Test 2 - Answer Key
Math 310
November 17, 2003

1. (Total 10 pts.) Consider the network in Figure 1.

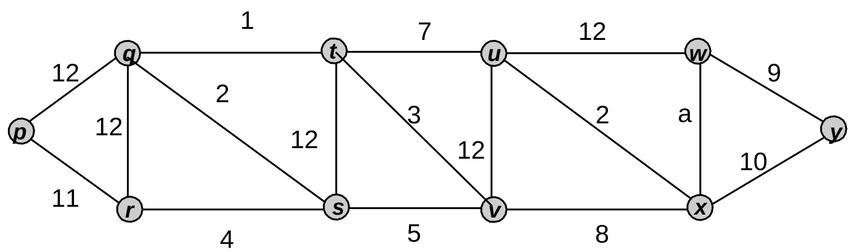

Figure 1: Network N.

(a) (3 pts.) Determine all values of a for which N contains the edge labelled a in one of its minimal spanning trees.

$$a \leq 10.$$

(b) (3 pts.) Determine all values of a for which N contains a unique minimal spanning tree.

$$a \neq 10.$$

(c) (4 pts.) For $a = 4$ list the edges of a minimal spanning tree. You may identify them in Figure 2.

$$1, 2, 2, 3, 4, 4, 7, 9, 11$$

2. (Total 10 pts.)

Let M be the network from Figure 3.

(a) (2 pts.) For $a = 7$ determine the shortest paths in M from the leftmost vertex p to any other vertex.

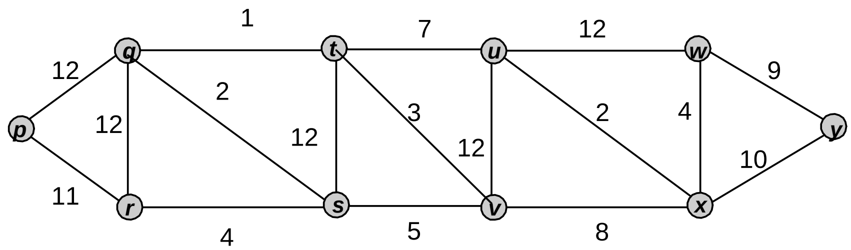

Figure 2: Mark the edges of a minimal spanning tree for $a = 4$ in N.

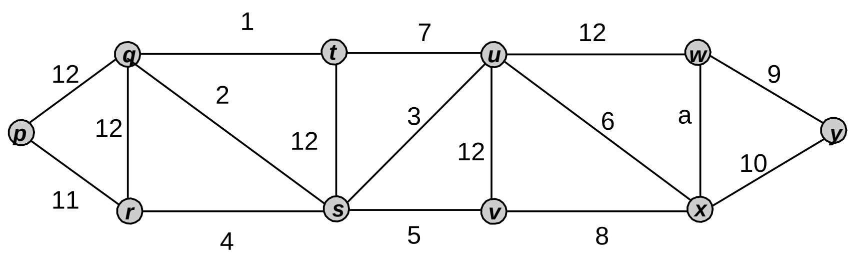

Figure 3: Network M.

The corresponding tree contains the following edges:

$$(p, q), (p, r), (q, s), (q, t), (s, u), (s, v), (u, w), (u, x), (x, y).$$

(b) (2 pts.) Is the shortest path tree obtained in (a) unique? Explain!

YES, there are no ties in the run of Dijkstra's algorithm.

(c) (3 pts.) Determine all values of a positive integer a for which the edge labelled a lies on a shortest path of M from leftmost vertex p to the rightmost vertex y.

$$a = 1.$$

(d) (3 pts.) Determine all values of the positive integer a for which the edge labelled a lies on some shortest path of M.

$$a \leq 18.$$

3. (Total 10 pts.) We form "words" (sequences of letters with repetition) composed of the 26 letters of the alphabet.

   (a) (2 pts.) How many eight-letter "words" are there?

   $$26^8 = 208827064576$$

   (b) (2 pts.) How many eight-letter words are there with exactly one of the five vowels (but that vowel may be repeated)?

   $$5 \times (22^8 - 21^8)$$

   (c) (2 pts.) How many eight-letter words are there that are the same when the order of their letters is inverted and no vowel appears more than twice (e.g. `abbiibba`).

   $$21^4 + C(5,1)P(4,1) \times 21^3 + C(5,2)P(4,2) \times 21^2 + C(5,3)P(4,3) \times 21 + C(5,4)P(4,4)$$

   $$= 21^4 + 20 \times 21^3 + 120 \times 21^2 + 240 \times 21 + 120 = 437781$$

   (d) (4 pts.) Repeat all three subproblems for n-letter words with letters taken from an alphabet having v vowels and k consonants.

   $$(v+k)^n$$

   $$v((k+1)^n - k^n)$$

   $$m = \lceil n/2 \rceil, k^m + C(v,1) \times P(m,1) \times k^{m-1} + C(v,2) \times P(m,2) \times k^{m-2} + \cdots$$

4. (Total 10 pts.) How many ways are there to pick 3 different cards from a standard 52-card deck such that:

   (a) (3 pts.) The first card is not a King, the second card is a Queen, and the third card is an Ace?

   $$50 \times 4 \times 4 - 4 \times 4 \times 4 = 46 \times 4 \times 4 = 736$$

   (b) (3 pts.) The first card is a spade, the second card is a diamond, and the third card is neither a spade nor a diamond?

   $$13 \times 13 \times 26 = 4394$$

(c) (4 pts.) The first card is a spade, the second card is an Ace, and the third card is not a diamond?

$$38 + 2 \times 37 + 12 \times 38 + 12 \times 3 \times 37 = 1900$$

5. (Total 10 pts.) We are forming 10-digit sequences with digits 1,2,3,4,5,6,7,8.

(a) (2 pts.) How many sequences can be formed?

$$8^{10}.$$

(b) (2 pts.) How many sequences can be formed with exactly two 2s, three 3s, and four 4s?

$$C(10,2)C(8,3)C(5,4) \times (8-3) = 63000$$

(c) (3 pts.) How many sequences can be formed with exactly four different digits?

Let p_k denote the number of 10-digit sequences with exactly k digits and let q_k denote the number of 10-digit sequences with at most k digits. Clearly $q_k = k^{10}$. Also $p_1 = q_1 = 1$.

$$p_2 = q_2 - 2p_1 = 2^{10} - 2 = 1022$$

$$p_3 = q_3 - C(3,2)p_2 - C(3,1)p_1 = 3^{10} - 32^{10} + 3 = 55980$$

$$p_4 = q_4 - C(4,3)p_3 - C(4,2)p_2 - C(4,1)p_1 = 818520.$$

The answer is $C(8,4)p_4$.

(d) (3 pts.) How many nondecreasing sequences can be formed?

$$C(17,7) = 19448$$

6. (Total 10 pts.) In a bridge deal, what is the probability that:

(a) (2 pts.) West has five spades, three hearts, three diamonds, and two clubs?

The total number of deals is given by $T = 52!/(13!)^4 = 536...440000$.
Let $F = C(13,5)C(13,3)C(13,3)C(13,2)(52-13)!/(13!)^3 = 693...8400$.
The probability is $F/T = 0.0129307$.

(b) (2 pts.) North and South have five spades, West has three spades, and East has no spade?

Let

$S = C(13, 5)C(52{-}13, 8)C(8, 5)C(52{-}13{-}8, 8)C(3, 3)C(52{-}13{-}16, 10)$.

The probability is $S/T = 0.000746002$.

(c) (3 pts.) One player has all the Aces and all the Kings?

Let $A = 4C(52 - 8, 13 - 8)(52 - 13)!/(13!)^3$.
The probability is $A/T = 22/3215975 = 6.84085 \times 10^{-6}$.

(d) (3 pts.) East has no face cards (J,Q,K, or A)?

Let $Q = C(52 - 16, 13)(52 - 13)!/(13!)^3$.
The probability is $Q/T = 0.00363896$.

7. (Total 10 pts.) Evaluate the following sums, where n is a positive integer.

(a) (1 pt.) $C(n, 0) + C(n, 1) + C(n, 2) + C(n, 3) + ... = 2^n$

(b) (3 pts.) $C(n, 0) + C(n, 2) + C(n, 4) + C(n, 6) + ... = 2^{n-1}$

(c) (3 pts.) $C(n, 1) + C(n, 3) + C(n, 5) + C(n, 7) + ... = 2^{n-1}$

(d) (3 pts.) $C(n, 0) - 2C(n, 1) + C(n, 2) - 2C(n, 3) + ... = -2^{n-1}$

8. (Total 10 pts.) How many integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 = 2003$$

with:

(a) (3 pts.) $x_i \geq 0$.

$$C(4 + 2003 - 1, 4 - 1) = 1343358020$$

(b) (3 pts.) $x_i > 0$.

$$C(4 + 2003 - 4 - 1, 4 - 1) = 1335334000$$

(c) (4 pts.) Write the generating function for a_r the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = r$ with

$$x_1 \geq 0, x_2 \geq 3, x_3 \geq 5, x_4 \geq x_2.$$

$$x^{11}/[(1 - x)^4(1 + x)] = x^{11} + 3x^{12} + 7x^{13} + ...$$

9. (Total 10 pts.) A partition is *self-conjugate* if the Ferrers diagram of the partition is equal to its own transpose.

   (a) (2 pts.) List all partitions of the integers 6 and 7.

$$6, 5{+}1, 4{+}2, 4{+}1{+}1, 3{+}3, 3{+}2{+}1, 3{+}1{+}1{+}1, 2{+}2{+}2, 2{+}2{+}1{+}1,$$

$$2+1+1+1+1, 1+1+1+1+1+1$$

$$7, 6{+}1, 5{+}2, 5{+}1{+}1, 4{+}3, 4{+}2{+}1, 4{+}1{+}1{+}1, 3{+}3{+}1, 3{+}2{+}2,$$

$$3+2+1+1, 3+1+1+1+1, 2+2+2+1, 2+2+1+1+1,$$

$$2+1+1+1+1+1, 1+1+1+1+1+1+1$$

   (b) (2 pts.) Write the first six terms of the generating function for a_r the number of partitions of r.

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + ...$$

   (c) (2 pts.) List all self-conjugate partitions of integers $r = 1, 2, 3, 4, 5, 6, 7$.

$$1, 2+1, 2+2, 3+1+1, 3+2+1, 4+1+1+1$$

   (d) (4 pts.) Write the generating function for b_r, the number of partitions of r into distinct odd integers.

$$(1+x)(1+x^3)(1+x^5)... = 1 + x + x^3 + x^4 + x^5 + x^6 + x^7 + ...$$

10. (Total 10 pts.) Let L_r denote the mirror L-shaped network composed of $2r - 1$ blocks, see Figure 4 for the case $r = 3$. The lower left corner is denoted A and the upper right corner is denoted B. Let a_r denote the number of walks (shortest paths) from A to B in L_r.

   (a) (2 pts.) Determine a_3, the number of shortest paths from corner A to corner B in L_3.

$$a_3 = 10$$

   (b) (2 pts.) Determine the values of a_r, for $r = 0, 1, 2, 3, 4, 5$.

$$a_0 = 1, a_2 = 2, a_3 = 10, a_4 = 17, a_5 = 26$$

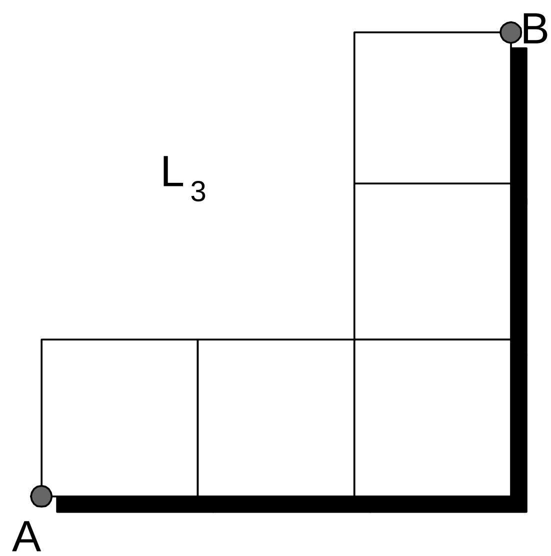

Figure 4: Block-walking network L_3.

(c) (3 pts.) Determine the formula for a_r.

$$a_r = r^2 + 1$$

(d) (4 pts.) Write the generating function for a_r.

$$(2x^2 - x + 1)/(1 - x)^3$$

11. (Total 10 pts.) Let $a_r = r^2$ and let $s_n = 1^2 + 2^2 + ... + n^2$.

(a) (3 pts.) Find a generating function for a_r.

$$(x^2 + x)/(1 - x)^3$$

(b) (3 pts.) Find a generating function for s_n.

$$(x^2 + x)/(1 - x)^4$$

(c) (4 pts.) Evaluate the sum $s_n = 1^2 + 2^2 + ... + n^2$.

$$s_n = n(n + 1)(2n + 1)/6.$$