From:

"Nonlinear Dynamics, Mathematical Biology and Social Science"
By Joshua M. Epstein
Santa Fe Institute
Addison Wesley Publishing Company, 1997

LECTURE 1

On The Mathematical Biology of Arms Races, Wars, and Revolutions

In this opening lecture, I will attempt a unifying overview of certain social phenomena—war, arms racing, and revolution—from the perspective of mathematical biology, a field which, in my view, must ultimately subsume the social sciences.[3] Unfortunately, few social scientists are exposed to mathematical biology, specifically the dynamical systems perspective pioneered by Alfred Lotka, Vito Volterra, and others. In turn, few mathematical biologists have considered the application of mathematical biology to problems of human society.[4]

Particularly in areas of interstate and intrastate conflict is there a need to explore formal analogies to biological systems. On the topic of animal behavior and human warfare, the anthropologist Richard Wrangham observes,

---

[3] The perspective taken here, however, is quite distinct from that taken by Edward O. Wilson, in his book *Sociobiology* (1980). Specifically, I do not discuss the role of genes in the control of human social behavior. Rather, the argument is that macro social behaviors such as war, revolution, arms races, and the spread of drugs may conform well to equations of mathematical biology—ecology and epidemiology in particular. Perhaps "socioecology" would be a suitable name for this level of analysis.

[4] For a notable exception, see Cavalli-Sforza and Feldman (1981). See also the innovative and understudied works, Rashevsky (1947) and Rashevsky (1949).

> "The social organization of thousands of animals is now known in considerable detail. Most animals live in open groups with fluid membership. Nevertheless there are hundreds of mammals and birds that form semi-closed groups, and in which long-term intergroup relationships are therefore found. These intergroup relationships are known well. In general they vary from benignly tolerant to intensely competitive at territorial borders. The striking and remarkable discovery of the last decade is that only two species other than humans have been found in which breeding males exhibit systematic stalking, raiding, wounding and killing of members of neighboring groups. They are the chimpanzee (*Pan troglodytes*) and the gorilla (*Pan gorilla beringei*) (Wrangham, 1985). In both species a group may have periods of extended hostility with a particular neighboring group and, in the only two long-term studies of chimpanzees, attacks by dominant against subordinate communities appeared responsible for the extinction of the latter.
>
> "Chimpanzees and gorillas are the species most closely related to humans, so close that it is still unclear which of the three species diverged earliest (Ciochon & Chiarelli, 1983). The fact that these three species share a pattern of intergroup aggression that is otherwise unknown speaks clearly for the importance of a biological component in human warfare" (Wrangham, 1988, p.78).

Although man has engaged in arms racing, warring, and other forms of organized violence for all of recorded history, we have comparatively little in the way of formal theory. Mathematical biology may provide guidance in developing such a theory. Wrangham writes, "Given that biology is in the process of developing a unified theory of animal behavior, that human behavior in general can be expected to be understood better as a result of biological theories, and that two of our closest evolutionary relatives show human patterns of intergroup aggression, there is a strong case for attempting to bring biology into the analysis of warfare. At present, there are few efforts in this direction."[5] I would like to see more effort, specifically more mathematical effort, in this direction and hope to stimulate some interest among you. To convince you that there might conceivably be some "unified field theory" worth pursuing, I want to share some observations with you. To set them up, a little background is required.

The fundamental equations in the mathematical theory of arms races are the so-called Richardson equations, named for the British applied mathematician and social scientist Lewis Frye Richardson, who first published them in 1939.[6] The fundamental equations in the mathematical theory of combat (warfare itself, as against peacetime arms racing) were published in 1916 by Frederick William Lanchester.[7]

---

[5] Wrangham (1988, p.78).

[6] Richardson (1939) and (1960).

[7] See Lanchester (1916). For a contemporary discussion with references, see Epstein (1985).

The formal theory of interstate conflict, to the extent there is one, rests on these twin pillars, if you will. Meanwhile, the classic equations of mathematical ecology are the Lotka-Volterra equations.

In light of the remarks above, I find the following fact intriguing: The Richardson and Lanchester models of human conflict are, mathematically, specializations of the Lotka-Volterra ecosystem equations.

Before proceeding, I must make one point unmistakably clear. I do not claim that any of these models is really "right" in a physicist's sense. They are illuminating abstractions. I think it was Picasso who said, "Art is a lie that helps us see the truth." So it is with these simple models. They continue to form the conceptual foundations of their respective fields. They are universally taught; mature practioners, knowing full-well the models' approximate nature, nonetheless entrust to them the formation of the student's most basic intuitions. And this because, like idealizations in other sciences—idealizations that are ultimately "wrong"—they efficiently capture qualitative behaviors of overarching interest. That these ecosystem and, say, arms race equations should look at all alike is unexpected. That, on closer inspection, they are virtually identical is, to me, really quite interesting. Let me go a bit further.

Under yet other parameter settings, the Lotka-Volterra equations yield standard models of epidemics. And, in other lectures, I will argue that social revolutions and illicit drugs may well spread in a strictly analogous way or—at the very least—that an epidemiological perspective on such social processes is promising. Once more, the point is simply that social science might learn a lot from mathematical biology and, conceivably, might inherit some of its apparent unity.

Let me now introduce the Lotka-Volterra equations and show how the classic arms race and war models fall out as special cases. Then, I will explore the analogy between revolutions and epidemics. In subsequent lectures, we will move beyond these simple—too simple—models.

THE LOTKA-VOLTERRA WORLD

The Lotka-Volterra equations are as follows:

$$\begin{aligned} \dot{x}_1 &= x_1(r_1 - a_{11}x_1 + a_{12}x_2)\,, \\ \dot{x}_2 &= x_2(r_2 + a_{21}x_1 - a_{22}x_2)\,. \end{aligned} \tag{1.1}$$

In discussing these equations, I will freely invoke nonlinear dynamical systems terminology presented in lecture 6.[8] Turning now to system (1.1), $x_i(t)$ is the species i population at time t; the a's and r's are real parameters.

---

[8] Under the name, "quadratic model," equivalent equations and a number of specializations—including combat variants—are discussed in Beltrami (1987).

If all a_{ij}'s equal zero and $r_1, r_2 > 0$, we have unbounded exponential—so-called Malthusian—growth. Since, ultimately, there are limits, for instance, environmental carrying capacities, the terms $a_{11}, a_{22} > 0$ are preceded by a negative sign. Then, in the language of lecture 6, the species are self-inhibiting. Leaving r_1 and r_2 positive and still assuming $a_{12} = a_{21} = 0$, this assumption yields a logistic approach for each species to the positive phase plane equilibrium

$$(\bar{x}_1, \bar{x}_2) = \left(\frac{r_1}{a_{11}}, \frac{r_2}{a_{22}}\right),$$

a node sink.

Now, life really gets interesting only when species interact, and this involves the cross terms a_{12} and a_{21}.

MUTUALISM

Leaving everything else as is, let us now assume $a_{12}, a_{21} > 0$. In that case our species are said to be in a relationship of mutualism, or reciprocal activation; the population level of one feeds back positively on the growth rate of the other. Bees and flowers—pollinators and pollinatees, if you will—provide examples. There are many others.

Setting $\dot{x}_1 = \dot{x}_2 = 0$, the interior equilibrium conditions are

$$\begin{aligned} r_1 - a_{11}x_1 + a_{12}x_2 &= 0, \\ r_2 + a_{21}x_1 - a_{22}x_2 &= 0. \end{aligned} \tag{1.2}$$

Of course, these are also the equilibrium conditions for the linear system:

$$\begin{aligned} \dot{x}_1 &= r_1 - a_{11}x_1 + a_{12}x_2, \\ \dot{x}_2 &= r_2 + a_{21}x_1 - a_{22}x_2. \end{aligned} \tag{1.3}$$

But this is exactly the famous Richardson model of an arms race! The more bees, the more flowers, and vice versa. It's the same in (1.3), but not quite as idyllic. The more weaponry my adversary has, the more I want, and vice versa, up to some economic—or ecological—limit or carrying capacity.

Richardson's basic idea is that a state's arms race behavior depends on three overriding factors: the perceived external threat, the economic burden of military competition, and the magnitude of grievances against the other party. I discuss these at greater length in lecture 3. Suffice it to say here that $r_1, r_2 > 0$ represent fundamental grievances; $a_{12}, a_{21} > 0$ are the reciprocal activation coefficients (the rates at which each arsenal grows in response to the other); and a_{11}, a_{22} are the self-inhibiting, or damping, terms which Richardson identified with economic fatigue.

Mathematical biologists have long asked how mutualistic populations avoid exploding in what Robert May called an "orgy of mutual benefaction."[9] Likewise, we can ask what mechanism damps the upward action-reaction military dynamic represented in the Richardson model. In each case, self-inhibitory effects must somehow dominate reciprocal activation effects if a stable species equilibrium—or military "balance of power"—is to emerge. Stability analysis bears this out.

Clearly, we can write (1.2) in matrix form $r + Ax = 0, x \in \mathcal{R}^2$. The positive (or interior) equilibrium of system (1.1) and the sole equilibrium of (1.3) is therefore given by $\bar{x} = -A^{-1}r$. For each model, the stability of $\bar{x}$ can be evaluated by the methods of lecture 6.

By a simple translation, the Richardson equations (1.3) are globally asymptotically stable at $\bar{x}$ if and only if $\dot{y} = Ay$ is globally asymptotically stable at the origin, where $y = x - \bar{x}$. From lecture 6, we have the well-known stability criterion

$$\text{Tr } A < 0 \text{ and Det } A > 0. \tag{1.4}$$

Now, Richardson's economic fatigue *means* $a_{11}, a_{22} > 0$. So, we have

$$\text{Tr } A = -a_{11} - a_{22} < 0.$$

And we will have Det $A > 0$ precisely when $a_{11}a_{22} > a_{12}a_{21}$, which is to say that inhibition ($a_{11}a_{22}$) outweighs activation ($a_{12}a_{21}$), confirming our intuition.

One can demonstrate[10] that the eigenvalues of the Jacobian of (1.1) at $\bar{x}$ have negative real parts (indeed, are negative reals) when the same condition is met. An isocline analysis is also revealing. We recall that an isocline is a curve—here a line—where one side's rate of growth is zero; clearly, an equilibrium is a point where isoclines intersect. From (1.2), the isoclines are given by:

$$\begin{aligned} \phi_1(x_1) &= \frac{a_{11}}{a_{12}}x_1 - \frac{r_1}{a_{12}} \ \text{(the } x_1 - \text{isocline)}, \\ \phi_2(x_1) &= \frac{a_{21}}{a_{22}}x_1 + \frac{r_2}{a_{22}} \ \text{(the } x_2 - \text{isocline)}. \end{aligned} \tag{1.5}$$

For local stability of the equilibrium $\bar{x}$, we require the configuration of figure 1.1. But, this occurs only if the slope of ϕ_1 exceeds the slope of ϕ_2, which is to say $a_{11}/a_{12} > a_{21}/a_{22}$, or

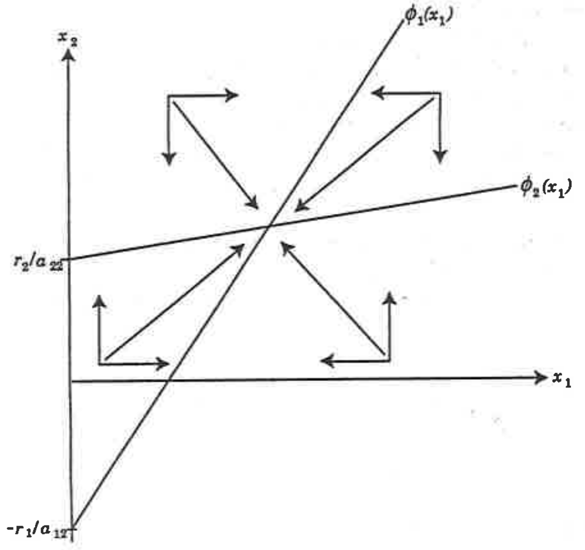
$$a_{11}a_{22} > a_{21}a_{12}.$$

Our intuition is again confirmed: stability requires self-inhibition to exceed reciprocal activation in this sense.

[9] May (1981).

[10] Goh (1979).

FIGURE 1.1 Mutualistic Stability



The main point, however, is that the classic Lotka-Volterra model of mutualistic species interaction embeds, in its equilibrium behavior, the classic Richardson arms race model.

AN ASIDE ON COEVOLUTION

In the models above, of course, the "phenotypes" do not change. In fact, ecosystem dynamics select against certain phenotypes. Roughly speaking, phenotypic frequencies and population levels have interdependent trajectories. This is very clear, for example, in immunology, where antigens and antibodies coevolve in a so-called "biological arms race." But, of course, real arms races work this way, too. Ballistic missiles beget antiballistic missile defenses, which beget various evasion and defense suppression technologies. The machine gun makes cavalry obsolete, giving rise to the "iron horse"—the tank—which begets antitank weapons, which beget special armor, and so on. Michael Robinson's analogy between moth-bat coevolution and the coevolution of World War II air war tactics is apposite.

> "Moths and their predators are in an arms race that started millions of years before the Wright brothers made the Dresden raids possible. Butterflies exploit the day, but their 'sisters' the moths dominate the insects' share of the night skies. Few vertebrates conquered night flying. Only a small fraction of bird species, mostly owls and goatsuckers, made the transition. Bats, of course, made it their realm. Many species of bats are skilled 'moth-ers': they pursue them at speed after detecting them with their highly attuned echolocation system. Some moths, however, have developed 'ears' capable of detecting the bat's ultrasonic cries. When they hear a bat coming, the moths take evasive action, including dropping below the bat's track. The parallels of the response of Allied bombers to the radar used by the Germans in World War II are interesting. If we visualize the bombers as the moths, and radars on the ground and in the night-fighter aircraft as bats (a reversal of sizes), the situation is similar. Bombers used rearward-listening radar to detect enemy night fighters. When they detected a fighter, they took evasive action. But heavy bombers, heavily laden, were not very maneuverable. They couldn't dodge about quite as well as moths. Some pilots tried to drop their aircraft into a precipitous dive. Moths also do this; it is easy for them to fold their wings and drop. The next stage in the night-battle escalation is predictable. The night fighter's radar was eventually tuned to detect the bomber's fighter-detector, and thus the bomber itself. Bats have not yet tuned in on moths' ears.
>
> "Bombers also used technological disruption. Night fighters came to be guided to bombers by long-distance radars on the ground. The fighters started winning. But nothing remains static. The ground radars could be jammed by various kinds of radio noise. The technological battle swung the other way. Then the fighters acquired radar. Much like a bat, a fighter emitted and listened to radar signals of its own. These, too, proved to be susceptible to countermeasures, however. The RAF could jam the fighters' radar or 'clutter' it with strips of aluminum foil. Each bomber in a formation dropped one thousand-strip bundle per minute, so that huge clouds of foil foiled the radar. Amazingly, there may be a similar counterweapon among moths. Some moths can produce ultrasonic sounds that fall within the bats' audio frequency. The moths' voice boxes are paired, one on each side of the thorax; double voices must be particularly confusing. Alien sounds in their waveband could confound the bats, exactly in the same way the foil confounded the fighters.
>
> "The next steps in the bat-versus-moth war may simply be awaiting discovery by some bright researcher; after all, we did not know a lot about echolocation in bats until after World War II. My guess would be that the detector will get more complex to meet the defenses. This may already have happened; bats specializing in moths with ears may have moved to a higher

frequency sound outside the moths' hearing range!" (Robinson, 1992, pp. 77–79).

Quite clearly, *levels* of armament (in the international system) and *levels* of population (in an ecosystem) interact, as in the Lotka-Volterra and Richardson models, but *phenotypes* themselves are also changing. In biology, there is a mathematical theory of coevolution.[11] In social science, there isn't. There probably could be, so I simply mention it as a promising direction.

Now, let us shift gears from the mutualistic/arms race variant of (1.1). Specifically, instead of assuming that a_{12} and a_{21} are positive, assume that they are negative.

COMPETITION

Rearranging slightly, the equations (1.1) take the form

$$\begin{aligned}\dot{x}_1 &= a_{12}x_1x_2 + r_1x_1\left(1 - \frac{x_1}{k_1}\right),\\ \dot{x}_2 &= a_{21}x_1x_2 + r_2x_2\left(1 - \frac{x_2}{k_2}\right),\end{aligned} \tag{1.6}$$

where $k_i \equiv (r_i/a_{ii}) > 0$ is the carrying capacity of the environment for each species. These equations were published in 1934 by the great Russian mathematical biologist G. F. Gause in his book *The Struggle for Existence*. Indeed, he termed a_{12} and a_{21} "coefficients of the struggle for existence."[12]

Now, examining (1.6), each species would exhibit logistic growth to its respective carrying capacity but for these interaction—struggle—terms. Including them, (1.6) gives a picture of uniform mixing of the populations x_1 and x_2, with contacts proportional to the product x_1x_2. Now, however, since the interaction coefficients are negative, each contact *kills* species 1 at rate a_{12} and species 2 at rate a_{21}. Quite clearly, a parallel to combat is suggested. But more is true.

In fact, unbeknownst to Gause, (1.6) is an exact form of the famous—and to this day ubiquitous—Lanchester model of warfare![13]

The transition from arms race to war, then, might be seen as a transition from the case of $a_{12}, a_{21} > 0$ to the case of $a_{12}, a_{21} < 0$. In the latter context, the well-known biological "principal of competitive exclusion" simply maps to the military principle that, usually, one side wins and the other side loses. Both these competitive exclusion behaviors reflect the mathematical fact that the interior $(x_1, x_2 > 0)$ equilibrium of (1.6) is a saddle. The stable equilibrium in the mutualistic—peacetime arms race—case was a node. To the extent these models are correct, then, we can say (*pacem* Poincaré) that war is topologically different from peace; the outbreak of war is a bifurcation from node to saddle.

---

[11] Roughgarden (1979).

[12] Gause (1934, p. 47).

[13] Lanchester (1916).

Thus far we have been exploring a mathematical biology of interstate relations; what about intrastate dynamics? Is there a Lotka-Volterra perspective on revolution, for instance? And, to what biological process might such social dynamics correspond?

REVOLUTIONS AND EPIDEMICS

Consider the following specialization of (1.1):

$$a_{12} = a_{21} > 0; r_1 = r_2 = a_{11} = a_{22} = 0\,. \tag{1.7}$$

Then (1.1) becomes

$$\begin{aligned}\dot{x}_1 &= -a_{12}x_1x_2\,,\\ \dot{x}_2 &= a_{12}x_1x_2\,,\end{aligned} \tag{1.8}$$

which is the simplest conceivable epidemic model. Now, rather than armament levels, x_1 represents the level of susceptibles, and x_2 the level of infectives, while the parameter a_{12} is the infection rate, expressing the contagiousness of the infection. Ideal homogeneous mixing, once more, is assumed. If population is constant at P_0, then $x_1 = P_0 - x_2$ and we obtain

$$\dot{x}_2 = a_{12}x_2(P_0 - x_2)\,, \tag{1.9}$$

our familiar friend the logistic differential equation. Here, $x_2 = 0$ is an unstable equilibrium; the slightest introduction of infectives, and the disease whips through the whole of society.

A traditional tactic for combating the spread of a disease is removal of infectives. Sometimes, nature does the removing, as with fatal diseases; often, society removes infectives from circulation by quarantine. The simplest possible assumption is that removal is proportional to the size of the infective pool, yielding the following variant of (1.1):

$$\begin{aligned}\dot{x}_1 &= -a_{12}x_1x_2\,,\\ \dot{x}_2 &= a_{12}x_1x_2 - r_2x_2\,,\end{aligned} \tag{1.10}$$

with $r_2 > 0$. This is the famous Kermack-McKendrick (1927) *threshold* epidemic model,[14] so-called because it exhibits the following behavior.

---

[14] Kermack and McKendrick (1927). For a contemporary statement, see Waltman (1974).

By definition, there is an epidemic outbreak only if $\dot{x}_2 > 0$. But this is to say $a_{12}x_1x_2 - r_2x_2 > 0$, or

$$x_1 > \frac{r_2}{a_{12}}. \tag{1.11}$$

The initial susceptible level $x_1(0)$ must exceed the threshold $\rho \equiv r_2/a_{12}$, sometimes called the relative removal rate, for an epidemic to break out. The fact that epidemics are threshold phenomena has important implications for public health policy and, I will argue below, for social science.

The public health implication, which was very controversial when first discovered, is that *less than* universal vaccination is required to prevent epidemics. By the threshold criterion (1.11), the fraction immunized need only be big enough that the unimmunized fraction—the actual susceptible pool—be below the threshold ρ. "Herd immunity," in short, need not require immunization of the entire herd. For instance, diphtheria and scarlet fever require 80-percent immunization to produce herd immunity.[15] Hethcote and Yorke argue that "a vaccine could be very effective in controlling gonorrhea. . .for a vaccine that gives an average immunity of 6 months, the calculations suggest that random immunization of 1/2 of the general population each year would cause gonorrhea to disappear."[16]

Mathematical epidemic models are discussed more fully in lecture 4. With the above as background, let us now consider the analogy between epidemics (for which a rich mathematical theory exists) and processes of explosive social change, such as revolutions (for which no comparable body of mathematical theory exists). Again, a more careful and deliberate development is given in lecture 4. Here, we simply offer the main idea. It will facilitate exposition to re-label the variables in (1.10). If $S(t)$ and $I(t)$ represent the susceptible and infective pools at time t and if r and γ are the infection and removal rates, the basic model is:

$$\begin{aligned} \dot{S} &= -rSI\,, \\ \dot{I} &= rSI - \gamma I\,, \end{aligned} \tag{1.12}$$

with epidemic threshold

$$S > \frac{\gamma}{r} = \rho. \tag{1.13}$$

The basic mapping from epidemic to revolutionary dynamics is direct. The infection or disease is, of course, the revolutionary idea. The infectives $I(t)$ are individuals who are actively engaged in articulating the revolutionary vision and in winning over ("infecting") the susceptible class $S(t)$, comprised of those who are receptive to the revolutionary idea but who are not infective (not actively engaged in transmitting the disease to others). Removal is most naturally interpreted as the political imprisonment of infectives by the elite ("the public health authority").

[15] Edelstein-Keshet (1988, p. 255).

[16] Hethcote and Yorke (1980, p. 47).

Many familiar tactics of totalitarian rule can be seen as measures to minimize r (the effective contact rate between infectives and susceptibles) or maximize γ (the rate of political removal). Press censorship and other restrictions on free speech reduce r, while increases in the rate of domestic spying (to identify infectives) and of imprisonment without trial increase γ.

Symmetrically, familiar revolutionary tactics—such as the publication of underground literature, or "samizdat"—seek to increase r. Similarly, Mao's directive that revolutionaries must "swim like fish in the sea," making themselves indistinguishable (to authorities) from the surrounding susceptible population, is intended to reduce γ.

GORBACHEV, DeTOQUEVILLE, AND THE THRESHOLD

Interpreting the threshold relation (1.13), if the number of susceptibles S_0 is, in fact, quite close to ρ, then even a slight reduction (voluntary or not) in central authority can push society over the epidemic threshold, producing an explosive overthrow of the existing order. To take the example of Gorbachev, the policy of Glasnost obviously produced a sharp increase in r, while the relaxation of political repression (e.g., the weakening of the KGB, the release of prominent political prisoners, and the dismantling of Stalin's Gulag system) constituted a reduction in γ. Combined, these measures evidently depressed ρ to a level below S_0, and the "revolutions of 1989" unfolded. Perhaps DeToqueville intuited the threshold relation (1.13), describing this phenomenon, when he remarked that "liberalization is the most difficult of political arts."

As a final element in the analogy, systematic social indoctrination can produce herd immunity to potentially revolutionary ideas. We even see "booster shots" administered at regular intervals—May 1 in Moscow; July 4 in America—on which occasions the order-sustaining myths ("The USSR is a classless workers' paradise"; "Everyone born in America has the same opportunities in life") are ritually celebrated.

Now, as I said before, all these analogies are doubtlessly terribly crude. I certainly do not claim either that any of the models are right or that the dynamical analogies among them are exact. Yet, the very fact that a single ecosystem model—the Lotka-Volterra equations—could specialize to equations that even caricature, however crudely, such basic and important social processes as arms racing, warring, and rebelling is, I believe, very interesting and serves to reinforce the larger point with which I began: social science is ultimately a subfield of biology.

CONCLUSION

Finally, let me conclude with an admission. I was surprised when I began to notice these connections. But why should we be surprised? In certain non-Western cultures, where our species is seen as "a part *of* nature," where gods—like the sphinx—can be part man and part lion, all these connections between ecosystems and social systems might appear quite unremarkable. But in Western cultures shaped by the Old Testament, where God creates *only* man—not the fishes, birds, and bushes—in *his* own image, man is seen as "apart *from* nature." And, accordingly, we are surprised when our models of fish—or worse yet, of viruses—turn out to be interesting models of man. Perhaps we are true Darwinians more in our heads than in our hearts. Creatures of habit, we are captive to a transmitted and slowly evolving culture. But, of course, this too is "only natural."

LECTURE 2

An Adaptive Dynamic Model of Combat

In this lecture I would like to give an introduction to some simple mathematical models of combat, including my own Adaptive Dynamic Model. Here, we are concerned with the *course* of war, rather than the arms races or crises that may precipitate war. Before discussing specifics, it may be well to consider the basic question: What are appropriate goals for a mathematical theory of combat at this point?

First and foremost, we need to be humble. Warfare is complex. Outcomes may depend, perhaps quite sensitively, on technological, behavioral, environmental, and other factors that are very hard to measure before the fact. Exact prediction is really beyond our grasp.

But, that's not so terrible. Theoretical biologists concerned with morphogenesis—the development of pattern—are, in some cases, situated similarly. For the particular leopard, we certainly cannot predict the exact size and distribution of spots. But, certain classes of partial differential equations—reaction-diffusion equations—will generate generic animal coat patterns of the relevant sort. So, we feel that this is the right body of mathematics to be exploring. The same sort of point holds for epidemiologists. Few would claim to be able to predict the exact onset point or severity of an epidemic. Theoreticians seek simple models that will generate a reasonable menu of core qualitative behaviors: threshold eruptions, persistence at endemic levels, recurrence in cycles, perhaps chaotic dynamics. The aim is to produce transparent, parsimonious models that will *generate the core menu of gross qualitative system behaviors.* This, it seems to me, is the sort of claim one would want to make for a mathematical theory of combat.

Now, in classical mechanics, the crucial variables are mass, position, and time. In classical economics, they are price and quantity. War, traditionally, is about territory and, unfortunately, death, or mutual attrition. A respectable model, at the very least, should offer a plausible picture of the relationship between the fundamental processes of attrition and withdrawl (i.e., territorial sacrifice). I will discuss attrition first.

LANCHESTER'S EQUATIONS

The big pioneer in this general area was Frederick William Lanchester (1868–1945). The eclectic English engineer made contributions to diverse fields, including automotive design and the theory of aerodynamics.[17] He is best remembered for his equations of war, appropriately dubbed the Lanchester equations. First set forth in his 1916 work, *Aircraft in Warfare*, these have a variety of forms, the most renowned of which is called—for reasons that will be given shortly—the Lanchester "square" model.[18] With no air power and no reinforcements, the Lanchester square equations are

$$\begin{aligned} \frac{dR}{dt} &= -bB\,, \\ \frac{dB}{dt} &= -rR\,. \end{aligned} \tag{2.1}$$

Here, $B(t)$ and $R(t)$ are the numbers of "Blue" and "Red" combatants—each of which is an idealized fire source—and $b, r > 0$ are their respective firing effectiveness per shot. Qualitatively, these equations say something intuitively very appealing, indeed, seductive: *The attrition rate of each belligerent is proportional to the size of the adversary.* In the phase plane, the origin is obviously the only equilibrium of (2.1) and the Jacobian of (2.1) at $\bar{x}$ is

$$DF(\bar{x}) = \begin{pmatrix} 0 & -b \\ -r & 0 \end{pmatrix}.$$

---

[17] Lanchester (1956).

[18] See Lanchester (1916). The same model was apparently developed independently by the Russian M. Osipov (1915).

The eigenvalues are clearly $\pm\sqrt{rb}$. Hence, the origin is a saddle, though the positive quadrant is all we care about. The system (2.1) is, of course, soluble exactly. With $B(0) = B_0$ and $R(0) = R_0$,

$$\begin{aligned} R(t) &= \frac{1}{2}\left[\left(R_0 - \sqrt{\frac{b}{r}}B_0\right)e^{\sqrt{rb}\,t} + \left(R_0 + \sqrt{\frac{b}{r}}B_0\right)e^{-\sqrt{rb}\,t}\right], \\ B(t) &= \frac{1}{2}\left[\left(B_0 - \sqrt{\frac{r}{b}}R_0\right)e^{\sqrt{rb}\,t} + \left(B_0 + \sqrt{\frac{r}{b}}R_0\right)e^{-\sqrt{rb}\,t}\right], \end{aligned} \tag{2.2}$$

with various trajectories for R and B over time. Depending on the parameters (b, r) and the initial values (B_0, R_0), either side can start ahead and lose, or start behind and win, as is observed historically.[19]

The most celebrated result of the theory is the so-called Lanchester Square Law, which is obtained easily. From (2.1), we have

$$\frac{dR}{dB} = \frac{bB}{rR}\,. \tag{2.3}$$

Separating variables and integrating from the terminal values $(R(t), B(t))$ to the higher initial values,

$$r\int_{R(t)}^{R_0} R\,dR = b\int_{B(t)}^{B_0} B\,dB\,,$$

we obtain the state equation

$$r(R_0^2 - R(t)^2) = b(B_0^2 - B(t)^2)\,. \tag{2.4}$$

Of course, stalemate occurs when $B(t) = R(t) = 0$, which yields the Lanchester Square Law:

$$\begin{aligned} bB_0^2 &= rR_0^2 \quad \text{or} \\ B_0 &= \sqrt{\frac{r}{b}}R_0\,. \end{aligned} \tag{2.5}$$

This equation is very important. It says that, to stalemate an adversary three times as numerous, it does not suffice to be three times as effective; you must be nine times as effective! This presumed heavy advantage of *numbers* is deeply embedded in virtually all Pentagon models. For decades, it supported the official dire assessments of the conventional balance in Central Europe, giving enormous weight

---

[19] Indeed, the numerically smaller force was the victor in such notable cases as Austerlitz (1805); Antietam (1862); Fredericksburg (1862); Chancellorsville (1863); the Battle of Frontiers (1914); the fall of France (1940); the invasion of Russia (Operation Barbarossa, 1941); the battle of Kursk (1943); the North Korean invasion (1950); the Sinai (1967); the Golan Heights (1967 and 1973); and the Falklands (1982), to name a few.

to sheer Soviet numbers and placing a huge premium on western technological supremacy. That, of course, had budgetary implications. But, the presumption of overwhelming Soviet *conventional* superiority also shaped the development of so-called theater-nuclear weapons and produced a widespread assumption that their early employment would be inevitable, which drove the Soviets to seek preemptive offensive capabilities, and so on, in an expensive and dangerous military coevolution (see the preceding lecture).

The whole dynamic, while driven by myriad political and military-industrial interests on all sides, was certainly supported by Lanchester's innocent-looking *linear* differential equations (2.1). But, the linearity itself implicitly assumes things that are implausible on reflection and it mathematically precludes phenomena that, in fact, are observed empirically. Moreover, anyone exposed to mathematical biology would have found the Lanchester variant (2.1) to be suspect immediately.

DENSITY

The equations, once again, are

$$\frac{dB}{dt} = -rR, \tag{2.6}$$

$$\frac{dR}{dt} = -bB. \tag{2.7}$$

In this framework, increasing density is a *pure benefit.* If the Red force R grows, a greater volume of fire is focused on the Blue force B, and in (2.6), the Blue attrition rate grows proportionally. At the same time, however, *no penalty* is imposed on Red in (2.7) when, in fact, if the battlefield is crowded with Reds, the Blue target acquisition problem is eased and Red's attrition rate should grow.

In warfare, each side is at once *both predator and prey.* Increasing density is a benefit for an army as predator, but it is a cost for that same army as prey. The Lanchester square system captures the predation benefit but completely ignores the prey cost of density. The latter, moreover, is familiar to us all. For instance, if a hunter fires his gun into a sky black with ducks, he is bound to bring down a few. Yet if a single duck is flying overhead, it takes extraordinary accuracy to shoot it down. For ducks, considered as prey, density carries costs.

And, as any ecologist would expect, the effect is indeed observed. Quoting Herbert Weiss, "the phenomenon of losses increasing with force committed was observed by Richard H. Peterson at the Army Ballistic Research laboratories in about 1950, in a study of tank battles. It was again observed by Willard and the present author [Weiss] has noted its appearance in the Battle of Britain data."[20] The work referred to is D. Willard's statistical study of 1500 land battles.[21]

To his credit, Lanchester actually offered a second, nonlinear variant of these equations, which is much more plausible in this ecological light. Here,

$$\frac{dR}{dt} = (-bB)R, \tag{2.8}$$

$$\frac{dB}{dt} = (-rR)B. \tag{2.9}$$

In parentheses are the Lanchester square terms reflecting the "predation benefit" of density, but they are now multiplied by a term (the prey force level) reflecting "prey costs," as it were. The Red attrition rate in (2.8) slows as the Red population goes to zero, reflecting the fact that, as the prey density falls, the predator's search ("foraging") requirements for the next kill increase. Equivalently, Red's attrition rate grows if, like the ducks in the analogy, its density grows. In summary, a density cost is present to balance the density benefit reflected in the parenthesized term.

If we now form the casualty-exchange ratio

$$\frac{dR}{dB} = \frac{b}{r},$$

separate variables, and integrate as before, we obtain the state equation

$$r(R_0 - R(t)) = b(B_0 - B(t)),$$

and the stalemate requirement

$$rR_0 = bB_0.$$

Now, as against the Lanchester Square Law, it *does* suffice to be three (rather than nine) times as good to stalemate an adversary three times as numerous.

AMBUSH AND ASYMMETRY

Further, asymmetrical, variants of the basic Lanchester equations have been devised. For example, the so-called ambush variant imputes the "square law" fire concentration capacity to one side (the ambushers) but denies it to the other (the ambushees). Here,

$$\frac{dB}{dt} = -rR,$$

$$\frac{dR}{dt} = -bBR,$$

---

[20] Weiss (1966).

[21] Willard (1962).

so that

$$\frac{dB}{dR} = \frac{r}{bB},$$

$$b(B_0^2 - B(t)^2) = r(R_0 - R(t)).$$

Now assuming a fight to the finish ($R(t) = B(t) = 0$) and equal firing effectiveness ($r = b$), a Blue force of B_0 can stalemate a Red force numbering B_0^2—a hundred can hold off ten thousand. It's Thermopolae.

REINFORCEMENT

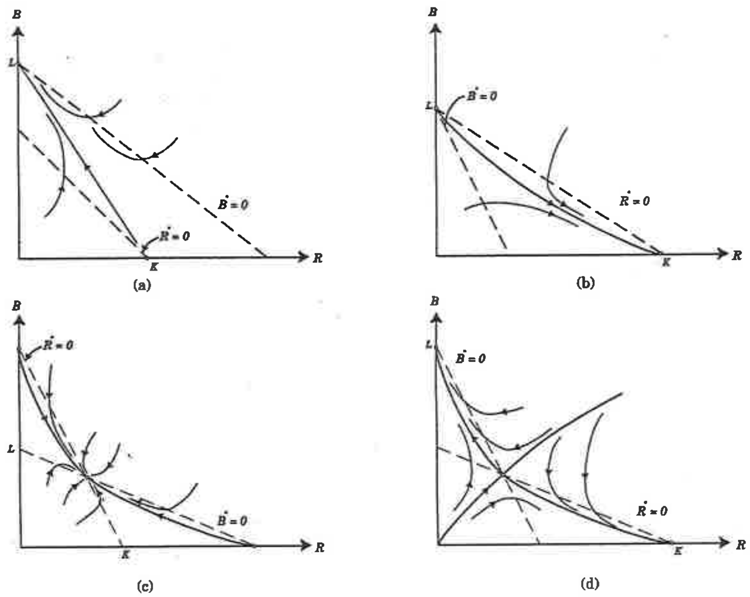
Thus far the discussion has concentrated on the dynamics of *engaged* forces. Often, however, there is some flow of reinforcements to the combat zone proper. But, there are limits to the number of forces one can pack into a given area—there are "force to space" constraints. One might therefore think of the combat zone as having a carrying capacity and, accordingly, posit logistic reinforcement. Attaching such a term to the Lanchester nonlinear attrition model produces

$$\begin{aligned} \frac{dR}{dt} &= -bRB + \alpha R\left(1 - \frac{R}{K}\right), \\ \frac{dB}{dt} &= -rBR + \beta B\left(1 - \frac{B}{L}\right), \end{aligned} \tag{2.10}$$

where α, β, K, and L are positive constants. As observed in the preceding lecture, this is *exactly* Gause's (1935) famous model of competition between two species, itself a form of the general Lotka-Volterra ecosystem equations.

Equations (2.10) admit four basic cases, corresponding to different "war histories." These are shown in the phase portraits in figure 2.1.

FIGURE 2.1 Phase portraits for Lanchester/Gause Model



Source: Based on Clark (1990, p. 194).

Cases (a) and (b) are clear instances of the biological "principle of competitive exclusion," or military principle that one or the other side usually wins. Case (c) shows the horrific stable node—the "permanent war" that neither side wins. Finally, we have case (d), a saddle equilibrium. Any perturbation (off the stable manifold) sends the trajectory to a Red or Blue triumph. There is, however, the interesting and important region below both isoclines. Each side feels encouraged in this zone; reinforcement rates exceed attrition rates so the forces are growing. But, for instance, as the trajectory crosses the $\dot{B} = 0$ isocline, matters start to sour for Blue; $\dot{B}$ goes negative while Red forces continue to grow. Expectations of Blue defeat may set in, Blue morale may collapse, and, as a result, the Blue force can "break" long before it is physically annihilated. Indeed, the general phenomenon of "breakpoints" is common.