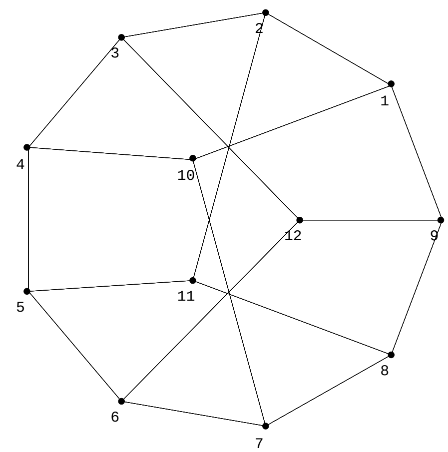

Figure 3: Graph G_3.

(a) (5 pts.) Find the length of the shortest circuit in G_3 from Figure 3.

(b) (5 pts.) Assume G_3 is planar. Determine the number of regions of any plane depiction of G_3.

(c) 5 pts. Prove that the graph G_3 is non-planar. [Hint: Use (a) and prove contradiction to (b).]

7. (Total 15 pts.) A **generalized Petersen graph** $P(n, r)$ has $2n$ vertices: $u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}$ and edges $u_i u_{i+1}, u_i v_i, v_i v_{i+r}$ (where addition is taken mod n).

   A **triangle** is a circuit of length 3.

   A graph is **cubic** if each of its vertices has degree 3.

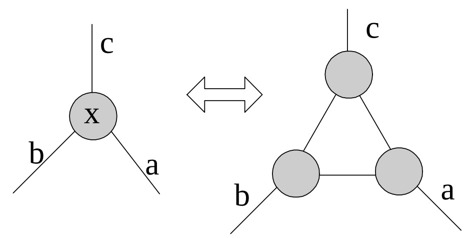

Figure 4: Truncation and triangle contraction.

   A **truncation** of a vertex x in a cubic graph G is a process in which x is replaced by a triangle to which the edges a, b, c that were incident to x are reattached.

   A **triangle contraction** in a cubic graph is the process opposite to a vertex truncation.

   (a) (1 pt.) Determine the graph that you obtain from $P(6, 2)$ after all triangle contractions.

   (b) (2 pts.) Determine all values of n and r for which the generalized Petersen graph contains 2 triangles.

(c) (2 pts.) Determine all values of n and r for which the generalized Petersen graph contains 3 triangles.

(d) (2 pts.) Prove that is $P(n, r)$ contains a triangle, then n must be a multiple of 3.

(e) 2 pts. Prove that a cubic graph is planar if and only if any of its vertex truncations is planar.

(f) 3 pts. Prove that any generalized Petersen graph containing at least three triangles is non-planar.

8. (Total 15 pts.) In the following a path P_n is labelled $1, 2, ..., n$, consecutively from one end-vertex to the other one. Please circle the source for the definition of your Prüfer sequence.

Notes Text

(a) (3 pts.) Determine the Prüfer sequence for the path P_5.

(b) (3 pts.) Find the general form for the Prüfer sequence for the labelled path P_n.

(c) (3 pts.) Prove that the formula you obtained in part (b) is true. [Hint: use induction.]

(d) (3 pts.) Determine the tree that has the following Prüfer sequence.

$$[1, 1, ..., 1](n - 2 \text{ - ones}).$$

(e) (3 pts.) How many leaves does the tree with the following Prüfer sequence have?

$$[2, 4, 6, 8, 2, 4, 6, 8, 2, 4, 6, 8].$$

9. (Total 15 pts.) A group of four people have to cross a bridge that will accommodate only two people at a time. It is night and there is one flashlight. Any party who crosses, either one or two people, must have a flashlight with them. Person A takes 1 minute, to cross the bridge, B takes 2 minutes, C takes 5 minutes, D takes 10 minutes. When two people cross, they must go with the speed of the slower person.

   (a) (3 pts.)Describe the state space associated with this puzzle. (States, initial state, goal states, rules)

   (b) (3 pts.) Determine the number of states in the whole state space. Explain.

(c) (3 pts.) Prove that the state space (if you ignore directions of edges) is a bipartite graph.

(d) (3 pts.) What is the minimal number of crossings that are needed to reach the goal state?

(e) (3 pts.) Find a solution that requires the minimal number of crossings but takes the longest time.

(f) (3 pts.) Find a solution that requires the minimal number of crossings and also takes the shortest possible time.

10. (Total 15 pts.) Let S be any finite set of binary strings: Let $G(S)$ be the graph obtained from the set S in the following way: the vertices are the binary strings from S. Two strings are adjacent if and only if one can be obtained from the other either by dropping the last digit or by adding a digit. This means, for instance, that `"000"` is adjacent to `"00"`, `"0000"`, and to `"0001"` but `"0001"` is not adjacent to `"00"` or `"0000"`.

   (a) (2 pts.) Let

   $$S = \{\text{""}, \text{"00"}, \text{"000"}, \text{"001"}, \text{"0010"}, \text{"0011"}, \text{"01"}, \text{"010"}, \text{"0101"}, \text{"1"}, \text{"10"}\}.$$

   Draw the graph $G(S)$.

   (b) (3 pts.) Add a single string x to the set S from (a) that will make the graph connected.

(c) (5 pts.) Prove that for any finite set S of binary strings, the graph $G(S)$ is a subgraph of a binary tree.

(d) (5 pts.) Let `gbadcfeh` be the preorder traversal of some binary search tree T and let `acefdbhg` be the postorder traversal of the same binary search tree. Find a set S of binary strings whose graph $G(S)$ is isomorphic to T.

EMPTY PAGE 1/2

EMPTY PAGE 2/2