Arithmetic Progressions, Quasi Progressions, and Gallai-Ramsey Colorings*

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Abstract

We investigate several functions related to the r-color off-diagonal van der Waerden numbers $\mathrm{w}(m_1,\ldots,m_r)$, where $\mathrm{w}(m_1,\ldots,m_r)$ is the minimal integer n such that every r-coloring of $\{1,2,\ldots,n\}$ admits an m_i-term arithmetic progression with all terms of color i for some $i \in \{1,2,\ldots,r\}$. We start by giving a new lower bound for these related numbers. Next, the exact values and bounds of numbers related to quasi-progressions and mixed quasi-progression-van der Waerden numbers are given. Then, inspired by the success of graph Gallai-Ramsey theory and rainbow arithmetic progressions, we introduce the concept of Gallai-van der Waerden numbers, and obtain some exact values and bounds for these numbers, some of which are derived by the probabilistic method and the Lovász Local Lemma.

Keywords: Ramsey Theory; Arithmetic Progression; Quasi Progression; Gallai-van der Waerden number; Lovász Local Lemma.

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1 Introduction and Brief Survey of Previous Results

Ramsey-type problems were introduced in 1930. This subject has been a hot topic in mathematics for decades now due to their intrinsic beauty, wide applicability, and overwhelming difficulty despite somewhat misleadingly simple statements; see [13] and [28].

This section introduces background and related results of the Ramsey-type problems we will be investigating.

1.1 Monochromatic Arithmetic Progressions and van der Waerden's Theorem

An *ℓ-term arithmetic progression* (simply, ℓ-AP) is a set S such that $S = \{a + id : 0 \leq i < \ell\} = \{a, a+d, a+2d, \ldots, a+(\ell-1)d\}$ for some integers a, d, and $d \neq 0$.

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In 1927, B. L. van der Waerden [15] published a proof of the following unexpected result.

Theorem 1.1 ([15]). *If the positive integers are partitioned into two classes, then at least one of the classes must contain arbitrarily long arithmetic progressions.*

There are two rather harmless looking modifications we make in the statement of van der Waerden's theorem, both of which have a major impact on the proof. The statement is as follows, where we introduce the standard notation $[1, n] = \{1, 2, \ldots, n\}$:

Theorem 1.2. (van der Waerden's Theorem) *For all r, ℓ, there exists n_0 so that, for $n \geq n_0$, if $[1, n]$ is r-colored there exists a monochromatic ℓ-AP.*

Definition 1.3. For all positive integers r and ℓ, the *van der Waerden number* $\mathrm{w}(r; \ell)$ is defined as the minimal integer such that for $n \geq \mathrm{w}(r; \ell)$, if $[1, n]$ is r-colored there exists a monochromatic ℓ-AP.

Definition 1.4. For all positive integers r and $m_1, \ldots, m_r$, the *off-diagonal van der Waerden number* $\mathrm{w}(m_1, \ldots, m_r)$ is defined as the minimal integer such that for $n \geq \mathrm{w}(m_1, \ldots, m_r)$, if $[1, n]$ is r-colored, then there exists an m_i-AP for some color i, where $1 \leq i \leq r$.

If $m_1 = m_2 = \cdots = m_r = m$, then we have $\mathrm{w}(m_1, \ldots, m_r) = \mathrm{w}(r; m)$. For more details on arithmetic progressions and van der Waerden numbers, we refer to the book [25] by Landman and Robertson and some papers [4, 2, 5, 7].

Recent progress has been made concerning lower bounds on van der Waerden numbers. In particular, Kozik and Shabanov [22] obtained the following lower bound.

Theorem 1.5. [22] *There exists a positive constant c such that for every $r \geq 2$ and $k \geq 3$, we have*

$$\mathrm{w}(r; k) > cr^{k-1}.$$

More recently, Green [17] deduced a non-polynomial lower bound for $\mathrm{w}(3, k)$, which was widely believed to have polynomial growth (perhaps quadratic, even).

Theorem 1.6. [17] *There exists a constant $c > 0$ such that for k sufficiently large,*

$$\mathrm{w}(3, k) > k^{c\left(\frac{\log k}{\log \log k}\right)^{1/3}}.$$

1.2 Quasi-progressions

Related to arithmetic progressions, but with less stringent criteria, are quasi-progressions.

Definition 1.7. Let k and s be integers with $k \geq 1$ and $s \geq 0$. A *k-term quasi-progression of diameter s* is a sequence of positive integers $\{x_1, \ldots, x_k\}$ for which there exists a positive integer d such that $d \leq x_i - x_{i-1} \leq d + s$ for $i = 2, \ldots, k$. We call the integer d a *low-difference* for the quasi-progression $X = \{x_1, \ldots, x_k\}$. We say that X is a *(k, s, d)-QP*.

Since quasi-progressions of diameter 0 are arithmetic progressions, and the set of quasi-progressions of diameter s is a subset of those of diameter t for any $t \geq s$, Theorem 1.1 allows us to make the following definition.

Definition 1.8. For positive integers s and k, denote by $Q(k, s)$ the least positive integer n such that for every 2-coloring of $[1, n]$ there is a monochromatic (k, d, s)-QP for some low-difference d. When we are not concerned with the low-difference, as is the case here, we will refer to the quasi-progressions as (k, s)-QPs.

Landman [23] gave a lower bound for $Q(k, k-i)$ in terms of k and i that holds for all $k > i \geq 1$ along with upper bounds for $Q(k, s)$ when $s \geq \lceil 2k/3 \rceil$. In particular, Landman showed that $Q(k, \lceil 2k/3 \rceil) = \frac{43}{324}k^2(1 + o(1))$. Exact formulae for $Q(k-1, k)$ and $Q(k-2, k)$, a table of computer-generated values of $Q(k, s)$ for small k and s, and several conjectures can also be found in [23].

For more details on the quasi-progressions, we refer to the book [25] and papers [5, 20, 21, 23, 24].

Definition 1.9. For positive integers s, r and k, denote by $\mathrm{Q}(r; k, s)$ the least positive integer n such that for every r-coloring of $[1, n]$ there is a monochromatic k-term quasi-progression of diameter s.

The following result is immediate by definition.

Theorem 1.10. *Let r, k, n be integers with $r \geq 2$, $k \geq 2$, and $s \geq 0$. Then $Q(r; k, s) \leq Q(r; k, 0) = \mathrm{w}(r; k)$, so that monochromatic (k, s)-QPs exist under any r-coloring of the positive integers.*

We will also be investigating the behavior of quasi-progressions and arithmetic progressions together.

Definition 1.11. For positive integers k and m, the *mixed quasi-progression-van der Waerden number* $\mathrm{QW}(m; k, s)$ is defined as the minimum integer such that for $n \geq \mathrm{QW}(m; k, s)$, if $[1, n]$ is 2-colored there exists a monochromatic m-AP of the first color or a monochromatic (k, s)-QP of the second color.

Existence of $\mathrm{QW}(m; k, s)$ is also implied by van der Waerden's Theorem.

For more than 2 colors, we have the following definition, with existence also implied by van der Waerden's Theorem.

Definition 1.12. For positive integers $a < r$, let $m_1, \ldots, m_r$ be integers with $3 \leq m_1 \leq \cdots \leq m_a$ and $3 \leq m_{a+1} \leq \cdots \leq m_r$. Let s be a nonnegative integer. The *mixed r-color quasi-progression-van der Waerden number*, denoted $\mathrm{QW}(m_1, \ldots, m_a; m_{a+1}, \ldots, m_r; s)$ is the minimal integer n such that every r-coloring of $[1, n]$ admits either an m_i-AP of color i for some $i \in [1, a]$ or a monochromatic (m_j, s)-QP of color j for some $j \in [a+1, r]$.

1.3 Gallai-Ramsey Numbers and Rainbow Arithmetic Progressions

Colorings of the edges of complete graphs that contain no rainbow triangle have a very interesting and somewhat surprising structure. In 1967, Gallai [16] first examined this structure under the guise of transitive orientations. The result was reproven in [19] in the terminology of graphs and can also be traced to [9].

If G and H are two graphs, we write $F \longrightarrow (G, H)$ to denote that G or H is a monochromatic subgraph of G in every 2-coloring of the edges of F. The *Ramsey number* $\mathrm{r}(G, H)$ of a graph F is defined as $\mathrm{r}(G, H) = \min\{n : K_n \longrightarrow (G, H)\}$. If G and H are two graphs, we write $F \overset{\mathrm{gr}_k}{\longrightarrow} (G, H)$ to denote that G is a rainbow subgraph or H is a monochromatic subgraph of F in every k-coloring of the edges of F. The *k-colored Gallai-Ramsey number* $\mathrm{gr}_k(G, H)$ of a graph F is defined as $\mathrm{gr}_k(G, H) = \min\{n : K_n \overset{\mathrm{gr}_k}{\longrightarrow} (G, H)\}$.

We refer the interested reader to [29] for a dynamic survey of small Ramsey numbers and [12] for a dynamic survey of rainbow generalizations of Ramsey theory, including topics like Gallai-Ramsey numbers.

In [14], Jungić *et al.* studied a rainbow counterpart of van der Waerden's theorem: Given positive integers r and ℓ, what conditions on r-colorings of $[1, n]$ guarantee the existence of a rainbow ℓ-AP? The *anti-van der Waerden number* $\mathrm{aw}(S, k)$ is the smallest r such that any r-coloring (that uses every color at least once) of S contains a rainbow k-term arithmetic progression. Note that this tautologically defines $\mathrm{aw}(S, k) = |S| + 1$ whenever $|S| < k$, and this definition retains the property that there is a coloring with $\mathrm{aw}(S, k) - 1$ colors that has no rainbow k-AP. Several important results on the existence of rainbow 3-APs implying information about $\mathrm{aw}([1, n], 3)$ and $aw(\mathbb{Z}_n, 3)$ have been established by Jungić, et al. [14]. For more details on the rainbow AP, we refer to [1, 8, 10, 14, 30].

Combining the above two concepts, we introduce a Gallai-Ramsey Version of van der Waerden's Theorem.

When dealing with rainbow structures, we need to be careful in certain situations regarding the number of colors truly used in a coloring. As such, we make use of the following definition from [3].

Definition 1.13. An r-coloring is *exact* if all colors are used at least once.

The following corollary, following from Theorem 1.2, can be regarded as the Gallai-Ramsey version of van der Waerden's Theorem.

Corollary 1.14. *For all r, k, ℓ, there exists n_0 so that, for $n \geq n_0$, every exact r-coloring of $[1, n]$ admits either a rainbow k-AP or a monochromatic ℓ-AP.*

As a generalization of the classical van der Waerden numbers, we propose the following two new concepts.

Definition 1.15. For all positive integers r, k, ℓ, the *Gallai-van der Waerden number* (simply, GW *number*) $\mathrm{GW}(r; k, \ell)$ is defined as the least integer so that, for all $n \geq \mathrm{GW}(r; k, \ell)$, every exact r-coloring of $[1, n]$ admits either a rainbow k-AP or a monochromatic ℓ-AP.

Remark 1.16. We need to be careful when applying Definition 1.15. For example, if we consider $\mathrm{GW}(3; 3, 3)$ we see that any exact 3-coloring of $[1, 3]$ admits a rainbow 3-AP. However, the exact 3-coloring of $[1, 4]$ with the colors of 2 and 3 being the same admits no rainbow nor monochromatic 3-AP.

We may also consider the situation where the colorings need not be exact. Since r-colorings of $[1,n]$ may be partitioned into exact colorings and colorings using less than r colors, the existence of the following numbers holds.

Definition 1.17. For all positive integers r,k,ℓ, define $\mathrm{GW}'(r;k,\ell)$ to be the least integer so that, for $n \geq \mathrm{GW}'(r;k,\ell)$, if $[1,n]$ is r-colored, then there exists either a rainbow k-AP or a monochromatic ℓ-AP.

Remark 1.18. Note that $\mathrm{GW}'(r;k,\ell) \leq \max\{\mathrm{GW}(i;k,\ell) : 1 \leq i \leq r\}$.

2 Improved Bounds for Off-diagonal van der Waerden Numbers

We can derive a lower bound by the result in [22].

Theorem 2.1. *There exists a positive constant c such that for every $r \geq 2$ and $m_1 \geq 3$, if $m_{i+1} - \sum_{j=1}^{i} m_j \geq 3$ for $1 \leq i \leq r-2$ and $m_r \geq \sum_{j=1}^{r-1} m_j$, then*

$$\mathrm{w}(m_1,\ldots,m_r) \geq \sum_{i=1}^{r-2} c(r-i)^{(m_{i+1}-\sum_{j=1}^{i} m_j)} + cr^{m_1-1} + m_r - \sum_{j=1}^{r-1} m_j + 1.$$

Proof. Define the intervals

$$L = \left[cr^{m_1-1}+1, cr^{m_1-1} + c(r-1)^{(m_2-m_1)}\right]$$

$$M_s = \left[\sum_{i=1}^{s} c(r-i)^{(m_{i+1}-\sum_{j=1}^{i} m_j)} + cr^{m_1-1}+1, \sum_{i=1}^{s+1} c(r-i)^{(m_{i+1}-\sum_{j=1}^{i} m_j)} + cr^{m_1-1}\right], \quad 1 \leq s \leq r-3$$

$$N = \left[\sum_{i=1}^{r-2} c(r-i)^{(m_{i+1}-\sum_{j=1}^{i} m_j)} + cr^{m_1-1}+1, \sum_{i=1}^{r-2} c(r-i)^{(m_{i+1}-\sum_{j=1}^{i} m_j)} + cr^{m_1-1} + m_r - \sum_{j=1}^{r-1} m_j + 1\right].$$

From Theorem 1.5, for any number of colors s, there exists an s-coloring of $[1, cs^{m_i-1}]$ containing no monochromatic m_i-AP. Accordingly, color the above-defined intervals using the following colors with a coloring that avoids monochromatic m_i-APs. Note that we are being loose with the constant c in the above intervals; however, clearly there exists a positive constant c that can work uniformly for all appeals the Theorem 1.5 (e.g., the minimum c used over all applications of Theorem 1.5).

- Color $[1, cr^{m_1}]$ with colors $1,2,\ldots,r$ avoiding monochromatic m_1-APs;
- Color L with colors $2,3,\ldots,r$ avoiding monochromatic (m_2-m_1)-APs;
- For each $s \in [1, r-3]$, color M_s with colors $s+2, s+3, \ldots, r$ avoiding monochromatic $(m_{s+2} - \sum_{j=1}^{s+1} m_j)$−APs;
- Color all $m_r - \sum_{j=1}^{r-1} m_j$ elements of N with color r.

By construction, there is no monochromatic m_i-AP of color i for any $i \in [1,r]$, thereby proving the bound. □

3 Quasi-progressions

The following two results can be easily obtained by the methods in [23].

Theorem 3.1. *Let r, k, s be integers with $r \geq 2$, $k \geq 2$, and $s \geq 1$. Then*

$$Q(r; k, s) \geq r(k-1) + 1.$$

Proof. Consider the r-coloring $\chi : [1, rk - r] \longrightarrow \{0, 1, \ldots, r-1\}$ defined by

$$\chi([(i-1)k - i + 2, ik - i]) = i - 1,$$

where $1 \leq i \leq r$. This coloring admits no monochromatic k-element set. In particular, it yields no k-term monochromatic quasi-progression of diameter s. Therefore, we have $Q(r; k, s) \geq r(k-1) + 1$. ☐

Theorem 3.2. *Let r and k be integers with $r \geq 2$ and $k \geq 2$. Then*

$$Q(r; k, (r-1)(k-1)) \leq r(k-1) + 1.$$

Proof. Let χ be an arbitrary r-coloring of $[1, r(k-1)+1]$. Clearly, there is some k-element set $X = \{x_1, \ldots, x_k\}$ where $x_1 < x_2 < \cdots < x_k$ that is monochromatic under χ. If for some j, $2 \leq j \leq k$, we have $x_j - x_{j-1} > (r-1)(k-1) + 1$, then

$$x_k - x_1 = \sum_{i=2}^{k} (x_i - x_{i-1}) > (r-1)(k-1) + 1 + (k-2) = r(k-1),$$

which is impossible. Thus, X is a monochromatic $(k, (r-1)(k-1), 1)$-progression, since $1 \leq x_i - x_{i-1} \leq (r-1)(k-1) + 1$ for $2 \leq i \leq k$. This shows that $Q(r; k, (r-1)(k-1)) \leq r(k-1) + 1$. ☐

The following corollary is immediate.

Corollary 3.3. *Let r, k, s be positive integers with $r \geq 2$ and $s \geq (r-1)(k-1)$. Then*

$$Q(r; k, s) = r(k-1) + 1.$$

For $Q(r; k, 1)$, we can give a lower bound better than in Theorem 3.1.

Theorem 3.4. *Let r, k, n be positive integers with $r \geq 2$, $k \geq 2$. Then*

$$Q(r; k, 1) \geq r(k-1)^2 + 1.$$

Proof. Define the r-coloring χ of $[1, r(k-1)^2]$ by the string

$$\underbrace{0\cdots0}_{k-1}\underbrace{1\cdots1}_{k-1}\cdots\underbrace{(r-1)\cdots(r-1)}_{k-1}\underbrace{0\cdots0}_{k-1}\underbrace{1\cdots1}_{k-1}\underbrace{(r-1)\cdots(r-1)}_{k-1}\cdots\underbrace{0\cdots0}_{k-1}\underbrace{1\cdots1}_{k-1}\underbrace{(r-1)\cdots(r-1)}_{k-1}$$

where each of the $(r-1)(k-1)$-element blocks

$$\underbrace{00\cdots 0}_{k-1}\underbrace{11\cdots 1}_{k-1}\ldots\underbrace{(r-1)(r-1)\cdots(r-1)(r-1)\cdots(r-1)}_{k-1}$$

appears $k-1$ times. To prove this theorem, it suffices to show that under this coloring there is no k-term monochromatic quasi-progression of diameter 1.

By way of contradiction, let $m = r(k-1)^2$, and assume that $X = \{x_1, \ldots, x_k\} \subseteq [1, m]$ is a quasi-progression of diameter 1 that is a monochromatic under χ. By the symmetry of χ, without loss of generality, we may assume that $\chi(X) = 1$. Since each monochromatic block of color 1 has $k-1$ elements, there is some i, $2 \le i \le k$, where x_i and x_{i-1} belong to two different such blocks. For this i, we have $x_i - x_{i-1} \ge (r-1)(k-1)+1$. Since X has diameter 1, this implies that X has a low-difference of at least $k-1$. Thus, each of the blocks of $k-1$ consecutive 1's contains no more than one member of X. Hence, X must have length at most $k-1$, a contradiction. $\square$

In the following, we obtain a lower bound for some specific instances of $Q(r;k,s)$.

Theorem 3.5. *Let r, k, n be positive integers with $r \ge 2$, $(j-2)(k-1)+1 \le i < (j-1)(k-1)+1$, and $2 \le j \le r - \lfloor \frac{r-1}{2} \rfloor$. Let $m = 1 + \left\lfloor \frac{k-2}{i-(j-2)(k-1)} \right\rfloor$. Then*

$$Q\left(r;k,\left(r-1-\left\lfloor\frac{r-1}{2}\right\rfloor\right)(k-1)+\left\lfloor\frac{r-1}{2}\right\rfloor y+1-i+(j-2)(k-1)\right) \ge r\left(\left\lfloor\frac{k-1}{m}\right\rfloor(k-1)+y\right)+1,$$

where $y = (i-(j-2)(k-1))\left((k-1) - m\left\lfloor\frac{k-1}{m}\right\rfloor\right)$.

Proof. Let

$$s = r\left(\left\lfloor\frac{k-1}{m}\right\rfloor(k-1)+y\right).$$

Define the r-coloring χ of $[1, s]$ by the string

$$\left(\left\lfloor\frac{r-1}{2}\right\rfloor+1\right)^y\left(\left\lfloor\frac{r-1}{2}\right\rfloor+2\right)^y\ldots(r-1)^y\left(0^{k-1}1^{k-1}\ldots(r-1)^{k-1}0^{k-1}1^{k-1}\ldots(r-1)^{k-1}0^{k-1}1^{k-1}\ldots(r-1)^{k-1}\right)0^y1^y\ldots\left\lfloor\frac{r-1}{2}\right\rfloor^y,$$

where, within the parentheses, each of the blocks $0^{k-1}1^{k-1}\ldots(r-1)^{k-1}$ occurs $\lfloor\frac{k-1}{m}\rfloor$ times. Note that this is, in fact, a string of length s. It is sufficient to show that, under χ, $[1, s]$ contains no monochromatic k-term quasi-progression of diameter $(r-1-\lfloor\frac{r-1}{2}\rfloor)(k-1)+\lfloor\frac{r-1}{2}\rfloor y+1-i$. We proceed by contradiction.

Assume that $X = \{x_1, \ldots, x_k\} \subseteq [1, s]$ is a quasi-progression of diameter $(r-1-\lfloor\frac{r-1}{2}\rfloor)(k-1)+\lfloor\frac{r-1}{2}\rfloor y+1-i$ that is monochromatic under χ. By the symmetry of χ, we may assume that $\chi(X) = 1$.

Since $m\lfloor\frac{k-1}{m}\rfloor \geq k-m$, it follows that

$$
\begin{aligned}
y &= (i-(j-2)(k-1))\left(k-1-m\left\lfloor\frac{k-1}{m}\right\rfloor\right) \\
&\leq (i-(j-2)(k-1))((k-1)-(k-m)) \\
&= (i-(j-2)(k-1))\left\lfloor\frac{k-2}{i-(j-2)(k-1)}\right\rfloor \\
&\leq k-2.
\end{aligned}
$$

Hence, there is no block of more than $k-1$ consecutive 1s. Thus, for some $j \in \{2,3,\ldots,k\}$, we have $x_j - x_{j-1} \geq (r-1-\lfloor\frac{r-1}{2}\rfloor)(k-1)+\lfloor\frac{r-1}{2}\rfloor y+1$, which implies that X can not have a low-difference that is less than $i-(j-2)(k-1)$.

Since the low-difference of X is at least $i-(j-2)(k-1)$, the first block of 1s (having length y), contains at most $\frac{y}{i-(j-2)(k-1)} = k-1-m\left\lfloor\frac{k-1}{m}\right\rfloor$ members of X. Similarly, in any block of $k-1$ consecutive 1s, there are at most $1+\lfloor\frac{k-2}{i-(j-2)(k-1)}\rfloor = m$ members of X. There are $\lfloor\frac{k-1}{m}\rfloor$ blocks of $k-1$ consecutive 1s, we see that X has at most

$$
k-1-m\left\lfloor\frac{k-1}{m}\right\rfloor + m\left\lfloor\frac{k-1}{m}\right\rfloor = k-1
$$

elements, a contradiction. □

We also have the following lower bound for $Q\,(r;k,(r-1)(k-1)+1-i)$.

Theorem 3.6. *Let r,k,i be positive integers with $r \geq 2$, and let $m = 1+\lfloor\frac{k-2}{i}\rfloor$. Then*

$$
Q\,(r;k,(r-1)(k-1)+1-i) \geq \left\lfloor\frac{k-1}{m}\right\rfloor(rk-r-2m)+2i(k-1)+1\,.
$$

Proof. Let

$$
s = \left\lfloor\frac{k-1}{m}\right\rfloor(rk-r-2im)+2i(k-1).
$$

Define the r-coloring χ of $[1,s]$ by the string

$$
(r-1)^y\left(0^{k-1}1^{k-1}\ldots(r-1)^{k-1}0^{k-1}1^{k-1}\ldots(r-1)^{k-1}0^{k-1}1^{k-1}\ldots(r-1)^{k-1}\right)0^y
$$

where within the parentheses each of the blocks $0^{k-1}1^{k-1}\ldots(r-1)^{k-1}$ occurs $\lfloor\frac{k-1}{m}\rfloor$ times, and where $y = i\left((k-1)-m\left\lfloor\frac{(k-1)}{m}\right\rfloor\right)$. Note that this is, in fact, a string of length s. It is sufficient to show that, under χ, the interval $[1,s]$ contains no monochromatic k-term quasi-progression of diameter $(r-1)(k-1)+1-i$. We proceed by contradiction.

Assume that $X = \{x_1,\ldots,x_k\} \subseteq [1,s]$ is a quasi-progression of diameter $(r-1)(k-1)+1-i$ that is monochromatic under χ. By the symmetry of χ, we may assume that $\chi(X)=1$. Since $m\lfloor\frac{k-1}{m}\rfloor \geq k-m$, it follows that

$$
y = i\left(k-1-m\left\lfloor\frac{k-1}{m}\right\rfloor\right) \leq i((k-1)-(k-m)) = i\left\lfloor\frac{k-2}{i}\right\rfloor \leq k-2.
$$

Hence, there is no block of more than $k-1$ consecutive 1s. Thus, for some $j \in \{2,3,\ldots,k\}$, we have $x_j - x_{j-1} \geq (r-1)(k-1)+1$, which implies that X can not have a low-difference that is less than i.

Since the low-difference of X is at least i, the first block of 1s (having length y), contains at most $\frac{y}{i} = k-1-m\left\lfloor\frac{k-1}{m}\right\rfloor$ members of X. Similarly, in any block of $k-1$ consecutive 1s, there are at most $1+\left\lfloor\frac{k-2}{i}\right\rfloor = m$ members of X. There are $\left\lfloor\frac{k-1}{m}\right\rfloor$ blocks of $k-1$ consecutive 1s, we see that X has at most

$$k-1-m\left\lfloor\frac{k-1}{m}\right\rfloor + m\left\lfloor\frac{k-1}{m}\right\rfloor = k-1$$

elements, a contradiction. □

The following corollary is immediate from Theorem 3.6.

Corollary 3.7. *Let r,k,i be positive integers with $r \geq 2$. The following hold.*

(1) *If $k \equiv 0 \pmod{i}$, then $Q\left(r;k,(r-1)(k-1)+1-i\right) \geq (ir+2i-r)(k-1)-2ki+2k+1$.*

(2) *If $k \equiv 1 \pmod{i}$, then $Q\left(r;k,(r-1)(k-1)+1-i\right) \geq ir(k-1)+1$.*

The last theorem allows us to provide an equality for certain instances of $Q(r;k,s)$.

Theorem 3.8. *Let r,k be positive integers with $r \geq 2$. Then*

$$Q\left(r;k,(r-1)(k-1)\right) = r(k-1)+1$$

Proof. From Theorem 3.6 (or Theorem 3.1) we obtain $Q\left(r;k,(r-1)(k-1)\right) \geq r(k-1)+1$. To obtain a matching upper bound, notice that any k integers in $[1, r(k-1)+1]$ form a $(k,(r-1)(k-1),1)$-QP. By the pigeonhole principle, there exist at least k integers of the same color under any r-coloring of $[1, r(k-1)+1]$. □

Theorem 3.9. *Let r,k,n be positive integers with $r \geq 2$. Then*

$$Q\left(r;k,(r-1)(k-1)-1\right) = \begin{cases} (r+2)(k-1)-1 & \text{if } k \text{ is even,} \\ 2r(k-1)+1 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. The lower bound follows from Corollary 3.7 with $i=2$. To obtain the upper bounds, let $\chi : \mathbb{Z}^+ \longrightarrow \{0,1,\ldots,r-1\}$ be any r-coloring. We will show that if k is even, then there is a monochromatic k-term quasi-progression with diameter $(r-1)(k-1)-1$ in $[1,(r+2)(k-1)-1]$, and that if k is odd then there exists such a progression in $[1, 2r(k-1)+1]$. For each case we assume, for a contradiction, that no such monochromatic quasi-progression exists.

To avoid a monochromatic k-term quasi-progression with diameter $(r-1)(k-1)-1$, we see that the number of elements of each color in $[1, r(k-1)]$ must be exactly $k-1$. Noting that quasi-progressions are translation invariant, we see that any interval of length $r(k-1)$ must contain exactly $k-1$ elements of each color. Comparing the colors of integers of $[1, r(k-1)]$ with $[2, r(k-1)+1]$ we see that $\chi(1) = \chi(r(k-1)+1)$. Comparing $[2, r(k-1)+1]$ with $[3, r(k-1)+2]$ we obtain $\chi(2) = \chi(r(k-1)+1$. Continuing in this fashion, we deduce that $[1, 2k-1]$ and $[r(k-1)+1, (r+2)(k-1)-1]$ are colored in exactly the same manner. We will use this fact in both cases.

`Case 1.` k is even.

Without loss of generality, we assume that $\chi(r(k-1)) = r-1$ and that for some $a \in [1, k-1]$ we have $\chi(r(k-1)+a) = 0$. Hence, we can conclude that $[1, a-1]$ (which may be empty) contains only integers of color $r-1$ while $\chi(a) = 0$. If $a < k-1$ then there exists an integer in $[a+1, r(k-1)-1]$ of color $r-1$. From this we can conclude that the $k-1$ integers in $[1, r(k-1)]$ of color $r-1$ form a $(k-1)$-term quasi-progression with low-difference 2 and diameter $(r-1)(k-1)-1$. Consequently, any integer of color $r-1$ in $[r(k-1)+1, (r+2)(k-1)-1]$ when appended to this $(k-1)$-term quasi-progression would create a monochromatic k-term quasi-progression with low-difference 2 and diameter $(r-1)(k-1)-1$. If $a > 1$, then such an integer exists. Hence, if $a < k-1$ then we must have $a = 1$.

We finish this case by considering two subcases: $a = 1$ and $a = k-1$.

`Subcase i.` $a = 1$: $\chi(1) = \chi(r(k-1)+1) = 0$

To avoid a monochromatic k-term quasi-progression with diameter $(r-1)(k-1)-1$, all integers of color 0 in $[1, r(k-1)]$ must form the interval $[1, k-1]$. Consequently, $[r(k-1)+1, (r+1)(k-1)]$ contains only integers of color 0. We are done with this subcase by noting that the progression

$$\left\{2i-1 : 1 \le i \le \frac{k}{2}\right\} \cup \left\{r(k-1)+2i-1 : 1 \le i \le \frac{k}{2}\right\}$$

is a monochromatic (of color 0) quasi-progressions with low-difference 2 and diameter $(r-1)(k-1)-1$ contained in $[1, (r+2)(k-1)-1]$.

`Subcase ii.` $a = k-1$: $\chi(k-1) = \chi((r+1)(k-1)) = 0$.

(This subcase is essentially the same as Subcase i translated by $k-2$.) To avoid a monochromatic k-term quasi-progression with diameter $(r-1)(k-1)-1$, all integers of color 0 in $[k-1, (r+1)(k-1)]$ must form the interval $[k-1, 2k-3]$. Consequently, $[(r+1)(k-1), (r+2)(k-1)-1]$ contains only integers of color 0. We are done with this subcase by noting that the progression

$$\left\{k+2i-3 : 1 \le i \le \frac{k}{2}\right\} \cup \left\{(r+1)(k-1)+2i-2 : 1 \le i \le \frac{k}{2}\right\}$$

is a monochromatic (of color 0) quasi-progressions with low-difference 2 and diameter $(r-1)(k-1)-1$ contained in $[1, (r+2)(k-1)-1]$.

`Case 2.` k is odd. Each of $[1, r(k-1)]$ and $[r(k-1)+1, 2r(k-1)]$ must contain exactly $k-1$ integers of each color. Without loss of generality, we may assume that $\chi(2r(k-1)+1) = 0$. As argued in Case 1, this implies that $[r(k-1)+1, (r+1)(k-1)]$ contains only integers of color 0. In turn, since $\chi(r(k-1)+1) = 0$, we see that $[1, k-1]$ contains only integers of color 0. We are done with this case by noting that

$$\left\{2i : 1 \le i \le \frac{k-1}{2}\right\} \cup \left\{r(k-1)+2i : 1 \le i \le \frac{k-1}{2}\right\} \cup \{r(k-1)+1\}$$

is a monochromatic (of color 0) quasi-progressions with low-difference 2 and diameter $(r-1)(k-1)-1$ contained in $[1, 2r(k-1)+1]$. □

3.1 Mixed Quasi-progression-van der Waerden Numbers

We now investigate the mixed quasi-progression-van der Waerden numbers (see Definitions 1.11 and 1.12). We start with a simple result.

Theorem 3.10. *For any positive integers k, r, and s with $s \geq r-1$, we have* $\mathrm{QW}(k; 2, 2, \ldots, 2; s) = k + r - 1$*, where the number of* $2s$ *is* $r - 1$.

Proof. Let $0, 1, \ldots, r-1$ be our colors, with color 0 tagged to the quasi-progression. First, note that any r-coloring of $[1, k+r-2]$ with exactly $k-1$ elements of color 0 and exactly one element of each of the other $r-1$ colors avoids monochromatic 2-APs and k-term quasi-progressions of color 0 with diameter s for any positive s. Hence, $\mathrm{QW}(k; 2, 2, \ldots, 2; s) > k + r - 2$. Next, consider an arbitrary r-coloring of $[1, k+r-1]$. If the interval contains at least 2 elements of any color other than 0 then we have a monochromatic 2-AP and are done. Hence, we have at least k elements of color 0, with the largest possible difference between any 2 consecutive blue elements being r. Hence, these k blue elements form a k-term quasi-progression with low-difference 1 and diameter s for any integer $s \geq r - 1$. □

Once we consider mixed quasi-progression-van der Waerden numbers with true arithmetic progressions (i.e., of length 3 or more), the situation becomes much more difficult and is related to many previously-studied functions. We introduce one such function next, which has its genesis in a paper by Rabung [27] and whose existence is implied by van der Waerden's theorem.

Definition 3.11. Let k and m be positive integers. We denote by $\Gamma(k; m)$ the minimal integer n such that any sequence $a_1 < a_2 < \cdots < a_n$ of n integers satisfying $a_j - a_{j-1} \leq m$ for $2 \leq j \leq n$ contains a k-AP.

As we can see, $\Gamma(k; m)$ is only concerned with the arithmetic progressions, while the considered mixed quasi-progression-van der Waerden numbers further consider the quasi-progressions. We have seen, thus far, that numbers associated with monochromatic quasi-progressions tend to have polynomial growth when the diameter is not too restrictive (it is known, however, that $Q(k, 1)$ is exponential; see [25]). On the other hand, it is known that $\Gamma(k; m)$ always has at least exponential growth. This was shown by Brown and Hare in [6] using the Lovász Local Lemma, a lemma which fundamentally improves bounds in probabilistic arguments in many instances. Hence, it is natural to investigate a lower bound for $\mathrm{QW}(k; m; s)$ by using the Lovász Local Lemma [11]. To state the lemma, we have need of a definition.

Definition 3.12. Let $A_1, \ldots, A_n$ be events in a probability space Ω. We say that a graph with vertex set $\{A_1, A_2, \ldots, A_n\}$ is a *dependency graph* precisely when, for all $i \neq j$ we have that

$$\{A_i, A_j\} \text{ is an edge} \iff A_i \text{ and } A_j \text{ are dependent events.}$$

Theorem 3.13. (**Lovász Local Lemma** [11]) *Let $A_1, \ldots, A_n$ be events in a probability space Ω with dependence graph Γ. Suppose that there exists $x_1, \ldots, x_n$ with $0 < x_i \leq 1$ such that*

$$\Pr[A_i] < (1 - x_i) \prod_{\{i,j\} \in \Gamma} x_j, \ 1 \leq i \leq n.$$

Then $\Pr[\bigwedge_i \overline{A_i}] > 0$.

A slightly more convenient form of the local lemma results from the following observation. Set

$$y_i = \frac{1 - x_i}{x \Pr[A_i]},$$

so that

$$x_i = \frac{1}{1 + y_i \Pr[A_i]}.$$

Since $1 + z \leq \exp(z)$, we have the following consequence.

Corollary 3.14. [13] *Suppose that* $A_1, \ldots, A_n$ *are events in a probability space having dependence graph* Ω*, and there exist positive* $y_1, y_2, \ldots, y_n$ *satisfying*

$$\log y_i > \sum_{\{i,j\} \in \Gamma} y_j \Pr[A_j] + y_i \Pr[A_i],$$

for $1 \leq i \leq n$*. Then* $\Pr[\bigwedge \overline{A_i}] > 0$.

We will have need of the following results when applying Corollary 3.14.

Lemma 3.15. *The number of* m*-APs in* $[1, N]$ *that contain* x *is at most* $N - 1$.

Proof. Let $x \in [1, N]$ be fixed. Let $A_1 = \{x, x + d, x + 2d, \ldots, x + (m - 1)d\}$ be the m-AP such that x is in the first position of this m-AP. If x is in the first position of a m-AP, then the number of m-APs in $[1, N]$ that contain x is at most $(N - x)/(m - 1)$. Similarly, if x is in the last position of a m-AP, then the number of m-APs in $[1, N]$ that contain x is at most $(x - 1)/(m - 1)$. For each i $(2 \leq i \leq m - 1)$, let $A_i = \{x - (i - 1)d, x - (i - 2)d, \ldots, x, x + d, \ldots, x + (m - i)d\}$ be the m-AP such that x is in the i-th position of this m-AP. Note that $x - (i - 1)d \geq 1$ and $x + (m - i)d \leq N$. Then $d \leq \min\{\frac{x-1}{i-1}, \frac{N-x}{m-i}\}$ for each i $(2 \leq i \leq m - 1)$, and hence the number of m-APs in $[1, N]$ that contain x is at most

$$f(x) = \sum_{i=2}^{m-1} \min\left\{\frac{x-1}{i-1}, \frac{N-x}{m-i}\right\} + \frac{N-x}{m-1} + \frac{x-1}{m-1} = \sum_{i=2}^{m-1} \min\left\{\frac{x-1}{i-1}, \frac{N-x}{m-i}\right\} + \frac{N-1}{m-1}.$$

Note that $f(x) \leq N - 1$. □

Lemma 3.16. *The number of* k*-term quasi-progressions of diameter* s *in* $[1, N]$ *that contain a given* $x \in [1, N]$ *is less than* $(s + 1)^{k-1} N$.

Proof. Let d be the common difference of a given k-AP and consider the number of k-term quasi-progressions of diameter s with low-difference is d. This is at most

$$\sum_{i=1}^{k-1} \binom{k-1}{i} s^i = (s + 1)^{k-1} - 1.$$

To see this, from the $k-1$ gaps between elements of the given k-AP, choose i of them to be different from d. This difference is the range of the diameter, so that instead of d, these gaps become one of $d+1, d+2, \ldots, d+s$. We adjust the gaps while making sure to fix x. Summing over possible values of i, we see that for each k-AP in $[1, N]$ that contains x, we have less than $(s+1)^{k-1}$ quasi-progressions of length k and diameter s in $[1, N]$ that contain x. Since there are less than N such k-APs, the result follows. □

We now present a lower bound on the mixed quasi-progression-van der Waerden numbers. As mentioned in the first section, it was recently shown that the van der Waerden number $w(3, k)$ grows faster than any polynomial. Before this result by Green [17], the best-known lower bound was of the order $\left(\frac{k}{\log k}\right)^2$ and many conjectured that k^2 may have been the correct order of growth. This result is a particular case of a result by Li and Shu [26], which has a very similar form to our next theorem. The fact that we can achieve this same growth rate for the mixed quasi-progression-van der Waerden numbers perhaps offers some insight into why (but not how) Green was able to improve the bound on $w(3, k)$. The proof of the next theorem is a minor modification of the proof found in [26].

Theorem 3.17. *Let k, m, s be positive integers. For k and s fixed and m sufficiently large, there exists a constant $c = c(k, s) > 0$ such that*

$$\mathrm{QW}(m; k, s) \geq c\left(\frac{m}{\log m}\right)^{k-1}.$$

Proof. Let $N = c\left(\frac{m}{\log m}\right)^{k-1}$, with c to be determined later. Color each integer of $[1, N]$ either red or blue. We will let the color blue be tagged to the m-AP and the color red be tagged to the k-QP. Let the probability that $i \in [1, N]$ is colored red be

$$p = \frac{\left(s+1+\frac{s+2}{k}\right) k \log m}{(s+1)m}.$$

For each (k, s)-QP S in $[1, N]$, let A_S denote the event that S consists of only red elements. We refer to this event as *type A*. For each m-AP T in $[1, N]$, let B_T denote the event that T is monochromatically blue. We refer to this event as *type B*. We will use

$$\Pr[B_T] = (1-p)^m = \left(1 - \frac{\left(s+1+\frac{s+2}{k}\right) k \log m}{(s+1)m}\right)^m \approx e^{-\left(k+\frac{s+2}{s+1}\right)\log m} = \left(\frac{1}{m}\right)^{k+\frac{s+2}{s+1}}.$$

Consider the dependency graph on all events of types A and B. Let N_{AA} denote the number of edges from a given type A vertex to other type A- vertices. Define N_{AB} to be the number of edges from a given type A vertex to type B vertices. Define N_{BA} and N_{BB} analogously.

From Lemma 3.15, the number of m-APs in $[1, N]$ that contain x is less than N.

Applying these, we now bound N_{AA}, N_{AB}, N_{BA}, and N_{BB}. For N_{AA}, fix a k-term quasi-progression of diameter s, say $A = \{a_1, a_2, \ldots, a_k\}$. Since any other (k, s)-QP can only intersect A in one of k terms, we apply Lemma 3.16, noting only k choices for x to obtain $N_{AA} \leq k(s+1)^k N$. Similarly, we obtain $N_{BA} \leq m(s+1)^k N$ as there are only m choices for x in Lemma 3.16. Using Lemma 3.15, we have $N_{AB} \leq kN$ and $N_{BB} \leq mN$.

To apply Corollary 3.14, it suffices to determine $a, b > 0$ such that

$$\log a \geq ak(s+1)^k Np^k + bkN(1-p)^m$$

and

$$\log b \geq am(s+1)^k Np^k + bmN(1-p)^m$$

are both satisfied.

Consider

$$b = \frac{1}{mN(1-p)^m} \qquad \text{and} \qquad a = b^{\frac{k}{m}}.$$

We have $\log a = \frac{k}{m} \log b$ so it suffices to show that

$$\log b \geq am(s+1)^k Np^k + bmN(1-p)^m,$$

which by definition of b reduces to showing

$$\log b \geq b^{\frac{k}{m}} m(s+1)^k Np^k + 1.$$

We have $b \approx m^{\frac{s+2}{s+1}} (\log m)^{k-1}$ so that $\log b > \left(1 + \frac{1}{s+1}\right) \log m$. Since k is fixed, for m sufficiently large we have $b^{\frac{k}{m}} < 2$. Using this, along with $k \geq 3$, by taking $c < \left(\frac{1}{(4s+5)k}\right)^k$ we see that $\log b \geq b^{\frac{k}{m}} m(s+1)^k Np^k + 1$ holds, finishing the proof. □

We now present a lower bound on the mixed quasi-progression-van der Waerden numbers for an arbitrarily number of colors. This is a generalization of the result in Theorem 3.18. We use the notation from Definition 1.15.

Theorem 3.18. *For positive integers $a < r$, let $m_1, \ldots, m_r$ be integers with $3 \leq m_1 \leq \cdots \leq m_a$ and $3 \leq m_{a+1} \leq \cdots \leq m_r$, and define $m = \min\{m_1, m_{a+1}\}$. For m_a sufficiently large, there exists a constant $c > 0$ such that*

$$\mathrm{QW}(m_1, \ldots, m_a; m_{a+1}, \ldots, m_r; s) \geq c \left(\frac{m_a(r-1)}{(m-1)\log m_a(r-1)} \right)^{m-1}$$

for any positive integer s.

Proof. Color each integer of $[1, N]$ by colors $1, 2, \ldots, r$ independently, in which each integer is colored i with probability p_i, where $\sum_{i=1}^r p_i = 1$. For each S_i of m_i-AP of $[1, N]$, let A_{S_i} denote the event that S_i is monochromatic of color i, where $1 \leq i \leq a$. For each T_i of m_j-term quasi-progression of diameter s of $[1, N]$, let A_{T_j} denote the event that T_j is monochromatic of color j, where $a+1 \leq j \leq r$.

Let Γ denote the dependency graph on the events $\{A_{S_i}\} \cup \{B_{T_j}\}$, so that $\{A_{S_i}, A_{S_j}\}$ is an edge of Γ if and only if $|S_i \cap S_j| \geq 1$ (i.e., the events A_{S_i} and A_{S_j} are dependent), $\{A_{S_i}, B_{S_j}\}$ is an edge of Γ if and only if $|S_i \cap T_j| \geq 1$, and $\{B_{S_i}, B_{S_j}\}$ is an edge of Γ if and only if $|T_i \cap T_j| \geq 1$. We will refer to the vertices $\{A_{S_i}\}$ as A-type vertices; we refer to the vertices $\{B_{T_i}\}$ as B-type vertices

Let $N_{A_iA_j}$ denote the number of edges containing A_{S_i} and connected to another A_j-type vertex. Define $N_{A_iB_j}$, $N_{B_jA_i}$ and $N_{B_iB_j}$ analogously. Using Lemmas 3.16 and 3.15 as explained in the proof of Theorem 3.18, we obtain the following bounds:

$$N_{A_iA_j} < m_iN \qquad\qquad N_{B_iA_j} < m_iN$$
$$N_{A_iB_j} < m_iN\left((s+1)^{m_j-1}-1\right) \qquad\qquad N_{B_iB_j} < m_iN\left((s+1)^{m_j-1}-1\right).$$

By Corollary 3.14, if there exist positive integers p_i, y_i, p_j, y_j $(1 \leq i \leq a,\ a+1 \leq j \leq r)$ such that

$$\log y_i > y_i p_i^{m_i} m_i N + \sum_{k=1,\ k\neq i}^{a} y_k p_k^{m_k} m_i N + \sum_{k=a+1}^{r} y_k p_k^{m_k} m_i N\left((s+1)^{m_k-1}-1\right) \tag{1}$$

and

$$\log y_j > y_j p_j^{m_j} m_j N\left((s+1)^{m_j-1}-1\right) + \sum_{k=a+1,\ k\neq j}^{r} y_k p_k^{m_k} m_k N\left((s+1)^{m_k-1}-1\right) + \sum_{k=1}^{a} y_k p_k^{m_k} m_j N \tag{2}$$

then $\mathrm{QW}(m_1, \ldots, m_a; m_{a+1}, \ldots, m_r; s) > N$.

For any given N, let

$$p = c_1(r-1)N^{-\frac{1}{m-1}}$$

where c_1 is a positive constant to be determined later, and define

$$p_1 = \cdots = p_{a-1} = p_{a+1} = \cdots = p_r = \frac{p}{r-1}, \quad p_a = 1-p.$$

With N given, we consider m_a as

$$m_a = \frac{c_2}{r-1} N^{\frac{1}{m-1}} \log N,$$

for some constant c_2 to be chosen later. Finally, let

$$y_1 = \cdots = y_{a-1} = y_{a+1} = \cdots = y_r = 1+\epsilon \ \ (\epsilon = o(1)), \text{ and } \ y_a = N^{c_3},$$

for some to-be-determined constant c_3.

It is sufficient to show that Inequalites (1) and (2) hold in all instances by showing that

$$\log y_a > (a-1)y_1 p_1^m m_a N + (r-a) y_1 p_1^m m_a N\left((s+1)^{m_r-1}-1\right) + y_a(1-p)^{m_a} m_a N, \tag{3}$$

and

$$\log y_{a+1} > (a-1)y_1 p_1^m m' N + (r-a) y_1 p_1^m m' N\left((s+1)^{m_r-1}-1\right) + y_a(1-p)^{m_a} m_a N \tag{4}$$

where $m' = \max\{m_{a-1}, m_r\}$.

Now choose constants c_1, c_2, and c_3 so that $c_3 - c_1^m c_2(s+1)^{m_r-1} > 0$ and $c_3 - c_1c_2 + 2 < 0$, i.e.,

$$\frac{c_3+2}{c_1} < c_2 < \frac{c_3}{c_1^m(s+1)^{m_r-1}}.$$

This is possible provided there exist positive constants c_1 and c_3 such that $\frac{c_3+2}{c_1} < \frac{c_3}{c_1^m(s+1)^{m_r-1}}$. By considering $c_3 = .7$ and $c_1 = (s+1)^{-\frac{2m_r}{m}}$ we see that such constants do exist.

With these choices of constants, and letting m_a be sufficiently large, a routine (but tedious) calculation shows that Inequalities (3) and (4) both hold so that $\mathrm{QW}(m_1, \ldots, m_a; m_{a+1}, \ldots, m_r; s) > N$. To determine our lower bound for N, with $c_2 > 0$ now chosen, we have

$$m_a(r-1) = c_2 N^{\frac{1}{m-1}} \log N$$

and we can conclude that there exists a positive constant c so that

$$N > c\left(\frac{m_a(r-1)}{(m-1)\log(m_a(r-1))}\right)^{m-1}$$

for m_a sufficiently large, thereby finishing the proof. □

4 Gallai-van der Waerden Numbers

Recall that by Definition 1.15 we are now dealing with *exact* colorings, i.e., r-colorings for which all colors are used at least once.

The following observation is immediate.

Observation 4.1. *For all positive integers r, k, ℓ, we have*

$$\mathrm{GW}(r; k, \ell - 1) \leq \mathrm{GW}(r; k, \ell) \text{ and } \mathrm{GW}(r; k-1, \ell) \leq \mathrm{GW}(r; k, \ell).$$

4.1 Exact Values

We start with a basic result.

Proposition 4.1. *The following hold:*

(1) $\mathrm{GW}(1; k, \ell) = \ell$

(2) *If $k = 2$, then* $\mathrm{GW}(r; 2, \ell) = r$.

(3) *If $r < k$, then* $\mathrm{GW}(r; k, \ell) = \mathrm{w}(r; \ell)$.

(4) *If $k \leq r$, then* $\mathrm{GW}(r; k, 2) = r$.

Proof. The proofs of (1), (2), and (3) are easily seen. We only give the proof of (4). For $k \leq r$, since we consider only exact colorings, it follows that $\mathrm{GW}(r; k, 2) \geq r$. To see that $\mathrm{GW}(r; k, 2) \leq r$ consider an arbitrary exact r-coloring of $[1, n]$, with $n \geq r$. If $n = r$, then $1, 2, \ldots, k$ is a rainbow k-AP. If $n > r$, then some color appears twice and we have a monochromatic 2-AP. □

Lemma 4.1. *Every* 3*-coloring* χ *of* $[1, 9]$ *admits a rainbow or monochromatic* 3*-AP.*

Proof. Using red, blue, and green as the colors, we consider the possible ways in which the integers 3 and 5 may be colored.

`Case 1.` $\chi(3) = \chi(5)$.

Without loss of generality, suppose that $\chi(3) = \chi(5)$ is red. Since $(1, 3, 5)$ cannot be monochromatic, $\chi(1)$ is blue or green. Likewise, since neither $(3, 4, 5)$ nor $(3, 5, 7)$ can be red, $\chi(4)$ and $\chi(7)$ must be blue or green. If $(1, 4, 7)$ is a blue (or green) 3-AP, then we get a contradiction. So there are two colors, blue and green, appearing in $(1, 4, 7)$. Without loss of generality, we may assume that $\chi(1)$ is blue.

Suppose that $\chi(1), \chi(4), \chi(7)$ are blue, blue, green, respectively. If $\chi(2)$ is red, then to avoid two rainbow 3-APs $(5, 6, 7)$ and $(4, 5, 6)$, $\chi(6)$ is red. To avoid two rainbow 3-APs $(2, 5, 8)$ and $(6, 7, 8)$, $\chi(8)$ is red. To avoid a red 3-AP $(2, 5, 8)$ and a rainbow 3-AP $(6, 7, 8)$, $\chi(8)$ is green. To avoid a red 3-AP $(3, 6, 9)$ and a green 3-AP $(7, 8, 9)$, $\chi(9)$ is blue. Then $(5, 7, 9)$ is a rainbow 3-AP, a contradiction. If $\chi(2)$ is blue, then $\chi(6), \chi(8), \chi(9)$ are red, red, green, respectively. Then $(1, 5, 9)$ is a rainbow 3-AP, a contradiction.

Suppose that $\chi(1), \chi(4), \chi(7)$ are blue, green, green, respectively. Clearly, $\chi(2)$ must be red. If $\chi(6)$ is green, then $\chi(8), \chi(9)$ are blue, green, respectively. Then $(1, 5, 9)$ is a rainbow 3-AP, a contradiction. If $\chi(6)$ is red, then $\chi(8), \chi(9)$ are green, red, respectively. Then $(5, 7, 9)$ is a rainbow 3-AP, a contradiction.

Suppose that $\chi(1), \chi(4), \chi(7)$ are blue, green, blue, respectively. Clearly, $\chi(2)$ and $\chi(6)$ must be red. If $\chi(8)$ is blue, then $(4, 6, 8)$ is a rainbow 3-AP, a contradiction. If $\chi(8)$ is red, then $(2, 5, 8)$ is a red 3-AP, a contradiction. If $\chi(8)$ is green, then $(5, 7, 8)$ is a rainbow 3-AP, a contradiction.

`Case 2.` $\chi(3) \neq \chi(5)$.

Without loss of generality, suppose that $\chi(3)$ is red and $\chi(5)$ is blue. Since $(1, 3, 5)$ cannot be rainbow, $\chi(1)$ is red or blue. Likewise, since neither $(3, 4, 5)$ nor $(3, 5, 7)$ can be rainbow, $\chi(4)$ and $\chi(7)$ must be red or blue. If $(1, 4, 7)$ is a red (or blue) 3-AP, then we get a contradiction. So there are two colors, red and blue, appearing in $(1, 4, 7)$.

Suppose that $\chi(1), \chi(4), \chi(7)$ are blue, blue, red, respectively. Clearly, $\chi(2)$ is not green. If $\chi(2)$ is red, then $\chi(6), \chi(8), \chi(9)$ are red, blue, blue, respectively, and hence $(1, 5, 9)$ is a blue 3-AP, a contradiction. If $\chi(2)$ is blue, then $\chi(6), \chi(8), \chi(9)$ are red, green, green, respectively, and hence $(5, 7, 9)$ is a rainbow 3-AP, a contradiction.

Suppose that $\chi(1), \chi(4), \chi(7)$ are blue, red, red, respectively. Clearly, $\chi(2)$ is blue. If $\chi(6)$ is red, then $\chi(8), \chi(9)$ are green, respectively, and hence $(1, 5, 9)$ is a rainbow 3-AP, a contradiction. If $\chi(6)$ is blue, then $\chi(8), \chi(9)$ are red, blue, respectively, and hence $(1, 5, 9)$ is a blue 3-AP, a contradiction.

Suppose that $\chi(1), \chi(4), \chi(7)$ are blue, red, blue respectively. Clearly, $\chi(2)$ is blue and $\chi(6)$ is red. If $\chi(8)$ is red, then $(4, 6, 8)$ is a red 3-AP, a contradiction. If $\chi(8)$ is blue, then $(2, 5, 8)$ is a blue 3-AP, a contradiction. If $\chi(8)$ is green, then $(6, 7, 8)$ is a rainbow 3-AP, a contradiction.

Suppose that $\chi(1), \chi(4), \chi(7)$ are red, red, blue respectively. Clearly, $\chi(2)$ is not red. If $\chi(2)$ is red, then $\chi(6), \chi(8)$ are red, blue, respectively, and hence $(2, 5, 8)$ is a blue 3-AP, a contradiction. If $\chi(2)$

is green, then $\chi(6), \chi(8), \chi(9)$ are red, blue, green, respectively, and hence $(1, 5, 9)$ is a rainbow 3-AP, a contradiction.

Suppose that $\chi(1), \chi(4), \chi(7)$ are red, blue, red, respectively. Clearly, $\chi(2), \chi(6), \chi(8)$ are blue, red, blue, respectively, and hence $(2, 5, 8)$ is a blue 3-AP, a contradiction.

Suppose that $\chi(1), \chi(4), \chi(7)$ are red, blue, blue, respectively. Clearly, $\chi(2)$ is blue. If $\chi(6)$ is red, then $\chi(8), \chi(9)$ are red, blue, respectively, and hence $(5, 7, 9)$ is a blue 3-AP, a contradiction. If $\chi(6)$ is green, then $\chi(8), \chi(9)$ are green, and hence $(1, 5, 9)$ is a rainbow 3-AP, a contradiction. □

Theorem 4.2. $\mathrm{GW}(3; 3, 3) = 9$.

Proof. To show $\mathrm{GW}(3; 3, 3) \geq 9$, it suffices to exhibit an exact 3-coloring of $[1, 8]$ with neither a rainbow 3-AP nor monochromatic 3-AP. One such coloring χ is the following: $\chi(1) = \chi(2) = \chi(4)$ are red, $\chi(3) = \chi(5) = \chi(6) = \chi(8)$ are blue, and $\chi(7)$ is green. It is easy to check that this coloring admits neither a rainbow 3-AP nor a monochromatic 3-AP.

To show $\mathrm{GW}(3; 3, 3) \leq 9$, from Lemma 4.1, we know that every 3-coloring of $[1, 9]$ admits a rainbow or monochromatic 3-AP. We must consider an arbitrary exact 3-coloring χ of $[1, n]$, with $n \geq 10$, and show that it admits a rainbow or monochromatic 3-AP. If all three colors appear in $[1, 9]$, we are done. If there there are only two colors appearing in $[1, 9]$, then there is a monochromatic 3-AP since it is known that $\mathrm{w}(2; 3) = 9$. □

For large r, we can derive the following result.

Theorem 4.3. *For $r \geq 12$, we have* $\mathrm{GW}(r; 3, 3) = r$.

Proof. Since we consider only exact colorings, it follows that $\mathrm{GW}(r; 3, 3) \geq r$. To show $\mathrm{GW}(r, 3, 3) \leq r$, we must prove that every exact r-coloring of $[1, n]$, with $n \geq r$, admits a rainbow or monochromatic 3-AP. We assume, for a contradiction, that there exists an exact r-coloring χ of $[1, n]$, with neither a rainbow 3-AP nor a monochromatic 3-AP.

We start by showing that within the interval $[3, 7]$, there exist two adjacent integers receiving the same color. To justify this, assume, to the contrary and assume that 3 and 4 have colors i_1 and $i_2 \neq i_1$, respectively. To avoid a rainbow 3-AP, 5 is colored either i_1 or i_2. However, we assume that it cannot be the same color as 4. Hence, $\chi(5) = i_1$. From here we can deduce that $\chi(6) = i_2$, and, consequently, $\chi(7) = i_1$. Then $(3, 5, 7)$ is a monochromatic 3-AP, a contradiction.

Let $a, a + 1 \in [3, 7]$, be two integers receiving the same color, say i_1. Note that $a \geq 3$ so that the integers $a - 2$ and $a - 1$ are positive integers. Before considering cases, we can deduce that, to avoid a monochromatic 3-AP, $\chi(a + 2)$ must receive a new color, say i_2.

`Case 1.` $\chi(a + 3) = i_2$. To avoid a rainbow and monochromatic 3-AP, we have $\chi(a + 4) = i_1$. Similarly, $\chi(a-1) = i_2$, which implies that $\chi(a+5) = i_1$ and $\chi(a-2) = i_1$. If $\chi(a+6) = i_1$, then $a+4, a+5, a+6$ is a monochromatic 3-AP, a contradiction. If $\chi(a+6) = i_2$, then $a-2, a+2, a+6$ is a monochromatic 3-AP, a contradiction. If $\chi(a+6)$ is neither i_1 nor i_2, then $a+2, a+4, a+6$ is a rainbow 3-AP, a contradiction.

`Case 1.` $\chi(a+3) \neq i_2$. To avoid a rainbow 3-AP, we must have $\chi(a+3) = i_1$. To avoid a rainbow 3-AP, we see that $\chi(a+4)$ is i_1 or i_2.

`Subcase i.` $\chi(a+4) = i_1$. By considering $(a+2, a+4, a+6)$ and $(a, a+3, a+6)$ we must have $\chi(a+6) = i_2$. By considering $(a+3, a+4, a+5)$ and $(a+4, a+5, a+6)$ we must also have $\chi(a+5) = i_2$. To avoid a rainbow 3-AP we must have $\chi(a-2)$ be i_1 or i_2. However, we obtain a contradiction if $\chi(a-2) = i_2$ by considering $(a-2, a+2, a+6)$. Hence, $\chi(a-2) = 1$, which means $(a-2, a+1, a+4)$ is a monochromatic 3-AP, a contradiction.

`Subcase ii.` $\chi(a+4) = i_2$. To avoid a rainbow and monochromatic 3-AP, we must have $\chi(a+5) = i_2$. To avoid $(a-1, a+2, a+5)$ and $(a-1, a, a+1)$ being monochromatic, we must have $\chi(a-1) = i_3 \notin \{i_1, i_2\}$. This implies that $\chi(a-2)$ is either i_1 or i_3; otherwise $(a-2, a-1, a)$ is a rainbow 3-AP. But if $\chi(a-2) = i_3$ then $(a-2, a, a+2)$ is a rainbow 3-AP. Hence, $\chi(a-2) = i_1$. By considering the 3-APs $(a-2, a+2, a+6)$ and $(a+4, a+5, a+6)$ we deduce $\chi(a+6) = 1$. But then $(a, a+3, a+6)$ is a monochromatic 3-AP, a contradiction.

Note that the largest integer used in the proof is $a+6$. Since $a \leq 6$ we can conclude that for $n \geq a+6 \geq 12$, every exact r-coloring of $[1, n]$ admits either a rainbow or monochromatic 3-AP. □

4.2 Lower Bounds

In [2], Behrend obtained the following result.

Lemma 4.4. [2] *If p is prime, then* $\mathrm{w}(2; p+1) \geq p2^p$.

We can derive the following result using Lemma 4.4.

Theorem 4.5. *Let r, k be positive integers with $r \geq k \geq 7$, and let p be a prime integer. Let*

$$x = \left\lfloor \frac{k^2 - 7k - r + 4}{2k - 12} \right\rfloor.$$

If $p \geq \frac{r-2x}{(k-5)x}$ and $r \leq k^2 - 9k + 16$, then

$$\mathrm{GW}(r; k, p+1) \geq xp(p2^p - 1) + \left\lfloor \frac{r - 2x}{k - 5} \right\rfloor \cdot \left\lfloor \frac{p2^p + 2}{3} \right\rfloor.$$

Proof. Let $i \equiv \left\{\frac{i}{x}\right\} \pmod{x}$. For each color pair

$$\begin{cases} \left(2\left\{\frac{i}{x}\right\}, 2\left\{\frac{i}{x}\right\} + 1\right) & \text{if } 0 \leq i \leq \frac{xp-2}{2}, \text{ and } x \text{ is even,} \\ \left(2\left\{\frac{i}{x}\right\}, 2\left\{\frac{i}{x}\right\} + 1\right) & \text{if } 0 \leq i \leq \frac{xp-p-2}{2}, \text{ and } x \text{ is odd,} \end{cases}$$

by Lemma 4.4, there exists an interval

$$X_i = [i(p2^p - 1) + 1, (i+1)(p2^p - 1)]$$

containing neither a rainbow k-AP nor a monochromatic $(p+1)$-AP under the above $2x$-coloring.

Let $m = \left\lfloor \frac{r-2x}{k-5} \right\rfloor$. For each $j \in [1, m]$, let

$$Y_j = \left[(j-1) \left\lfloor \frac{p2^p + 2}{3} \right\rfloor + 1, j \left\lfloor \frac{p2^p + 2}{3} \right\rfloor \right].$$

be an interval colored by $(k-5)$ previously unused colors such that there is no monochromatic $(p+1)$-AP in Y_j. Trivially, Y_j contains no rainbow k-APs.

Since $p \geq \frac{r-2x}{(k-5)x}$, we can insert $Y_1, Y_2, \ldots, Y_m$ into $X_0, X_1, \ldots, X_{xp-1}$ as follows:

$$Z = Y_1, X_0, Y_2, X_1, \ldots, Y_m, X_{m-1}, X_m, \ldots, X_{xp-1}.$$

Note that we have used at most r colors. Hence, we let

$$\varphi : Z \to \left[1, xp(p2^p - 1) + m \left\lfloor \frac{p2^p + 2}{3} \right\rfloor \right]$$

be the r-coloring defined by Z.

The proof is finished by showing that there is no rainbow k-AP under φ. Since the total number of colors used in the X_i's is at most $2x$, it follows that $|A_k \cap (\bigcup_{i=0}^{xp-1} X_i)| \leq 2x$. Next, we have, $|A_k \cap Y_j| \leq 1$ for each j $(1 \leq j \leq m)$. Since $x = \left\lfloor \frac{k^2-7k-r+4}{2k-12} \right\rfloor$, it follows that $m + 2x \leq k - 1$, and hence there is no rainbow k-APs in Z. □

Using the basic probability method, the following result for $\mathrm{GW}(r; \ell, k)$ can be derived.

Theorem 4.6. *Let r, k, ℓ be positive integers with $k \leq r$. Let $m = \min(k, \ell)$. Then*

$$\mathrm{GW}(r; k, \ell) \geq \frac{\sqrt{2(m-2)}}{\sqrt{\left(\frac{r-(k-1)/2}{r}\right)^k + r^{1-\ell}}}$$

Proof. Randomly r-color $[1, n]$, each i being colored c_j with probability $\frac{1}{r}$, where $1 \leq j \leq r$. For each S of k-*AP*, let A_S be the event "S is rainbow". For each T of ℓ-*AP* , let B_T be the event "T is monochromatic". It is clear that

$$\Pr[A_S] = \frac{r(r-1)\cdots(r-k+1)}{r^k} \text{ and } \Pr[B_T] = r^{1-\ell}.$$

Then

$$\begin{aligned}
\Pr\left[\left(\bigvee_{|S|=k} A_S\right) \bigvee \left(\bigvee_{|T|=\ell} B_T\right)\right] &\leq \Pr\left[\left(\bigvee_{|S|=k} A_S\right)\right] + \Pr\left[\left(\bigvee_{|T|=\ell} B_T\right)\right] \\
&\leq \sum_{|S|=k} \Pr[A_S] + \sum_{|T|=\ell} \Pr[B_T] \\
&\leq \frac{n^2}{2(k-2)} \frac{r(r-1)(r-2)\cdots(r-k+1)}{r^k} + \frac{n^2}{2(\ell-2)} r^{1-\ell} \\
&< \frac{n^2}{2(m-2)} \left[\left(\frac{r-(k-1)/2}{r}\right)^k + r^{1-\ell}\right].
\end{aligned}$$

Let

$$q = \frac{n^2}{2(m-2)}\left[\left(\frac{r-(k-1)/2}{r}\right)^k + r^{1-\ell}\right].$$

Setting $q < 1$ we have

$$n < \frac{\sqrt{2(m-2)}}{\sqrt{\left(\frac{r-(k-1)/2}{r}\right)^k + r^{1-\ell}}},$$

and hence

$$\Pr\left[\left(\bigwedge_{|S|=k} \overline{A_S}\right) \bigwedge \left(\bigwedge_{|T|=\ell} \overline{B_T}\right)\right] > 0,$$

meaning there exists an r-coloring of $[1, n]$ that avoids the monochromatic and rainbow structures considered, giving the stated bound. □

Corollary 4.7. *Let $k > 2r \log r$. For r sufficiently large,*

$$\mathrm{GW}(r; k, k) \geq \sqrt{k-2} \cdot e^{\frac{k(k-1)}{4r}} > \sqrt{k-2} \cdot r^{\frac{k-1}{2}}.$$

Proof. First note that

$$\left(\frac{r-(k-1)/2}{r}\right)^k = \left(\left(1 - \frac{k-1}{2r}\right)^r\right)^{k/r} \approx e^{\frac{-k(k-1)}{2r}}.$$

For the given bound on k we have

$$e^{\frac{k(k-1)}{2r}} < r^{k-1}.$$

Using the bound in Theorem 4.6 with $k = \ell$ we have

$$\frac{\sqrt{2(k-2)}}{\sqrt{\left(\frac{r-(k-1)/2}{r}\right)^k + r^{1-k}}} \approx \frac{\sqrt{2(k-2)}}{\sqrt{e^{-k(k-1)/2r} + r^{1-k}}} > \frac{\sqrt{2(k-2)}}{\sqrt{2e^{-k(k-1)/2r}}} = \sqrt{k-2} \cdot e^{\frac{k(k-1)}{4r}}.$$

□

Since the Lovász Local Lemma is successful in improving bound for many Ramsey-type numbers, we investigate that next. The result is similar to the bound obtained in [26] as the argument is similar, but the number of colors used and the probability of rainbow arithmetic progressions needs to be addressed.

Theorem 4.8. *Let k, ℓ, r be positive integers with $\ell \geq 3$ and $r > k \geq 9$. For any absolute constant $c < 1$, we have*

$$\mathrm{GW}(r; k, \ell) > c\left(\frac{(r-k)(\ell-1)}{(r-1)\ln(\ell-1)}\right)^{(k-1)/4}.$$

Proof. Let the integers in $[1,n]$ be independently r-colored with the probability that a number in $[1,n]$ is colored by c_i $(1 \leq i \leq r-1)$ equal to $\frac{p}{r-1}$, and the probability of it being colored by c_r equal to $1-p$. To each S of k-AP associate the event A_S that all k-APs in S have colored rainbow. To each T of ℓ-AP associate the event B_T that all the ℓ-APs in T have colored monochromatic.

For each S of k-AP, let A_S be the event "S is rainbow". For each T of ℓ-AP , let B_T be the event "T is monochromatic". It is clear that

$$\begin{aligned}\Pr[A_S] &= (r-1)(r-2)\cdots(r-k)\left(\frac{p}{r-1}\right)^k + k(r-1)(r-2)\cdots(r-k+1)(1-p)\left(\frac{p}{r-1}\right)^{k-1} \\ &= \frac{(r-1)!}{(r-k-1)!}\left(\frac{p}{r-1}\right)^{k-1}\left[\left(\frac{p}{r-1}\right)+\frac{k(1-p)}{r-k}\right]\end{aligned}$$

so that with

$$N = \frac{(r-1)!}{(r-k-1)!}\left(\frac{1}{(r-1)^{k-1}}\right)\left[\left(\frac{p}{r-1}\right)+\frac{k(1-p)}{r-k}\right]$$

we have

$$\Pr[A_S] \leq Np^{k-1}.$$

Provided that $p \leq \frac{r-1}{r}$, we have

$$\begin{aligned}\Pr\left[B_T\right] &= (r-1)\left(\frac{p}{r-1}\right)^{\ell} + (1-p)^{\ell} = p\left(\frac{p}{r-1}\right)^{\ell-1} + (1-p)^{\ell} \\ &\leq p\,(1-p)^{\ell-1} + (1-p)^{\ell} = (1-p)^{\ell-1}.\end{aligned}$$

In order to use Corollary 3.14, we need some preliminary results (which are standard for applications of the Lovász Local Lemma to arithmetic progressions). Consider the dependency graph on all possible A_S and B_T. Let N_{AA} denote the number of vertices of the form A_S for some S joined to some other vertex of this form, and let N_{AB}, N_{BA} and N_{BB} be defined analogously. It is routine (see, e.g., [26]) to derive the following bounds:

$$N_{AB} \leq \frac{\ell k n}{\ell-1}; \quad N_{AA} \leq \frac{k^2 n}{k-1}; \quad N_{BB} \leq \frac{\ell^2 n}{\ell-1}; \quad N_{BA} \leq \frac{\ell k n}{k-1}.$$

By Corollary 3.14, if there exist positive p, y, z such that

$$\log y > y\Pr[A_S]N_{AA} + z\Pr[B_T]N_{AB}, \qquad \log z > y\Pr[A_S]N_{BA} + z\Pr[B_T]N_{BB}, \tag{5}$$

then $\mathrm{GW}(r;k,\ell) > n$. Set

$$p = c_1 n^{-4/(k-1)} N^{-1/(k-1)}; \qquad z = \exp(c_3(\log n)), \qquad y = 1+\epsilon,$$

and note that for n sufficiently large we have $p \leq \frac{r-1}{r}$.

Let $c_2 > \frac{4(r-1)}{(k-1)(r-k)}$ and choose $c_1 > 0$ and $c_3 > 0$ so that $c_3 - c_1c_2 + 3 < 0$. For n sufficiently large, a bit of algebra shows that the following inequalities hold:

$$\log y > y \cdot Np^{k-1} \cdot \frac{k^2 n}{k-1} + z \cdot (1-p)^{\ell-1} \cdot \frac{\ell k n}{\ell-1}$$

and

$$\log z > y \cdot Np^{k-1} \cdot \frac{\ell k n}{k-1} + z \cdot (1-p)^{\ell-1} \cdot \frac{\ell^2 n}{\ell - 1}.$$

By choice of c_2 we have

$$\begin{aligned}
\ell - 1 &\leq c_2 (\log n) n^{4/(k-1)} N^{1/(k-1)} \\
&\leq c_2 \frac{(k-1)}{4} \left(\log n^{4/(k-1)}\right) n^{4/(k-1)} N^{1/(k-1)} \\
&\leq c_2 \frac{(k-1)}{4} \log \left((\ell-1) N^{(-1)/(k-1)}\right) n^{4/(k-1)} N^{1/(k-1)}.
\end{aligned}$$

It follows that

$$n^{4/(k-1)} > \frac{4(\ell-1) \cdot N^{(-1)/(k-1)}}{c_2 (k-1) \ln((\ell-1) N^{(-1)/(k-1)})}.$$

Note that by choice of c_2, we have

$$n^{4/(k-1)} > c \left(\frac{r-k}{r-1}\right) \frac{(\ell-1) \cdot N^{(-1)/(k-1)}}{\ln((\ell-1) N^{(-1)/(k-1)})}.$$

for any $c < 1$.

Next, we see that for any $m > \frac{1}{\ell-1}$ we have $\frac{m(\ell-1)}{\log((\ell-1)m)} \geq \frac{\ell-1}{\log(\ell-1)}$. To see this, note that as a function of m, the expression on the left is minimized (over positive values of m) at $m = \frac{e}{\ell-1}$ and the inequality follows since $e \geq 1 + \frac{1}{\log(\ell-1)}$ since $\ell \geq 3$. We will show that we may take $m = N^{(-1)/(k-1)}$. Noting that

$$\frac{(r-1)!}{(r-k-1)!} \left(\frac{1}{(r-1)^{k-1}}\right) \leq r - 1$$

so that $\frac{1}{N} \geq \frac{r-k}{k(r-1) - rp(k-1)} > \frac{r-k}{k(r-1)}$ it remains to show that $\frac{1}{N} > \frac{1}{(\ell-1)^{k-1}}$, which is satisfied when

$$\frac{r-k}{k(r-1)} > \frac{1}{(\ell-1)^{k-1}}.$$

Since $r > k$, it suffices to have $r < \frac{(\ell-1)^{k-1}}{k}$, an expression that is significantly larger than the bound given in the theorem's statement. Hence, since $r < \mathrm{GW}(r; k, \ell)$ is trivially true, we may indeed take $m = N^{(-1)/(k-1)}$ and, consequently, apply Corollary 3.14, with

$$n = c \left(\frac{(r-k)(\ell-1)}{(r-1)\ln(\ell-1)}\right)^{(k-1)/4}$$

to prove that there exists an r-coloring of $[1, n]$ that does not admit either a monochromatic ℓ-AP or a rainbow k-AP, thereby finishing the proof. □

To end this section, we consider the situation where we are not restricted to using all colors (i.e., we are not restricted to exact colorings). It is easy to see that $\mathrm{GW}'(r; k, \ell) > (\ell-1)^2(k-1)$ by considering the $(k-1)$-coloring

$$1^{\ell-1} 2^{\ell-1} 3^{\ell-1} ... (k-1)^{\ell-1}$$

repeated $\ell - 1$ times. However, we can do significantly better than this.

Theorem 4.9. *Let r, k, ℓ be integers. There exists a positive constant c such that*

$$\mathrm{GW}'(r; k, \ell) \geq c(k-1)^{\ell-1}.$$

Proof. This follows by noting that with $n = \mathrm{w}(k-1; \ell) - 1$ there exists a $(k-1)$-coloring of $[1, n]$ with no monochromatic ℓ-term arithmetic progression. Since we do not use enough colors to have a rainbow arithmetic progression of k terms, we have $\mathrm{GW}'(r; k, \ell) \geq \mathrm{w}(k-1; \ell)$. The result follows by applying Theorem 1.5. □

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