RAMSEY PROPERTIES FOR INTEGER SEQUENCES WITH RESTRICTED GAPS

Bruce M. Landman
Department of Mathematics, Adjunct Professor, University of Georgia, Athens, Georgia

Aaron Robertson
Department of Mathematics, Colgate University, Hamilton, New York

Quinn Robertson
Technology Services Department, SUNY Morrisville, Morrisville, New York

Abstract

For $D \subseteq \mathbb{Z}^+$, a sequence $x_1 < x_2 < \ldots < x_k$ is called a *D-diffsequence* if $x_i - x_{i-1} \in D$ for all $i \in \{2, \ldots, k\}$. Given a positive integer r, we say that $D \subseteq \mathbb{Z}^+$ is *r-accessible* if every r-coloring of $\mathbb{Z}^+$ admits arbitrarily long monochromatic D-diffsequences. If every r-coloring of $\mathbb{Z}^+$ admits arbitrarily long monochromatic arithmetic progressions $x, x+d, x+2d, \ldots, x+(k-1)d$ with $d \in D$, we say that D is *r-large*. We prove some new results on accessibility. We also define a property that is stronger than accessibility but weaker than largeness, and study the connections among these various properties. The paper also includes bounds, as well as tables of exact values, for the associated Ramsey functions for some specific choices of D.

1. Introduction

A celebrated theorem of van der Waerden [12] in the area of Ramsey theory states that for each $r \in \mathbb{Z}^+$, every r-coloring of $\mathbb{Z}^+$ contains arbitrarily long monochromatic arithmetic progressions. An equivalent form of van der Waerden's theorem says that for all $k, r \in \mathbb{Z}^+$, there exists a positive integer $w(k; r)$ such that for every r-coloring of $[1, w(k; r)]$ there is a k-term monochromatic arithmetic progression. Results analogous to van der Waerden's theorem may be considered, where the family of arithmetic progressions, AP, is replaced by some other family of integer sequences. One type of family that has been studied in this context is a proper subfamily of AP consisting of only those arithmetic progressions such that the gap between consecutive terms of the progression must belong to a specified proper subset of $\mathbb{Z}^+$, for which we have the following definition.

Definition 1. For $S \subseteq \mathbb{Z}^+$ and $r \in \mathbb{Z}^+$, we say that S is *r-large* if every r-coloring of $\mathbb{Z}^+$ admits arbitrarily long monochromatic arithmetic progressions $\{x, x+d, x+2d, \ldots, x+(k-1)d$ where the gap, d, belongs S; such a progression may be called an *S-ap*. A set that is r-large for all $r \geq 1$ is called *large*.

Results regarding large sets, including necessary conditions for a set to be large as well as sufficient conditions, may be found in [2, 3, 4, 5, 6, 10]. One of the most interesting unsolved problems is the so-called "2-large conjecture," which states that every 2-large set is large [3].

In [7], Ramsey properties for sequences where the gaps between consecutive elements must belong to a prescribed set were again considered, but without the requirement that the sequence be an arithmetic progression. That is, given a set $D \subseteq \mathbb{Z}^+$, a sequence $x_1 < x_2 < \ldots < x_k$ is called a *D-diffsequence* if $x_i - x_{i-1} \in D$ for all $i \in \{2, \ldots, k\}$. Regarding these sequences, the following definition was given in [7].

Definition 2. For a positive integer r, a set $D \subseteq \mathbb{Z}^+$ is *r-accessible* if every r-coloring of $\mathbb{Z}^+$ has arbitrarily long monochromatic D-diffsequences. If D is r-accessible for all positive integers r, we say that D is *accessible*. If a set D is not accessible, then the *degree of accessibility* of D, denoted $\text{doa}(D)$, is the largest r such that D is r-accessible.

Many interesting results pertaining to accessibility may be found in [1, 4, 7, 9, 10, 11]. In [1], Ramsey properties related to the set of Fibonacci numbers are studied, and there it is shown that

$$2 \leq \text{doa}(F) \leq 5, \tag{1}$$

where $F = \{f_n\}_{n=1}^{\infty}$ is the set of Fibonacci numbers. On the other hand, in [4] it is proven that F is not 2-large.

In Section 2, we present some new results on accessible sets. Section 3 deals with a new property of sets that is stronger than accessibility but weaker than largeness, which we call super-accessibility. We present several results as well as computer calculations on the size of the associated Ramsey numbers. In Section 4 we consider a refinement of the notion of accessibility, whereby each color is associated with its own specific set of differences. We conclude the article with some ideas for future research.

2. Some New Results on the Degree of Accessibility

Lowering the upper bound given in Inequalities 1 (if possible) may be rather difficult. We can produce, however, a new result for the set of even Fibonacci numbers.

Theorem 3. *We have* $\text{doa}(F_E) \leq 2$*, where F_E is the set of even Fibonacci numbers.*

Proof. We prove this by providing a 3-coloring of $\mathbb{Z}^+$ with no monochromatic 2-term F_E-diffsequence.

First, recall that $F_E = \{f_{3n}\}_{n=1}^{\infty}$. Let $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio). For $x \in \mathbb{R}$, let $\{x\}$ be the fractional part of x and let $\|x\| = \min(\{x\}, 1 - \{x\})$ represent the distance from x

to the nearest integer, so that $0 \leq \|x\| \leq \frac{1}{2}$. By Binet's formula, for all $n \geq 1$ we have

$$\begin{aligned} f_{n+1} - \phi f_n = \frac{\phi^{n+1} - (-\phi)^{n+1}}{\sqrt{5}} - \frac{\phi^{n+1} + (-\phi)^{n-1}}{\sqrt{5}} &= \frac{-1}{\sqrt{5}} \left(\frac{1}{(-\phi)^{n-1}} + \frac{1}{(-\phi)^{n+1}} \right) \\ &= \frac{-(\phi^2+1)}{\sqrt{5}} \cdot \frac{1}{(-\phi)^{n+1}} \\ &= \frac{(-1)^n}{\phi^n}. \end{aligned}$$

Hence, $\|F_{3n+1} - \phi F_{3n}\| \searrow 0$. It follows that, since F_{3n+1} is odd,

$$\left\| \frac{\phi}{2} F_{3n} \right\| \nearrow \frac{1}{2}.$$

Direct calculation gives us that $\left\| \frac{\phi}{2} F_3 \right\| \approx .38 > \frac{1}{3}$ so that we can conclude that

$$\left\| \frac{\phi}{2} F_{3n} \right\| > \frac{1}{3}$$

for all $n \in \mathbb{Z}^+$.

We now define a 3-coloring of $\mathbb{Z}^+$ by letting $C_1 = [0, \frac{1}{3})$, $C_2 = [\frac{1}{3}, \frac{2}{3})$, and $C_3 = [\frac{2}{3}, 1)$, and coloring each $i \in \mathbb{Z}^+$ with color j where $\left\{ \frac{\phi}{2} i \right\} \in C_j$.

The proof is completed by noting that if a and $a+f$ are two integers with $f \in F_E$, then $\|\frac{\phi}{2}(a+f) - \frac{\phi}{2}a\| = \|\frac{\phi}{2}f\| > \frac{1}{3}$ so that $a+f$ and a cannot reside in the same C_j, meaning that a and $a+f$ have different colors. □

Remark 4. Since $F \setminus F_E$ contains only odd integers, we know that $\text{doa}(F \setminus F_E) = 1$ by considering the 2-coloring of $\mathbb{Z}^+$ where the even numbers are red and the odd numbers are blue. The same coloring shows that the degree of accessibility of $F_E \cup \{f_{3n+1}\}_{n \in \mathbb{Z}^+}$ is also at most 2 (with a proof very much like that of Theorem 3). The coloring in the proof does, however, allow arbitrarily long monochromatic $\{f_{3n+2}\}_{n \in \mathbb{Z}^+}$-diffsequences since $\left\{ \frac{\phi}{2} f_{3n+2} \right\} \to 0$ and $\frac{\phi}{2} f_{3n+2}$ is alternately larger and smaller than $\frac{1}{2} f_{3n+3}$.

Theorem 3 can also be obtained from the following theorem by noting that the even Fibonacci numbers, $\{e_n\}$, satisfy the recurrence $e_{n+1} = 4e_n + e_{n-1}$ so that $e_{n+1} \geq 4e_n$ for all $n \in \mathbb{Z}^+$.

Theorem 5 ([11], Theorem 2)**.** *Let $D = \{d_1 < d_2 < \cdots\}$ be a set of positive integers such that $d_{n+1} \geq 4d_n$ for all $n \in \mathbb{Z}^+$. Then* $\text{doa}(D) \leq 2$. *In general, if $d_{n+1} \geq (2+\epsilon)d_n$ for some $\epsilon > 0$, then* $\text{doa}(D) < 2 + \frac{2}{\epsilon}$.

Remark 6. Using Theorem 3, we can offer a simple proof that $\text{doa}(F) \leq 5$ (see [1] for the original proof). Let $\chi : \mathbb{Z}^+ \to \{0, 1, 2\}$ be a 3-coloring of $\mathbb{Z}^+$ that avoids arbitrarily long monochromatic $\{F_{3n}\}$-diffsequences. Now consider the 6-coloring $\gamma : \mathbb{Z}^+ \to \{0, 1, \ldots, 5\}$

defined by $\gamma(i) = \chi(i)$ for i even and $\gamma(i) = \chi(i) + 3$ for i odd. We claim that γ avoids arbitrarily long monochromatic F-diffsequences. To see this, note that any γ-monochromatic diffsequence must have even differences by construction. Hence, any γ-monochromatic F-diffsequence must be a γ-monochromatic $\{f_{3n}\}$-diffsequence. By definition of γ, this implies that there are arbitrarily long monochromatic $\{f_{3n}\}$-diffsequences under χ, a contradiction.

Using the coloring construction of Remark 6, the following result holds.

Theorem 7. *Let $S \subseteq \mathbb{Z}^+$ and let S_E be the set of even integers in S. If* $\mathrm{doa}(S_E) < r$*, then* $\mathrm{doa}(S) < 2r$*.*

Although we have not been able to determine whether or not F_E is 2-accessible, computer output suggests that it is not. Specifically, we have determined[1] that if it is 2-accessible then the associated Ramsey-type number for 3-term F_E-diffsequences would exceed 55070. Based on this, we make the following conjecture.

Conjecture. The set of even Fibonacci numbers is not 2-accessible.

Remark 8. If the above conjecture is true, then by Theorem 7 we would have $\mathrm{doa}(F) \leq 3$, a significant improvement over the known upper bound of [1]. Note that the method used for the proof of Theorem 3 would not be useful in trying to prove the conjecture, as every 2-coloring of $[1, 35]$ has a monochromatic 2-term F_E-diffsequence (assuming otherwise, the colors must alternate along the sequence $1, 9, 17, 25, 27, 29, 31, 33, 35$, but then 1 and 35 have the same color, forming the requisite sequence). Also, our attempt to extend one of the 2-colorings of $[1, 55070]$ that avoids 3-term monochromatic F_E-diffsequences has not proven fruitful.

Remark 9. The set of even Fibonacci numbers contains multiples of every integer so that any coloring of $\mathbb{Z}^+$ based on residue classes will contain arbitrarily long monochromatic F_E-diffsequences. To see this, let e_n be the nth even Fibonacci numbers so that $e_n = 4e_{n-1} + e_{n-2}$ and $e_0 = 0$ and $e_1 = 2$. Let $m \in \mathbb{Z}^+$ and consider $e_n \equiv 4e_{n-1} + e_{n-2} \pmod{m}$. Of course the set of residues modulo m is finite, so the possible number of pairs $(e_k \pmod{m}, e_{k-1} \pmod{m})$ is also finite. Hence, if we have more than m^2 pairs, two will be the same, meaning that solutions to the congruence are cyclic. Since $(2, 0) = (e_1 \pmod{m}, e_0 \pmod{m})$ is a pair for any $m \geq 3$ and the solutions are cyclic, this pair will eventually occur again, meaning we have another even Fibonacci number divisible by m (actually we have infinitely many).

Although we have not been able to verify the conjecture, we are able to give results in Theorems 11 and 12 concerning the degree of accessibility of sets closely associated with F_E.

We will make use of the following lemma. We mention that an analogous result, where accessibility is replaced by largeness, can be shown in the same way (it is mentioned without proof in [3] so we include a proof here for completeness).

[1] using a computer program written by the third author, available at `https://github.com/SpeederFlash/Colorings`

Lemma 10. *Let $c \in \mathbb{Z}^+$ and let $S \subseteq \mathbb{Z}^+$. If cS is r-accessible, then S is r-accessible.*

Proof. Assume cS is r-accessible, and let γ be an r-coloring of $\mathbb{Z}^+$. We will show that γ admits arbitrarily long monochromatic S-diffsequences. Let α be the r-coloring of $\mathbb{Z}^+$ defined as follows:

$$\text{for every } x \in [c(j-1)+1, cj] \text{ let } \alpha(x) = \gamma(j) \text{ for each } j \in \mathbb{Z}^+.$$

By hypothesis, α admits arbitrarily long monochromatic cS-diffsequences. Hence, under α there must be some $i \in \{0, 1, \ldots, c-1\}$ such that the congruence class $i + c\mathbb{Z}^+$ contains arbitrarily long monochromatic cS-diffsequences. Now, for any cS-diffsequence $\{i + cj_1, i + cj_2, \ldots, i + cj_k\}$ that is monochromatic under α, we see from the definition of α that the sequence $\{j_1, j_2, \ldots, j_k\}$ is a monochromatic S-diffsequence under γ. Hence γ admits arbitrarily long monochromatic S-diffsequences. □

Using Lemma 10, we can offer the following result related to the Fibonacci numbers.

Theorem 11. *Let $S = \{f_{3n+3} + f_{3n} : n \in \mathbb{Z}^+\}$ and let $T = \{f_{3n+3} - f_{3n} : n \in \mathbb{Z}^+\}$. Then neither S nor T is 2-accessible.*

Proof. It is easy to see that $S = \{2f_{3n+2} : n \in \mathbb{Z}^+\}$ and $T = \{2f_{3n+1} : n \in \mathbb{Z}^+\}$. Since neither $\{f_{3n+2} : n \in \mathbb{Z}^+\}$ nor $\{f_{3n+1} : n \in \mathbb{Z}^+\}$ contains an even number, they are not 2-accessible. By the contrapositive of Lemma 10, we see that neither S nor T is 2-accessible. □

The next result and its proof offer some insight into the interplay between S and cS with respect to 2-accessibility.

Theorem 12. *Let $S = \{b_i\}_{i \in \mathbb{Z}^+}$ be generated by the recurrence $b_n = cb_{n-1} + b_{n-2}$ for some $c \in \mathbb{Z}^+$ and some non-negative initial conditions. Then $S \cup cS$ is 2-accessible. In particular, with $c = 4$ and $b_1 = 0, b_2 = 2$, we obtain $S = \{f_{3n}\}_{n=1}^{\infty}$ so that $\{f_{3n}\}_{n=1}^{\infty} \cup \{4f_{3n}\}_{n=1}^{\infty}$ is 2-accessible.*

Proof. Assume, for a contradiction, that $S \cup cS$ is not 2-accessible. Then there exists a 2-coloring of $\mathbb{Z}^+$ with the longest monochromatic S-diffsequence having k terms. Let $x_1, x_2, \ldots, x_k$ be one such monochromatic, say red, S-diffsequence. Then $X = \{x_k + b_{2n-1} : n \in \mathbb{Z}^+\}$ contains only blue integers. But then X is an infinitely long blue cS-diffsequence, a contradiction. □

Remark 13. In general, we cannot conclude that the set of integers generated by the recurrence $b_n = cb_{n-1} + b_{n-2}$ is 2-accessible. Consider, for example, $b_n = 2b_{n-1} + b_{n-2}$ with $b_1 = b_2 = 1$. It is easy to see that the corresponding set S contains only odd integers, so that the coloring $010101\ldots$ provides a 2-coloring without 2-term monochromatic S-diffsequences.

We end this section with some related small Ramsey-type values. For k and ℓ positive integers, let $n = n(k, \ell)$ be the least positive integer such that every 2-coloring of $[1, n]$ admits either a k-term $(\{f_{3n}\} \cup \{4f_{3n}\})$-diffsequence of the first color or an ℓ-term one of the second color. We have determined the following using a computer program[2]: $n(2, 2) = 9$, $n(3, 2) = 39$, $n(4, 2) = 43$, $n(5, 2) = 75$, $n(3, 3) = 73$, $n(4, 3) = 107$, $n(3, 5) = 143$, and $n(4, 4) > 200$.

3. Between Accessibility and Largeness

Motivated by our desire to learn more about largeness, and because answers to questions about the degree of accessibility have come much easier than answers regarding the degree of largeness, we will consider a property that lies between the concepts of largeness and accessibility.

Definition 14. For $r \in \mathbb{Z}^+$, a set $S \subseteq \mathbb{Z}^+$ is called *r-super-accessible* if every r-coloring of $\mathbb{Z}^+$, using the colors $1, 2 \ldots, r$, admits arbitrarily long S-aps of color 1 or arbitrarily long S-diffsequences of color j for some $j \in \{2, 3, \ldots, r\}$. If S is r-super-accessible for all $r \in \mathbb{Z}^+$, we say S is *super-accessible*.

It is obvious from the above definition that, for each r, every r-large set is r-super-accessible, and every r-super-accessible set is r-accessible. Somewhat surprisingly, as we show in Theorem 17, any set that is accessible (r-accessible for all r) is also super-accessible. Before proving this result, let us recall that a set $S = \{s_1 < s_2 < s_3 < \cdots\}$ is called *syndetic* if there exists $n \in \mathbb{Z}^+$ such that $s_{i+1} - s_i \leq n$ for all $i \in \mathbb{Z}^+$. Refining this notion, we will say that a set $S = \{s_1 < s_2 < s_3 < \cdots\}$ is *m-syndetic* if $s_{i+1} - s_i \leq m$ for all $i \in \mathbb{Z}^+$.

Proposition 15. *Let $A \subseteq \mathbb{Z}^+$. The set A is accessible if and only if every syndetic set admits arbitrarily long A-diffsequences.*

Proof. First assume that A is accessible. For a contradiction, assume that C is an m-syndetic set without arbitrarily long A-diffsequences. Let $C_1 = C$ and define, for $i = 2, 3, \ldots, m$, in order, the sets

$$C_i = \{i + c : c \in C\} \setminus \bigcup_{j=0}^{i-1} C_j.$$

Since diffsequences are translation invariant, we see that each C_i also avoids arbitrarily long A-diffsequences. However

$$\mathbb{Z}^+ = \bigsqcup_{j=1}^{m} C_i$$

[2]written by the third author; available at `https://github.com/SpeederFlash/Colorings`, and also at `http://math.colgate.edu/∼aaron`

provides an m-coloring of $\mathbb{Z}^+$ without arbitrarily long monochromatic A-diffsequences, contradicting the accessibility of A.

Conversely, assume that every syndetic set admits arbitrarily long A-diffsequences. We prove, by induction on r, that every r-coloring of $\mathbb{Z}^+$ admits arbitrarily long monochromatic A-diffsequences. For $r = 1$, since $\mathbb{Z}^+$ is itself syndetic, the result holds trivially. We now assume the result for r and show that it holds for $r + 1$. If one of the color classes is a syndetic set, then we are done. Hence, we may assume that none of the color classes is syndetic. By the Compactness Principle, there exists an integer $n(A, k; r)$ such that every r-coloring of an interval of $n(A, k; r)$ consecutive integers admits a k-term monochromatic A-diffsequence. Consider one of the color classes, say red. Since the set of red elements is not syndetic, there are arbitrarily long intervals between consecutive red elements. Consider any such gap of at least $n(A, k; r)$ integers. None of these integers are colored red, so we have an r-coloring of an interval of length $n(A, k; r)$. By induction, we are done. □

Remark 16. We see that Proposition 15 is similar to a result about large sets [10]: *A set L is large if and only if every syndetic set admits arbitrarily long L-aps.*

Theorem 17. *If A is accessible, then A is super-accessible.*

Proof. Let A be accessible. We will show that A is r-super-accessible for all $r \in \mathbb{Z}^+$ by induction on the number of colors r used to color $\mathbb{Z}^+$. For the base case, $r = 2$ (the $r = 1$ case is trivially true), let the color classes be C_1 and C_2. By Proposition 15, if C_2 is syndetic then we are done. Hence, we may assume that it is not, so that there are arbitrarily large gaps between consecutive elements of C_2. Letting $m = \min(a \in A)$, we see that any gap of size at least $km + 1$ means we have at least $km + 1$ consecutive elements from C_1. This guarantees a k-term A-ap in C_1.

We now assume that A is r-super-accessible, where $r \geq 2$, and will show that it is $(r+1)$-super-accessible. By the Compactness Principle there exists a minimal integer $n(A, k; r)$ such that every r-coloring of $[1, n(A, k; r)]$ admits a monochromatic k-term A-ap of color C_1 or a monochromatic k-term A-diffsequence of some other color.

Consider any $(r + 1)$-coloring of $\mathbb{Z}^+$ with color classes $C_1, C_2, \ldots, C_{r+1}$. Appealing to Proposition 15 again, we may assume that C_{r+1} is not syndetic. In particular, there exists a gap of at least $n(A, k; r)$ integers between two consecutive integers of C_{r+1}. These $n(A, k; r)$ integers use only r colors and by the inductive hypothesis, we are done. □

From [10], there exists an accessible set that is not 2-large. Hence, the following is immediate.

Corollary 18. *There exists a super-accessible set that is not 2-large.*

Just as one can consider the degree of largeness and the degree of accessibility of a set, we have the following definition.

Definition 19. Let $S \subseteq \mathbb{Z}^+$. The maximal number of colors r for which S is r-super-accessible is called the *degree of super-accessibility* of S and we denote this by $\text{dosa}(S)$.

We now consider the degree of super-accessibility for various sets.

For $m \in \mathbb{Z}^+$, let $V_m = \{i \in \mathbb{Z}^+ : m \nmid i\}$. In [7] it is shown that for each $m \geq 2$, we have $\text{doa}(V_m) = m - 1$. Hence, for every $r \geq 1$ there exists a set with degree of accessibility r; as will be shown below, this property is also true for the degree of super-accessibility but not for the same reason.

Theorem 20. *Let $m \geq 2$ and let p be the least prime divisor of m. Then $dosa(V_m) = \frac{m}{p}$.*

Proof. Let $\ell = \frac{m}{p}$. We first show that V_m is ℓ-super-accessible. For a contradiction, let α be an ℓ-coloring of $\mathbb{Z}^+$ with colors $1, 2, \ldots, \ell$, and assume that there is no arbitrarily long V_m-ap with color 1 and no arbitrarily long monochromatic V_m-diffsequence in any of the other colors. For $i = 0, 1, \ldots, \ell - 1$, let $R_i = \{j \in \mathbb{Z}^+ : j \equiv i \pmod{\ell}\}$. As each R_i is infinite, in each R_i there must be some color that occurs infinitely many times. Furthermore, for each i, some color other than 1 must occur infinitely many times in R_i; otherwise, $\alpha(R_i \cap [n, \infty]) = 1$ for some $n \in \mathbb{Z}^+$, which would imply that there is an infinitely long V_m-ap (with common difference $\ell \in V_m$) of color 1.

Since we have partitioned $\mathbb{Z}^+$ into the ℓ residue classes and each contains one of $\ell - 1$ colors infinitely many times, two residue classes, say R_x and R_y, must contain the same color, say b, infinitely many times. Let B_x and B_y be the sets of integers in R_x and R_y, respectively, that have color $b \neq 1$. Now take $a_1 \in B_x$ and $a_2 \in B_y \cap [a_1 + 1, \infty)$. We then alternately select between B_x and B_y by taking $a_{2j+1} \in B_x \cap [a_{2j} + 1 \cap \infty)$ and $a_{2j+2} \in B_y \cap [a_{2j+1} + 1, \infty)$ for $j = 1, 2, \ldots$. From the defintions of R_x and R_y, the sequence of a_i's is an infinitely long V_m-diffsequence with color b, a contradiction.

To complete the proof, it suffices to find an $(\ell + 1)$-coloring of $\mathbb{Z}^+$, with color classes $C_0, C_1, \ldots, C_\ell$, without arbitrarily long V_m-aps in C_0 and without arbitrarily long monochromatic V_m-diffsequences in any of the other color classes. To this end, for $1 \leq i \leq \ell$, let $C_i = \{j \in \mathbb{Z}^+ : j \equiv i \pmod{m}\}$, and let $C_0 = \mathbb{Z}^+ \setminus \bigcup_{i=1}^{\ell} C_i$. It is clear that each C_i fails to contain a 2-term V_m-diffsequence for $1 \leq i \leq \ell$.

We will now show that C_0 does not contain an m-term arithmetic progression with gap belonging to V_m. Let $A = \{a, a + d, \ldots, a + (m - 1)d\}$ be a V_m-ap. Let $g = \gcd(m, d)$. Since $d \in V_m$, we know that $g < m$. Let H be the subgroup of the group $(\mathbb{Z}_m, +)$ that is generated by g see so that H contains $\frac{m}{g}$ elements. Note that for any interval $[j, k] \subseteq [1, m]$ with $k - j = g - 1$, there exists an $x \in [j, k]$ such that some member of H is congruent to x modulo m. Hence, some member of $a + H$ contains an integer congruent to t modulo m where $t \in [1, g] \subseteq [1, \ell]$. Since H is also generated by d we see that some member of A lies in $\bigcup_{i=1}^{\ell} C_i$. Hence C_0 cannot contain an m-term V_m-ap. □

As a consequence of Theorem 20, we have the following corollary.

Corollary 21. *For every positive integer r, there is a set whose degree of super-accessibility equals r.*

Proof. Let $r \in \mathbb{Z}^+$. Then $\text{dosa}(V_{2r}) = r$ since, if $r > 1$, by Theorem 20 we have $\text{dosa}(V_{2r}) = r$, while $\text{dosa}(V_2) = 1$ is trivial. □

Let $\mathcal{L}$, $\mathcal{S}$, and $\mathcal{A}$ denote the families of all large sets, super-accessible sets, and accessible sets, respectively. Also, for $r \in \mathbb{Z}^+$, let $\mathcal{S}_r$ and $\mathcal{A}_r$ denote the families of all r-super-accessible sets and all r-accessible sets, respectively. As noted earlier, there exists an accessible set that is not 2-large. Also, by Theorem 20 and the fact that $\text{doa}(V_m) = m - 1$, we see that for any $m \geq 3$, $\text{dosa}(V_m) < \text{doa}(V_m)$. This, along with Theorem 17, immediately gives us the following consequence.

Corollary 22. *For any $r \geq 3$, we have $\mathcal{L} \subsetneq \mathcal{S} = \mathcal{A} \subsetneq \mathcal{S}_r \subsetneq \mathcal{A}_r$.*

Also from Theorem 20 we have the following structural result about 2-super-accessible sets.

Corollary 23. *If S is 2-super-accessible, then S contains a multiple of every prime.*

Proof. Assume otherwise and let S be a 2-super-accessible set that contains no multiple of some prime p. Then $S \subseteq V_p$, but this contradicts the fact that V_p is not 2-super-accessible. □

Remark 24. This corollary shows more similarity with 2-large sets than with 2-accessible sets. We say this because it is known [3] that every 2-large set contains a multiple of every positive integer, while V_3 is 2-accessible (but has no multiple of 3). Likewise, the set $\{2^{n-1} : n \in \mathbb{Z}^+\}$ is known to be 2-accessible but by Corollary 23 it is not 2-super-accessible.

By Corollary 23 and Theorem 17, we see that any accessible set contains a multiple of every prime. However, we actually know that, like 2-large sets [3], accessible sets contain a multiple of every integer. To see this, we use the fact that $\text{doa}(V_{m+1}) = m$. If a set S does not contain a multiple of some integer, say n, then $S \subseteq V_n$ and so $\text{doa}(S) \leq n - 1$.

Corollary 25. *If S is accessible, then S contains a multiple of every integer.*

Remark 26. The converse of Corollary 25 is not true. It is well-known (see also Remark 9) that F, the set of Fibonacci numbers, contains a multiple of every integer while, as shown in [1], we know that $\text{doa}(F) \leq 5$.

As another example, consider the set $\{2^{n-1} : n \in \mathbb{Z}^+\}$. It is known that this set is not 2-large [3] but that its degree of accessibility is 2 [7]. By Corollary 23 we see that it is not 2-super-accessible.

An example of a set whose degree of strong accessibility lies strictly between its degree of accessibility and its degree of largeness is $S = \{2\} \cup \{2n - 1 : n \in \mathbb{Z}^+\}$. It is known [7] that $\text{doa}(S) = 3$, but it is not 2-large as it does not contain a multiple of 4. Regarding this set, we have the following result.

Proposition 27. *Let $S = \{2\} \cup \{2n - 1 : n \in \mathbb{Z}^+\}$. Then* $\text{dosa}(S) = 2$.

Proof. First, we show that S is not 3-super-accessible. Let χ be the 3-coloring of $\mathbb{Z}^+$ defined as follows: (i) $\chi(i) = 1$ if $i \equiv 1 \pmod 4$; (ii) $\chi(i) = 2$ if $i \equiv 2 \pmod 4$; (iii) $\chi(i) = 0$ if

$i \equiv 0$ or 3 (mod 4). Clearly, colors 1 and 2 are each void of 2-term S-diffsequences. Now consider $x, x+d$ of color 0, with d odd. If $x \equiv 3 \pmod 4$, then $x+d \equiv 0 \pmod 4$. But then $x + 2d \equiv 1 \pmod 4$, and hence it has color 1. Similarly, if $x \equiv 0 \pmod 4$, it follows that $x + 2d$ has color 2. In either case, there is no 3-term monochromatic arithmetic progression with odd gap of color 0. It is obvious that there is no pair of numbers both of color 0 with difference 2. Hence S is not 3-super-accessible.

To show that S is 2-super-accessible, let k be a positive integer and consider an arbitrary 2-coloring of $[1, 2k^2 - 2k + 1]$. We will show that we must have either a blue k-term S-ap or a red k-term S-diffsequence. Assume that in color blue there is no k-term S-ap. Then some member of $[1, k]$, say x_1, has color red. To avoid a k-term blue arithmetic progression with common difference 2, some member of $\{x_1 + 2j - 1 : 1 \leq j \leq k\}$, say x_2, must be red. Repeating this process $k - 2$ additional times, there is a red sequence $x_1, x_2, \ldots, x_k$ such that $x_{i+1} - x_i$ is odd for $1 \leq i \leq k - 1$. □

Remark 28. Proposition 27 provides us with an example of two sets A and B, neither of which is 2-super-accessible, for which $A \cup B$ is 2 super-accessible. By 2-coloring $\mathbb{Z}^+$ based on the parity of each integer, we see that B is not 2-super-accessible; the 2-coloring of $\mathbb{Z}^+$ given by 00110011... shows that A is not 2-super-accessible.

We now consider the order of magnitude for the 2-color Ramsey function for super-accessibility. That is, for a set $D \subseteq \mathbb{Z}^+$, define $s(D, k)$ to be the least positive integer s, if it exists, such that every 2-coloring of $[1, s]$ admits either a monochromatic k-term D-ap of the first color or a monochromatic k-term D-diffsequence of the second color. We can provide some bounds on $s(D, k)$ for some choices of D that we have shown are 2-super-accessible.

From the proof of Theorem 27, we have the following bound.

Proposition 29. *Let $D = \{2\} \cup \{2n-1 : n \in \mathbb{Z}^+\}$ and $k \in \mathbb{Z}^+$. Then $s(D, k) \leq 2k^2 - 2k + 1$.*

In an exercise in [8], it is noted that the 2-color Ramsey functions for V_3-diffsequences and V_4-diffsequences of length k have linear growth, and it seems likely that this is also true for V_m for all larger values of m. The following result shows that $s(V_m, k)$ has a greater order of magnitude.

Proposition 30. *Let $m \geq 2$ and let q be a prime. Then $s(V_m, q) \geq q^2 - q + 1$.*

Proof. It suffices to give a 2-coloring of $[1, q^2 - q]$ that has no q-term V_m-ap of one color and no q-term V_m-diffsequence of the other color. Define the 2-coloring $\chi : [1, q^2 - q] \longrightarrow \{0, 1\}$ as follows: let $\chi(j) = 0$ if $q \mid j$, and let $\chi(j) = 1$ otherwise. From the definition of χ, it is obvious that there are fewer than q members of its domain that have color 0, and hence there is no q-term diffsequence of color 0. Now let $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$ be any arithmetic progression in $[1, q^2 - q]$, where $a, d \geq 1$. Then we must have $d < q$. Hence, since $(d, q) = 1$, it follows that A is a complete residue system modulo q. Hence, some member of A is divisible by q and therefore has color 0. Thus, there is no q-term V_m-ap that is monochromatic. □

For the set of Fibonacci numbers, F, we are able to obtain a lower bound on $s(F,k)$ using the probabilistic method.

Theorem 31. *Let F be the set of Fibonacci numbers. For k sufficiently large, we have*

$$s(F,k) \geq e^{.7\sqrt{k}}.$$

Proof. Let $n \geq 2$ be an integer. Letting

$$x = \left(\frac{\log n}{\log \phi}\right)^{1-2/k}, \tag{2}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, we define

$$p = \frac{x}{1+x}.$$

Note that

$$p^k = \left(\frac{\log n}{\log \phi}\right)^{k-2} (1-p)^k. \tag{3}$$

Consider a random coloring of $[1,n]$ by coloring $i \in [1,n]$ red with probability p and blue with probability $1-p$. We will prove that one such coloring avoids red k-term F-aps and blue k-term F-diffsequences.

Let $F(m) = |F \cap [1,m]|$. It is easy to see that, within $[1,n]$, there are less than $n(F(n))$ arithmetic progressions of length k with common difference in F, and less than $n(F(n))^{k-1}$ k-term F-diffsequences.[3]

Hence, for any n satisfying

$$nF(n)p^k + n(F(n))^{k-1}(1-p)^k < 1$$

we have the existence of a 2-coloring of $[1,n]$ avoiding the prescribed monochromatic sequences. By Binet's formula we see that $F(n) \approx \frac{\log n}{\log \phi}$. Coupling this with Equation (3), our goal is to determine n such that

$$2nF(n)p^k < 1. \tag{4}$$

We next approximate p^k when n is sufficiently large. With x as defined in Equation (2), using $\log\left(1+\frac{1}{x}\right) \approx \frac{1}{x}$ gives

$$p^k = \left(\frac{x}{1+x}\right)^k \approx e^{-\frac{k}{x}}. \tag{5}$$

Using Binet's formula again, along with Equations (2) and (5), Inequality (4) becomes

$$\frac{2}{\log \phi} n \log n < e^{\frac{k(\log\phi)^{1-2/k}}{(\log n)^{1-2/k}}}.$$

[3]Although we could have bounded the number of arithmetic progressions by the stronger bound $nF(n/k)$, that bound is only useful if we can find an accurate approximation of the number of F-diffsequences in $[1,n]$, which has so far eluded the authors

Taking the logarithm of both sides gives

$$\log n + \log\log n + \log\left(\frac{2}{\log\phi}\right) < \frac{k(\log\phi)^{1-2/k}}{(\log n)^{1-2/k}}.$$

To satisfy this inequality, it is sufficient to show that, for any $\epsilon > 0$, we have

$$(1+\epsilon)\log n < \frac{k(\log\phi)^{1-2/k}}{(\log n)^{1-2/k}},$$

which reduces to showing that

$$(1+\epsilon)^k(\log n)^{2k-2} < k^k(\log\phi)^{k-2}.$$

This is satisfied for

$$n < e^{\sqrt{\frac{\log\phi}{1+\epsilon}k}} < e^{.7\sqrt{k}},$$

for ϵ sufficiently small, which finishes the proof. □

Remark 32. Although we do not know if the set of Fibonacci numbers is 2-super-accessible, limited computer evidence suggests it is (see Table 2). However, given the exponential growth rate of the Fibonacci sequence this may be a case of the "law of small numbers."

As has been done with other Ramsey-type numbers, one can consider the "mixed" (or "off-diagonal") Ramsey numbers for super-accessibility. That is, for a set $D \subseteq \mathbb{Z}^+$ and $\ell, k \in \mathbb{Z}^+$, define $s(D, \ell, k)$ to be the least positive integer s, if it exists, such that every 2-coloring of $[1, s]$ admits either a monochromatic ℓ-term D-ap of the first color or a monochromatic k-term D-diffsequence of the second color. Thus, $s(D, k)$ has the same meaning as $s(D, k, k)$.

In the tables below, we give the values of $s(D, \ell, k)$ for small values ℓ and k for various sets D, which were calculated using a thread-level parallel backtracking algorithm written by the third author[4]. The computation time for each table was limited to a total run time of approximately one day on a standard 8-core MacBook Pro laptop, although each subtable in Table 3 was completed in less than a minute. Note that we do not know if some of the sets are indeed 2-super-accessible, but, nevertheless, the values of $s(D, \ell, k)$ may exist for ℓ and k small enough.

4. Pairwise Accessibility

In this section we restrict our attention to the 2-color situation; the generalization to more colors should be clear.

Definition 33. Let $S, T \subseteq \mathbb{Z}^+$. We say that the ordered pair (S, T) is *pairwise-accessible* if every 2-coloring of $\mathbb{Z}^+$ admits arbitrarily long S-diffsequences in the first color or arbitrarily long T-diffsequences in the second color.

[4]available at `https://github.com/SpeederFlash/Colorings` and `http://math.colgate.edu/~aaron`

	k					
ℓ	2	3	4	5	6	7
2	3	4	6	7	8	10
3	6	7	8	11	14	15
4	8	15	19	21	26	28
5	10	19	24	29	36	39
6	12	23	32	41	50	53
7	14	27	40	53	60	71

Table 1: Small values of $s(D, \ell, k)$ where $D = \{2\} \cup \{2n-1 : n \in \mathbb{Z}^+\}$

	k				
ℓ	2	3	4	5	6
2	3	4	6	7	8
3	6	7	8	11	14
4	8	15	20	27	28
5	10	19	27	37	49
6	12	50	82	124	183

Table 2: Small values of $s(D, \ell, k)$ where D is the set of Fibonacci numbers

	k					
ℓ	2	3	4	5	6	7
2	3	4	6	7	8	10
3	6	7	8	11	14	15
4	8	15	19	21	24	27
5	10	19	24	29	36	39
6	12	23	32	37	44	49
7	14	27	40	53	59	69

(a) V_4 values

	k					
ℓ	2	3	4	5	6	7
2	3	4	5	6	8	9
3	6	7	8	10	12	15
4	7	11	12	14	16	18
5	10	15	20	21	22	23
6	12	18	28	35	38	40
7	14	21	34	41	46	49

(b) V_6 values

	k					
ℓ	2	3	4	5	6	7
2	3	4	5	6	7	8
3	6	7	8	10	12	15
4	7	11	12	14	16	18
5	10	15	20	21	22	23
6	11	16	21	27	28	30
7	14	21	28	35	42	43

(c) V_8 values

Table 3: Small values of $s(V_4, \ell, k)$, $s(V_6, \ell, k)$, and $s(V_8, \ell, k)$

It is easy to see that pairwise-accessibility is, in some natural sense, stronger than 2-accessibility, as the following lemma shows.

Lemma 34. *If (S,T) is pairwise-accessible, then $S \cup T$ is 2-accessible.*

Proof. For a contradiction, assume otherwise. Then there exists a 2-coloring of $\mathbb{Z}^+$ without monochromatic $(S \cup T)$-diffsequences of length k for some $k \in \mathbb{Z}^+$. This coloring serves as a counterexample to (S,T) being pairwise-accessible. □

Theorem 35. *Let $S,T \subseteq \mathbb{Z}^+$ such that either S contains arbitrarily long T-diffsequences or T contains arbitrarily long S-diffsequences. Then (S,T) is pairwise-accessible.*

Proof. Assume that S contains arbitrarily long T-diffsequences and assume, for a contradiction, that (S,T) is not pairwise-accessible. Then there exists a 2-coloring of $\mathbb{Z}^+$ with no red S-diffsequences and no blue T-diffsequences of length longer than k, for some $k \in \mathbb{Z}^+$. Let $a_1, a_2, \ldots, a_\ell$ be a maximal length red S-diffsequence. Consider $a_\ell + S$. By assumption, this translation of S must be entirely blue and, since diffsequences are translation-invariant, must also contain arbitrarily long (in particular, of length longer than k) blue T-diffsequences, a contradiction. □

Example 36. The following pairs (S,T) are pairwise-accessible, where F_n represents the nth Fibonacci number:

(1) S is the set of odd Fibonacci numbers, T is the set of even Fibonacci numbers;

(2) $S = \{F_{3n+2}\}$, $T = \{2F_{3n}\}$;

(3) $S = \{c\}$, $T = \{cn + d : n \in \mathbb{Z}^+\} \cap \mathbb{Z}^+$ for any $c \in \mathbb{Z}^+$ and $d \in \mathbb{Z}$;

(4) $S = \{b_i\}$ where $b_{n+1} = cb_n + b_{n-1}$, $T = cS$ for any $c \in \mathbb{Z}^+$.

To see that (1) and (2) hold, consider $S \supseteq \{x_1 < x_2 < x_3 < \cdots\}$ with $x_1 = F_4$, $x_2 = F_5$, $x_3 = F_7$, $x_4 = F_8$, etc. In general, we see that

$$x_n - x_{n-1} = F_{3\lceil \frac{n}{2} \rceil} \text{ and } x_{2n} - x_{2n-2} = 2F_{3n}$$

so we appeal to Theorem 35. Items (3) and (4) both hold since, in each case, T is an infinite S-ap.

In Example 36, item (3), we have pairwise-accessibility with neither set being individually 2-accessible. To see this, consider $c = 2$ and $d = 1$. Then S is not 2-accessible by considering the coloring $00110011\ldots$, while T is not 2-accessible by considering the coloring $010101\ldots$.

The following example shows that the converse of Lemma 34 is not true.

Example 37. Let $S = \{1\}$ and $T = \{2\} \cup \{2n+1 : n \in \mathbb{Z}^+\}$. Then $S \cup T$ is 2-accessible, but (S,T) is not pairwise-accessible. To see this, consider the 2-coloring $\chi : \mathbb{Z}^+ \to \{0,1\}$ given by $\chi(i) = 1$ if $i \equiv 0 \pmod 4$ and $\chi(i) = 0$ otherwise. Then the longest S-diffsequence of color 0 has length 3, and there is no 2-term T-diffsequence of color 1.

Since the converse of Lemma 34 does not hold, perhaps a good question to ask is: If A is 2-accessible, must there exist $S, T \subseteq A$ such that $S \cup T = A$ and (S, T) is pairwise-accessible? In general, this seems to be a difficult question; however, if A has a certain additive property, we can answer in the affirmative.

Theorem 38. *Let $A \subseteq \mathbb{Z}^+$. If there exists an infinite (multi)set of positive integers $D = \{d_1 \leq d_2 \leq \cdots\}$ and an integer $a \in A \setminus D$ such that*

$$\left\{ a + \sum_{i=1}^{j} d_i : j \in \mathbb{Z}^+ \right\} \subseteq A \setminus D,$$

then $(D, A \setminus D)$ is pairwise-accessible (and, consequently, A is 2-accessible).

Proof. Assume, for a contradiction, that there exists a 2-coloring χ of $\mathbb{Z}^+$ such that the longest monochromatic D-diffsequence of color red has at most k terms, and such that the longest monochromatic $(A \setminus D)$-diffsequence of color blue has at most k terms, for some $k \in \mathbb{Z}^+$. Let $a_1, a_2, \ldots, a_\ell$ be a maximal length blue $(A \setminus D)$-diffsequence ($\ell \leq k$) under χ. Consider

$$X = \left\{ a_\ell + a + \sum_{i=1}^{j} d_i : j \in \mathbb{Z}^+ \right\}.$$

By assumption, X does not contain any blue element for otherwise we would have a blue $(S \setminus D)$-diffsequence of length $\ell + 1$. Hence, X contains only red elements. But then we have an infinitely long red D-diffsequence, contradicting our assumption. □

Example 39. Applying Theorem 38 to the set of Fibonacci numbers $\{F_n\}$, we have that $(\{F_{2n}\}_{n \in \mathbb{Z}^+}, \{F_{2n+1}\}_{n \in \mathbb{Z}^+})$ is pairwise-accessible by the fact that $2 + \sum_{i=2}^{m} F_{2i} = F_{2m+1}$. Coupling this with part (1) of Example 36, we see that the partitioning of a 2-accessible set into a pairwise-accessible pair need not be unique (if it exists).

Theorem 38 reduces to the following special case when all d_i are equal.

Corollary 40. *Let $A \subseteq \mathbb{Z}^+$ with the property that there exist $a, d \in A$ with $a \neq d$ such that $a + (j-1)d \in A$ for all $j \in \mathbb{Z}^+$. Then there exist S and T with $S \cap T = \emptyset$ such that $S \sqcup T = A$ and (S, T) is pairwise-accessible (and, consequently, A is 2-accessible).*

Remark 41. Recall that for $S = \{s_1, s_2, \ldots\}$, the set of finite sums of S is $FS(S) = \{\sum_{i \in I} s_i : \emptyset \neq I \subseteq \mathbb{Z}^+, |I| < \infty\}$. From [3] we know that if $FS(\{d_i\}) \subseteq S$, then S is large. However, Theorem 38's hypothesis that some translation is in $S \setminus D$ is weaker. Consider $S = \{2\} \cup \{2n - 1 : n \in \mathbb{Z}^+\}$ and let $d_i = 2$ for all i with $a = 1$. Then the conditions of Theorem 38 are satisfied but S is not large (it is not 2-large because it has no multiple of 4).

5. Final Remarks and Open Questions

Among the concepts of accessibility, super-accessibility, and largeness, the most difficult questions (and probably the most intriguing) deal with largeness. Indeed, the idea of super-accessibility was conceived in the hopes of learning more about large sets; in particular, to shed some light on the 2-large conjecture. In this article, we have presented some results involving super-accessibility that more closely resemble properties of largeness than those of accessibility (for example, Corollary 23), but the open questions about large sets still remain. With this in mind, we can refine the definition of super-accessibility further as follows, which provides us with a notion that is closer to largeness than super-accessibility.

Definition 42. Let $r, t \in \mathbb{Z}^+$ with $0 \leq t \leq r$. We say that a set $S \subseteq \mathbb{Z}^+$ is (r,t)-*accessible* if every r-coloring, using the colors $1, 2 \ldots, r$, of $\mathbb{Z}^+$ admits arbitrarily long S-aps of color j for some $j \in \{1, 2, \ldots, t\}$ or arbitrarily long S-diffsequences of color j for some $j \in \{t+1, t+2, \ldots, r\}$.

Note that $(r,0)$-accessible has the same meaning as r-accessible, $(r,1)$-accessible has the meaning as r-super-accessible, and (r,r)-accessible has the same meaning as r-large.

Remark. Denote by $\mathcal{S}_{r,t}$ the family of sets that are (r,t)-accessible. Let $\mathcal{L}_r$ denote the family of r-large sets and let $\mathcal{A}_r$ denote the family of r-accessible sets. It is obvious that if A is not r-large, and $s \geq 1$, then A is not $(r+s, r)$-accessible. Also, from the above definition, it is clear that for $r \geq 1$,

$$\mathcal{L}_r \subseteq \mathcal{S}_{r,r-1} \subseteq \mathcal{S}_{r,r-2} \subseteq \cdots \subseteq \mathcal{S}_{r,1} \subseteq \mathcal{A}_r. \tag{6}$$

With the hope of getting closer to solving the 2-large conjecture, we would like to answer the following question.

1. Is there an r large enough so that $A \in \mathcal{S}_{r,r-1}$ implies $A \in \mathcal{S}_{t,t-1}$ for all $t \geq r$. In other words, we want to know if there is an r for which sets that are $(r, r-1)$-accessible (in a sense, "almost r-large") are "almost large"? This would be close to the 2-large conjecture.

We also pose the following open questions.

2. We know (Corollary 22) that $\mathcal{S}_{r,1} \subsetneq \mathcal{A}_r$. Which of the other inclusions in (6) are strict?
3. Is the set of Fibonacci numbers 2-super-accessible? We know it is 2-accessible but that it is not 2-large.
4. What is the degree of accessibility of the Fibonacci numbers?
5. Is it true that the degree of accessibility of the set of even Fibonacci numbers is 1?
6. Is there a more useful set of criteria than that of Theorem 38 for when a 2-accessible set can be partitioned into pairwise-accessible sets?

7. Does 2-large imply accessible?

8. Determine $s(D,k)$ for $D = \{2\} \cup \{2n-1 : n \in \mathbb{Z}^+\}$.

9. Is there an $(r,2)$-accessible set that is not large? If so, this would prove that $\mathcal{L} \neq \mathcal{L}_2$.

10. Determine the rate of growth of $s(D,3,k)$ for various choices of D.

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