Matrices:

Definition of: $\Rightarrow$ a short hand notation $\Leftarrow$

$$\begin{matrix} 2x + 3y = 5 \\ 4x - 6y = 3 \end{matrix} \iff \underbrace{\begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 5 \\ 3 \end{bmatrix}}_{\underline{b}} \iff \underline{\underline{A}}\,\underline{x} = \underline{b}$$

In general,

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{matrix} \iff \underline{\underline{A}}\,\underline{x} = \underline{b}$$

Multiplication, Addition, etc. **size:** $\underline{\underline{A}}$ is an "m by n" matrix if it has m rows and n columns

Matrices can be added or subtracted if they are the same size. Just add components.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$$

Matrix multiplication is like a dot product: $\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = [1\cdot 2 + 3\cdot 4] = [14] =$

1 by 1 matrix is a number

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}}_{B} = \begin{bmatrix} 2+8 & 3+10 \\ 6+16 & 9+20 \end{bmatrix} = \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix}$$

To find the i^{th} row, j^{th} column of the product take dot product of the i^{th} row with the j^{th} column.

WARNING $\underline{\underline{A}} \cdot \underline{\underline{B}}$ is not $\underline{\underline{B}} \cdot \underline{\underline{A}}$ most of the time. **ORDER MATTERS**

So division cannot use fraction notation! We must keep the order intact.

Division: **Note:** $\underline{\underline{I}} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$ is a matrix with ones on the main diagonal and zeros off this diagonal. It is called the Identity matrix because $\underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}}$ for all $\underline{\underline{A}}$

to solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$ for $\underline{x}$, we just divide by $\underline{\underline{A}}$ on the left. We write that as multiplying by $\underline{\underline{A}}^{-1}$

$\underline{\underline{A}}^{-1}\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$. To have this work as division, it must be that $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$

Define: $\underline{\underline{A}}^{-1}$ to be the matrix such that $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$ for a given $\underline{\underline{A}}$.

WARNING: Just as we cannot divide by zero, the inverse may not exist!!

The determinant of $\underline{\underline{A}}$ tells you if the inverse exists. $\det(\underline{\underline{A}}) \neq 0$ means $\underline{\underline{A}}$ has an inverse (is invertible)

Determinants: This is an operation that takes a matrix and gives a single number.

2x2 (2 by 2) matrices: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$ (multiply the diagonal elements and subtract the product of off diagonal elements)

3x3 matrices:
$$\det \begin{array}{c} \begin{array}{ccc} \oplus & \ominus & \oplus \end{array} \\ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \end{array} = a \cdot \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$
$$= aei - afh - bdi + bfg + cdh - ceg$$

Larger matrices are similarly defined in terms of smaller determinants:

n x n matrices:
$$\det(\underline{\underline{A}}) = a_{11} \cdot \det\left(\underline{\underline{A}} \text{ with row 1 and column 1 removed}\right) - a_{12} \cdot \det\left(\underline{\underline{A}} \text{ with row 1 and column 2 removed}\right) + a_{13} \cdot \det\left(\text{row 1 column 3 removed}\right)$$
$$\ominus \cdots \pm a_{1n} \det\left(\underline{\underline{A}} \text{ with row 1 column } n \text{ removed}\right)$$
minus if n odd, plus if n even.

Trace: The trace of $\underline{\underline{A}}$ is just the sum of the main diagonal. $tr\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1+5+9 = 15$

Eigen values; eigenvectors: An eigenvalue problem for $\underline{\underline{A}}$ involves finding a solution pair $(\lambda, \underline{u})$ such that $\boxed{\underline{\underline{A}}\,\underline{u} = \lambda \underline{u}}$

Note: If $\underline{\underline{A}}$ is $n \times n$ then $\underline{u}$ is $n \times 1$ so we have $n+1$ unknowns with n equations.

First find eigenvalues: solve $\underline{\underline{A}}\,\underline{u} - \lambda \underline{u} = \underline{0}$
$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{u} = \underline{0}$ but the solution $\underline{u}$ will be zero unless the inverse of $(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ does not exist.

Choose λ so that $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0.$
Then $\underline{u}$ can be non-zero.

Second: find eigenvectors: for each λ, plug in λ to $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{u} = \underline{0}$ and then solve for $\underline{u}$.

Note: any multiple of $\underline{u}$ is also an eigenvector, so you can set one element of $\underline{u}$ to be a convenient non-zero value like 1.

Example: $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \det\begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2$ so we find the roots $\lambda_{1,2} = \frac{5 \pm \sqrt{25+8}}{2}$

$\lambda = \frac{5}{2} + \frac{\sqrt{33}}{2} \Rightarrow$ find $\underline{u}$ $\begin{bmatrix} -\frac{3}{2} - \frac{\sqrt{33}}{2} & 2 \\ 3 & \frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{3+\sqrt{33}}{2} \end{bmatrix}$ works as an eigenvector for the eigen value $\frac{5}{2} + \frac{\sqrt{33}}{2}$