Final Exam
Math 310
December 10, 2003

NAME____________________________________
Please, print!

- This is a take-home open textbook exam.
- You may use textbook, class notes and your notes. No other sources should be used.
- You should not consult with others concerning this exam.
- You may use calculators and Matlab.
- The test is due Monday, December 15 by noon. If you turn in the test by 6 am Saturday, December 13, then you will receive 5 bonus points.
- In each True or False problem circle your answer.
- There are 150 points on this exam. It is written to be taken in 2 hours, and you have unlimited time.
- I hope you all do well. Good luck!

1. (Total 15 pts.) In the following consider the graphs determined by moves of chess pieces (chessmen): Queen (Q), King (K), Rook (R) and Bishop (B) on an $n \times n$ chessboard. The corresponding graphs are denoted by $Q(n), K(n), R(n), B(n)$. So $Q(n)$ has vertices squares of the chessboard and two squares s_1 and s_2 are adjacent if and only if a queen can move from s_1 to s_2 in a single move.

   (a) (5 pts.) Which of the graphs $Q(4), K(4), R(4), B(4)$ have a Hamilton circuit? Explain.

   (b) (5 pts.) Determine the chromatic number of $R(n)$.

   (c) (5 pts.) Determine all values of $n, n > 1$, for which the graph $K(n)$ is planar.

2. (Total 15 pts.)

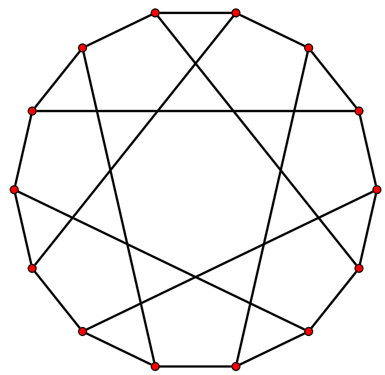

Figure 1: Graph G.

(a) (5 pts.) Find the length of the shortest circuit in G from Figure 1.

(b) (5 pts.) Assume G is planar. Determine the number of regions of any plane depiction of G.

(c) 5 pts. Prove that the graph G is non-planar. [Hint: Use (a) and prove contradiction to (b).]

3. (Total 15 pts.)

   Let G be a graph and let x and y be two non-adjacent vertices. Let G' denote the graph obtained from G by adding the edge xy. Let G'' denote the graph obtained from G by identifying the vertices x and y.

   (a) (3 pts.) Let $D(k)$ be the number of colorings of graph G with k colors such that the vertices x and y receive different colors. Express $D(k)$ in terms of $P_k(G')$.

   (b) (2 pts.) Let $S(k)$ be the number of colorings of an arbitrary graph G with k colors such that the vertices x and y receive the same color. Express $S(k)$ in terms of $P_k(G'')$.

   (c) (5 pts.) Prove that for any graph $G \neq K_n$ and for any pair of non-adjacent vertices x and y the following is true:

$$P_k(G) = P_k(G') + P_k(G'').$$

(d) (5 pts.) The graph DK_{2n} is obtained from the complete graph K_{2n} by removing n disjoint edges. For example, DK_4 is isomorphic to the circuit graph C_4. Determine the chromatic polynomial of DK_6.

4. (Total 15 pts.) Consider the arrangements of the word `COMBINATORICS`.

   (a) (5 pts.) How many different arrangements are there?

   (b) (5 pts.) How many, if no consonant appears between two vowels?

   (c) (5 pts.) Among all arrangements without any pair of consecutive vowels, what fraction have `C` adjacent to `O`?

5. (Total 15 pts.) We are forming 12-digit sequences with digits 0,1,2,3,4,5,6,7,8,9.

   (a) (5 pts.) How many sequences can be formed?

   (b) (5 pts.) How many sequences can be formed with exactly one 1, two 2s, three 3s, and four 4s?

   (c) (5 pts.) How many sequences can be formed, such that the subsequence formed by the first, third, fifth, .., digit is non-increasing, while the subsequence formed by the second, fourth, ... digit is non-decreasing?

6. (Total 15 pts.) In a poker deal, what is the probability that:

   (a) (5 pts.) Given player has three of a kind?

   (b) (5 pts.) The player gets (exactly) three of a kind, if the first two cards are a pair?

   (c) (5 pts.) The player gets full house (= three of a kind + pair) if the first two cards are a pair?

7. (Total 15 pts.) How many integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2004$$

with:

(a) (5 pts.) $x_i \geq 0$.

(b) (5 pts.) $x_i > 0$.

(c) (5 pts.) Write the generating function for a_r the number of integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = r$ with

$$x_1 \geq 0, x_2 \geq 3, x_3 \geq 5, x_4 \geq x_2, x_5 \geq 2x_4.$$

8. (Total 15 pts.) Let S_r denote a staircase shaped network composed of r layers with 2 blocks in each layer. See Figure 2 for the case $r = 4$. The lower left corner is denoted A and the upper right corner is denoted B. Let a_r denote the number of walks (shortest paths) from A to B in S_r.

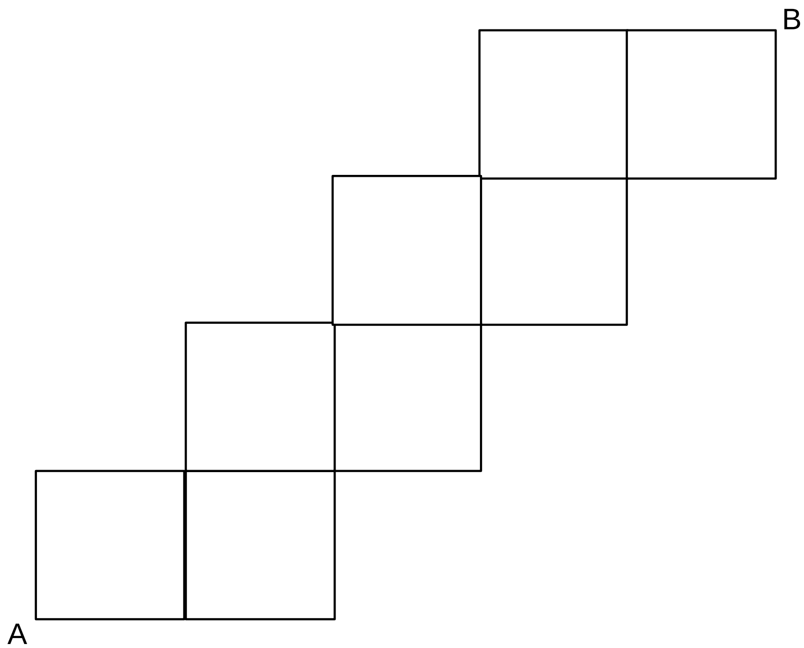

Figure 2: Block-walking network S_4.

(a) (2 pts.) Determine a_4, the number of shortest paths from corner A to corner B in S_4.

(b) (3 pts.) Set up a recurrence relation for a_r.

(c) (5 pts.) Determine the formula for a_r.

(d) (5 pts.) Write the generating function for a_r.

9. (Total 15 pts.) Let $a_r = 2(r-1)(r+1)$ and let

$$s_n = 2 \times (-1) \times 1 + 2 \times 0 \times 2 + 2 \times 1 \times 3 + ... + 2 \times (n-1) \times (n+1).$$

(a) (5 pts.) Find a generating function $g(x)$ for a_r.

(b) (5 pts.) Find a generating function $h(x)$ for s_n.

(c) (5 pts.) Evaluate the sum s_n.

10. (Total 15 pts.) Given the following inhomogeneous recurrence relation $a_n = 5a_{n-1} - 3a_{n-2} - 9a_{n-3} + 10$ with initial conditions $a_0 = a_1 = a_2 = 0$.

    (a) (5 pts.) Find a particular solution to the inhomogeneous recurrence relation.

    (b) (5 pts.) Find a general solution to the associated homogeneous recurrence relation. [Hint: check for integer roots of the characteristic polynomial.]

    (c) (5 pts.) Find a particular solution to the inhomogeneous recurrence relation that satisfies the initial conditions stated above.