HW #4 Solutions (Math 323)

11.2) a) $1, 1, 1, \ldots$; b_n itself; c_n itself, d_n itself b) $1, -1$; 0; ∞; $\frac{6}{7}$ c) $-1, 1$; $0, 0$; ∞, ∞; $\frac{6}{7}, \frac{6}{7}$

d) converge: b_n, d_n; diverge to ∞: a_n; diverge: a_n e) a_n, b_n, d_n

11.4) a) $4, 16, 64 \ldots$; $5, 5, 5, \ldots$; $0, 0, 0, \ldots$; $0, 0, 0, \ldots$ b) $\{\pm\infty\}$; $\{\frac{1}{5}, 5\}$; $\{0, 2\}$; $\{0, \pm\infty\}$ c) $\infty, -\infty$; $5, \frac{1}{5}$; $2, 0$; $\infty, -\infty$ d) None converge or diverge to $\pm\infty$.

11.5) a) $[0, 1]$ (we can get any real number in the interval as a subsequential limit (Example 3 in Section 11). To see that 0 is also a subsequential limit point, consider $(1/n)$. b) $\limsup q_n = 1$, $\liminf q_n = 0$.

11.7) We will find $r_{n_1} < r_{n_2} < \cdots$ with $n_1 < n_2 < \cdots$ such that $r_{n_k} > k$ for all k. This will be the desired subsequence. Clearly we can choose $r_{n_1} > 1$. So, assume we have $r_{n_1} < \cdots < r_{n_k}$ with $n_1 < n_2 < \ldots < n_k$ such that $r_{n_i} > i$. We will show how to choose $r_{n_{k+1}}$. Let $M = r_{n_k}$. Consider $M + 1, M + 2, \ldots$. For some $z \in \mathbb{Z}^+$ we have $M + z \notin \{r_1, r_2, \ldots, r_{n_k}\}$ (since this is a finite set). Let $r_{n_{k+1}} = M + z$. Then $r_{n_{k+1}} > r_{n_k}$ and $n_{k+1} > n_k$ (by construction). Lastly, $r_{n_{k+1}} = M + z \geq M + 1 = r_{n_k} + 1 > k + 1$ and we are done.

11.9) Assume, for a contradiction, that $(a_n) \to c$ with $c \notin [a, b]$ where all a_n are in $[a, b]$. Without loss of generality, let $c > b$. Let $\epsilon = c - b$. Then, there exists N such that for all $n > N$ we have $|a_n - c| < \epsilon = c - b$. Hence, $a_n > b$ for all $n > N$. But then not all a_n are in $[a, b]$, a contradiction.

11.10) a) S$= \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ b) $\limsup s_n = 1$, $\liminf s_n = 0$.

12.1) We have $L_1 = \liminf t_n = \lim_{N\to\infty} \inf(t_n : n > N\}$. If $L_1 = \infty$ we are done, so assume $L_1 \in \mathbb{R}$ (if $L = -\infty$ then we have a subsequence (t_{n_j}) with $t_{n_j} < -j$. Hence, $s_{n_j} < -j$ and $\liminf s_n = -\infty$). Hence, for all $\epsilon > 0$, there exists N_1 such that for all $m > N_1$, we have $|\inf(t_n : n > m) - L_1| < \epsilon$. Likewise, we have for all $\epsilon > 0$, there exists N_2 such that for all $m > N_2$, we have $|\inf(s_n : n > m) - L_2| < \epsilon$. Consider $\hat{N} = \max(N_0, N_1, N_2)$. If we assume $L_2 > L_1$, take $\epsilon = \frac{1}{2}(L_2 - L_1)$ to see that for all $m \geq \hat{N}$, $\inf(t_n : n > m) < \frac{L_1+L_2}{2}$ and $\inf(s_n : n > m) > \frac{L_1+L_2}{2}$. Let $t = \inf(t_n : n > \hat{N})$. Since $t < \frac{L_1+L_2}{2}$, by the definition of inf, we have $t \leq t_x < \frac{L_1+L_2}{2}$ for some $x > \hat{N}$. But we have $s_n > \frac{L_1+L_2}{2}$ for all $n > \hat{N}$, in particular, $s_x > \frac{L_1+L_2}{2}$, a contradiction (since we are given $s_x \leq t_x$). Hence, $L_2 \leq L_1$ as desired. The $\limsup$ case is very similar.

12.2) If $\lim s_n = 0$ then $\limsup s_n = 0$ so that $\limsup |s_n| = 0$. Now, if $\limsup |s_n| = 0$, since $|s_n| \geq 0$ for all n, we have $\liminf |s_n| \geq 0$. Using $0 \leq \liminf |s_n| \leq \limsup |s_n| = 0$, we have $\liminf |s_n| = \limsup |s_n| = \lim |s_n| = 0$.

12.3) a) 0; b) 1; c) 2; d) 3; e) 3; f) 2

12.4) Use the hint. Since the sequences are bounded, the sups are real numbers. Let $s_N = \sup(s_n : n > N)$ and $t_N = \sup(t_n : n > N)$. Recall Exercise 4.14. If $S = \{s + t : s \in (s_n), t \in (t_n)\}$ then $\sup(S) = \sup(s_n) + \sup(t_n)$. Now, consider $T = \{s_n + t_n : n \in \mathbb{Z}^+\}$. Then $T \subseteq S$. By Exercise 4.7, we have $\sup(T) \leq \sup(S)$. Hence, the inequality in the hint is true. Now take the limit as $N \to \infty$ of both sides to get the result. More technically, consider $(\sup(s_n + t_n) - (\sup(s_n) + \sup(t_n))_{n>N}$. From the hint's inequality, this is at most 0 for all N. Hence, the limit is at most 0.

12.6) a) Let $v_N = \sup(s_n : n > N)$. We already have that $\lim(kx_n) = k \lim(x_n)$ for any sequence. Well, v_N is a sequence so we are done.

b) Define $u_N = \inf(s_n : n > N)$ and use the reasoning in part (a). We don't need 11.8.

c) If $k < 0$, then $\liminf(ks_n) = k \limsup(s_n)$.

12.7) Use the reasoning in 12.6a.

12.9) a) By Corollary 11.4, there exists $t_{n_j} \to t > 0$. Hence, for some N_0, for all $j > N_0$ we have $t_{n_j} > t/2$. We also have for all $n > N_1$, $s_n > M$ (for any M). Let $N = \max(N_0, N_1)$, then $s_{n_j} t_{n_j} > Mt/2$, which can be arbitrarily large. Hence, $\limsup(s_n t_n) = \infty$.

b) The same proof as in (a) does not work since we would need the subsequences to "line up." We do have, however, by the definition of the limit, that for some N_1, for all $m > N_1$, $\sup(s_n : n > m) > M$ for any given $M > 0$. Also, for some N_2, for all $m > N_2$ we have $v_m = \inf(t_n : n > m) > 0$. So, let $N = \max(N_1, N_2)$. Then, $\sup(s_n t_n : n > m) > M \inf(t_n : n > m) = M \inf(v_n : n > m) = Mt$ (with $t > 0$). Since t is fixed, and M can be arbitrary, Mt can be arbitrarily large.

c) Nothing to do here.

12.10) If (s_n) is bounded, say $|s_n| < M$, then clearly $\limsup |s_n| \leq M$. Since $\limsup |s_n| < \infty$ (it can't be $-\infty$ because $|s_n| \geq 0$ for all n), let $\limsup |s_n| = s \in \mathbb{R}$. We must show (s_n) is bounded. Assume not. Then given $M > 0$, there exists n such that $|s_n| > M$. Take $M = s$ to get $|s_{n_1}| > s$. Next, take $M = |s_{n_1}|$ to get $|s_{n_2}| > |s_{n-1}| > s$. We make choose $n_2 > n_1$ for otherwise, (s_n) would be bounded by $\max(s_1, s_2, \ldots, s_{n1})$. Continue to get a sequence bounded *below* by s. Then $\limsup |s_n| > s$, a contradiction.

12.13) Let $c = \sup A$. If $d > c$, then $s_n \geq d$ for all but finitely many s_n is not true. Hence $d \neq \liminf s_n$ (since we have infinitely many s_n less than d). If $d < c$, then by construction, only finitely many s_n lie between d and c. Hence, c is a better lower bound than d for the statement $s_n \geq x$ for all but finitely many s_n. Thus, d would not be $\liminf s_n$. The only remaining choice is $c = d = \liminf s_n$.

12.14) a) Let $s_n = n!$. We have $\left|\frac{s_{n+1}}{s_n}\right| = n + 1$. Since this diverges to ∞, by Theorem 12.2 the given limit is ∞.

b) Let $t_n = \frac{n!}{n^n}$. We have $\left|\frac{s_{n+1}}{s_n}\right| = \left(\frac{n}{n+1}\right)^n$. Since $\left(\frac{n}{n+1}\right)^n = \frac{1}{1-1/(n+1)} \cdot \left(1 - \frac{1}{n+1}\right)^{n+1} \to 1 \cdot \frac{1}{e} = \frac{1}{e}$, we see that, by Theorem 12.1, the given limit is $\frac{1}{e}$.