A Complete Proof of the Robbins Conjecture

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1 Boolean algebra

The language of Boolean algebra consists of two binary function symbols $\cup$ and $\cap$, one unary function symbol $\overline{}$, and two constants 0 and 1. The axioms of Boolean algebra, as found in [3, pages 7–8], are:

(B_1)	$x \cup (y \cup z) = (x \cup y) \cup z$	(B_1')	$x \cap (y \cap z) = (x \cap y) \cap z$
(B_2)	$x \cup y = y \cup x$	(B_2')	$x \cap y = y \cap x$
(B_3)	$x \cup (x \cap y) = x$	(B_3')	$x \cap (x \cup y) = x$
(B_4)	$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$	(B_4')	$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$
(B_5)	$x \cup \overline{x} = 1$	(B_5')	$x \cap \overline{x} = 0$

I will refer to these axioms collectively as B. Axioms B_1 and B_1' are called the associativity of $\cup$ and $\cap$, respectively. Axioms B_2 and B_2' are called the commutativity of $\cup$ and $\cap$, respectively. Axioms B_3 and B_3' are called the absorption axioms. Axioms B_4 and B_4' are called the distributivity axioms. Axioms B_5 and B_5' are called the complementation axioms. If one term can be obtained from another simply through applications of B_1, B_1', B_2, and B_2', the two terms will be called AC identical. In most proofs, applications of associativity and commutativity will go without mention.

It will aid our discussion to have at our disposal some of the well-known properties of Boolean algebras.

Proposition 1 (Idempotence). $B \vdash x \cup x = x = x \cap x$.

Proof. By absorption, $x \cup x = x \cup (x \cap (x \cup x)) = x$, and $x \cap x = x \cap (x \cup (x \cap x)) = x$. □

Proposition 2. $B \vdash x \cup y = y \leftrightarrow x \cap y = x$.

Proof. If $x \cup y = y$, then by absorption $x \cap y = x \cap (x \cup y) = x$. Conversely, if $x \cap y = x$, then by absorption $x \cup y = (x \cap y) \cup y = y$. □

Proposition 3. $B \vdash x \cup 0 = x = x \cap 1$.

Proof. By B_5' and B_3, $x \cup 0 = x \cup (x \cap \overline{x}) = x$. By B_5 and B_3', $x \cap 1 = x \cap (x \cup \overline{x}) = x$. □

Proposition 4. $B \vdash x \cap 0 = 0 \wedge x \cup 1 = 1$.

Proof. By B_5' and idempotence, $x \cap 0 = x \cap x \cap \overline{x} = x \cap \overline{x} = 0$. By B_5 and idempotence, $x \cup 1 = x \cup x \cup \overline{x} = x \cup \overline{x} = 1$. □

Definition. Two elements x and y of a Boolean algebra are *complements* if $x \cup y = 1$ and $x \cap y = 0$.

Proposition 5. *$\overline{x}$ is the unique complement of x.*

Proof. The elements x and $\overline{x}$ are complements by B_5 and B_5'. Suppose y and z are both complements of x. Then

$$\begin{aligned} y &= y \cap 1 & z &= z \cap 1 && \text{(by 3)} \\ &= y \cap (x \cup z) & &= z \cap (x \cup y) && \text{(by hypothesis)} \\ &= (y \cap x) \cup (y \cap z) & &= (z \cap x) \cup (z \cap y) && \text{(by } B_4) \\ &= 0 \cup (y \cap z) & &= 0 \cup (z \cap y) && \text{(by hypothesis)} \\ &= y \cap z. & &= z \cap y. && \text{(by 3)} \end{aligned}$$

Hence $y = z$. □

Proposition 6. $B \vdash \overline{\overline{x}} = x$.

Proof. Both x and $\overline{\overline{x}}$ are complements of $\overline{x}$. □

Proposition 7. $B \vdash \overline{x} = \overline{y} \to x = y$.

Proof. If $\overline{x} = \overline{y}$, then by the previous proposition, $x = \overline{\overline{x}} = \overline{\overline{y}} = y$. □

Proposition 8. $B \vdash \overline{0} = 1 \wedge \overline{1} = 0$.

Proof. By Proposition 3, $0 \cup 1 = 1$ and $0 \cap 1 = 0$, so 0 and 1 are complements. □

Proposition 9 (De Morgan). $B \vdash \overline{x \cup y} = \overline{x} \cap \overline{y} \wedge \overline{x \cap y} = \overline{x} \cup \overline{y}$.

Proof. By distributivity, B_5, and Proposition 4 we have:

$$(x \cup y) \cup (\overline{x} \cap \overline{y}) = (x \cup y \cup \overline{x}) \cap (x \cup y \cup \overline{y}) = (y \cup 1) \cap (x \cup 1) = 1 \cap 1 = 1.$$

By distributivity, B_5', and Proposition 4 we have:

$$(x \cup y) \cap (\overline{x} \cap \overline{y}) = (x \cap \overline{x} \cap \overline{y}) \cup (y \cap \overline{x} \cup \overline{y}) = (0 \cup \overline{y}) \cap (0 \cap \overline{x}) = 0 \cap 0 = 0.$$

Thus $x \cup y$ and $\overline{x} \cap \overline{y}$ are complements. The dual argument shows that $x \cap y$ and $\overline{x} \cup \overline{y}$ are complements. □

It follows immediately from De Morgan's laws and Proposition 6 that $x \cup y = \overline{\overline{x} \cap \overline{y}}$ and $x \cap y = \overline{\overline{x} \cup \overline{y}}$.

2 Huntington algebra

In 1933, E.V. Huntington [2, 1] showed that the following three axioms, to which I shall refer as H, form a basis for Boolean algebra. That is, any theorem of Boolean algebra can be derived from the three, and none of the three can be derived from the other two.

(H_1) $x \cup (y \cup z) = (x \cup y) \cup z$ (associativity)
(H_2) $x \cup y = y \cup x$ (commutativity)
(H_3) $\overline{\overline{x} \cup \overline{y}} \cup \overline{\overline{x} \cup y} = x$ (Huntington equation)

The term $\overline{\overline{x} \cup \overline{y}} \cup \overline{\overline{x} \cup y}$ will be called the Huntington expansion of x by y. It is easy to prove that every Boolean algebra satisfies Huntington's axioms.

Theorem 10. $\mathrm{B} \vdash \mathrm{H}$.

Proof. Since H_1 and H_2 are identical to B_1 and B_2, all we need to prove is the Huntington equation. We can restate the Huntington equation as:

$$(x \cap y) \cup (x \cap \overline{y}) = x.$$

By distributivity, B_5, and Proposition 3,

$$(x \cap y) \cup (x \cap \overline{y}) = x \cap (y \cup \overline{y}) = x \cap 1 = x. \qquad \square$$

Observe that Huntington's axioms use only one binary function symbol $\cup$, and one unary function symbol $\overline{}$. Strictly speaking, to show that H is a basis for Boolean algebra, one must expand the language of Huntington algebra to include $\cap$, 0, and 1 by defining them in terms of $\cup$ and $\overline{}$. On occasion, we will use the abbreviation $nx = \underbrace{x \cup \ldots \cup x}_{n}$.

3 Robbins algebra

Shortly after Huntington proved his result, Herbert Robbins conjectured that the following three axioms, to which I shall refer as R, also form a basis for Boolean algebra.

(R_1)	$x \cup (y \cup z) = (x \cup y) \cup z$	(associativity)
(R_2)	$x \cup y = y \cup x$	(commutativity)
(R_3)	$\overline{\overline{x \cup y} \cup \overline{x \cup \overline{y}}} = x$	(Robbins equation)

The term $\overline{\overline{x \cup y} \cup \overline{x \cup \overline{y}}}$ will be called the Robbins expansion of x by y. It is equally easy to prove that every Boolean algebra satisfies Robbins' axioms.

Theorem 11. $\mathrm{B} \vdash \mathrm{R}$.

Proof. Since R_1 and R_2 are identical to B_1 and B_2, all we need to prove is the Robbins equation. We can restate the Robbins equation as:

$$(x \cup y) \cap (x \cup \overline{y}) = x.$$

By distributivity, B_5', and Proposition 3,

$$(x \cup y) \cap (x \cup \overline{y}) = x \cup (y \cap \overline{y}) = x \cup 0 = x. \qquad \square$$

The Robbins equation is simpler than the Huntington equation. It has one fewer occurrence of $\overline{}$. Despite the similarity of Huntington's and Robbins' axioms, Robbins and Huntington were unable to find a proof that all Robbins algebras are Boolean. The question "Are all Robbins algebras Boolean?" became known as the Robbins problem.

The problem remained unsolved for many years. According to McCune [4], the first major step toward the solution came in the 1980s when Steve Winker proved several conditions sufficient to make a Robbins algebra Boolean. That is, any Robbins algebra that satisfies one of Winker's conditions is a Boolean algebra. The problem was finally solved in 1997 by EQP, a theorem prover created at Argonne National Laboratory, which under the direction of William McCune proved that all Robbins algebras satisfy what is known as Winker's first condition.

The remainder of this paper will present a complete proof that all Robbins algebras are Boolean.

4 $\mathrm{H} \vdash \mathrm{B}$

First, we will prove some basic properties of $\overline{}$.

Proposition 12. $\mathrm{H} \vdash x \cup \overline{x} = \overline{x} \cup \overline{\overline{x}}$.

Proof. Use the Huntington equation to expand x and $\overline{x}$ by $\overline{\overline{x}}$:

$$x \cup \overline{x} = \left(\overline{\overline{x} \cup \overline{\overline{\overline{x}}}} \cup \overline{\overline{x} \cup \overline{\overline{x}}}\right) \cup \left(\overline{\overline{\overline{x}} \cup \overline{\overline{\overline{x}}}} \cup \overline{\overline{\overline{x}} \cup \overline{\overline{x}}}\right).$$

Likewise, use the Huntington equation to expand $\overline{x}$ and $\overline{\overline{x}}$ by $\overline{x}$:

$$\overline{x} \cup \overline{\overline{x}} = \left(\overline{\overline{\overline{x}} \cup \overline{\overline{x}}} \cup \overline{\overline{\overline{x}} \cup \overline{x}}\right) \cup \left(\overline{\overline{\overline{\overline{x}}} \cup \overline{\overline{x}}} \cup \overline{\overline{\overline{\overline{x}}} \cup \overline{x}}\right).$$

The right-hand sides of these two equations are AC identical. Therefore $x \cup \overline{x} = \overline{x} \cup \overline{\overline{x}}$. ◻

Proposition 13. $\mathrm{H} \vdash \overline{\overline{x}} = x$.

Proof. Use the Huntington equation to expand $\overline{\overline{x}}$ by $\overline{x}$, then simplify with Proposition 12 and the Huntington equation applied to x and $\overline{\overline{x}}$:

$$\overline{\overline{x}} = \overline{\overline{\overline{\overline{x}}} \cup \overline{\overline{x}}} \cup \overline{\overline{\overline{\overline{x}}} \cup \overline{x}} = \overline{\overline{x} \cup \overline{\overline{\overline{x}}}} \cup \overline{\overline{x} \cup \overline{\overline{x}}} = x.$$ ◻

Proposition 14. $\mathrm{H} \vdash \overline{x} = \overline{y} \rightarrow x = y$.

Proof. If $\overline{x} = \overline{y}$, then by the previous proposition, $x = \overline{\overline{x}} = \overline{\overline{y}} = y$. ◻

Next, we define $\cap$ and prove some useful propositions such as De Morgan's laws.

Definition. $x \cap y = \overline{\overline{x} \cup \overline{y}}$.

Proposition 15. $\mathrm{H} \vdash \overline{x \cup y} = \overline{x} \cap \overline{y} \wedge \overline{x \cap y} = \overline{x} \cup \overline{y}$.

Proof. By the definition of $\cap$ and Proposition 13, $\overline{x} \cap \overline{y} = \overline{\overline{\overline{x}} \cup \overline{\overline{y}}} = \overline{x \cup y}$. Similarly, $\overline{x \cap y} = \overline{\overline{\overline{x} \cup \overline{y}}} = \overline{x} \cup \overline{y}$. ◻

Proposition 16. $\mathrm{H} \vdash x \cup y = \overline{\overline{x} \cap \overline{y}}$.

Proof. By Propositions 13 and 15, $x \cup y = \overline{\overline{x \cup y}} = \overline{\overline{x} \cap \overline{y}}$. ◻

Proposition 16 shows that we could formulate the axioms of Huntington algebra in terms of $\overline{}$ and $\cap$ instead of $\overline{}$ and $\cup$. We can also give a more intuitive formulation of the Huntington equation in terms of all three symbols.

Proposition 17. $\mathrm{H} \vdash (x \cap y) \cup (x \cap \overline{y}) = x$.

Proof. $(x \cap y) \cup (x \cap \overline{y}) = \overline{\overline{x} \cup \overline{y}} \cup \overline{\overline{x} \cup \overline{\overline{y}}} = \overline{\overline{x} \cup \overline{\overline{y}}} \cup \overline{\overline{x} \cup \overline{y}} = x$. ◻

The associativity and commutativity of $\cap$ follow directly from the associativity and commutativity of $\cup$.

Theorem 18. $\mathrm{H} \vdash \mathrm{B}_1'$.

Proof. By Proposition 15, $x \cap (y \cap z) = \overline{\overline{x} \cup \overline{y \cap z}} = \overline{\overline{x} \cup (\overline{y} \cup \overline{z})} = \overline{(\overline{x} \cup \overline{y}) \cup \overline{z}} = \overline{\overline{x \cap y} \cup \overline{z}} = (x \cap y) \cap z$. ◻

Theorem 19. $\mathrm{H} \vdash \mathrm{B}_2'$.

Proof. $x \cap y = \overline{\overline{x} \cup \overline{y}} = \overline{\overline{y} \cup \overline{x}} = y \cap x$. ☐

In any Huntington algebra, the function defined by $f(x) = x \cup \overline{x}$ is constant.

Proposition 20. $\mathrm{H} \vdash x \cup \overline{x} = y \cup \overline{y}$.

Proof. Use the Huntington equation to expand x and $\overline{x}$ by $\overline{y}$:

$$x \cup \overline{x} = \left(\overline{\overline{x} \cup \overline{\overline{y}}} \cup \overline{\overline{x} \cup \overline{y}}\right) \cup \left(\overline{\overline{\overline{x}} \cup \overline{\overline{y}}} \cup \overline{\overline{\overline{x}} \cup \overline{y}}\right).$$

Likewise, use the Huntington equation to expand y and $\overline{y}$ by $\overline{x}$:

$$y \cup \overline{y} = \left(\overline{\overline{y} \cup \overline{\overline{x}}} \cup \overline{\overline{y} \cup \overline{x}}\right) \cup \left(\overline{\overline{\overline{y}} \cup \overline{\overline{x}}} \cup \overline{\overline{\overline{y}} \cup \overline{x}}\right).$$

The right-hand sides of these two equations are AC identical. Therefore, $x \cup \overline{x} = y \cup \overline{y}$. ☐

We can now extend the language of Huntington algebra to include the constants 0 and 1.

Definition. $1 = x \cup \overline{x}$.

Definition. $0 = \overline{1} = \overline{x \cup \overline{x}}$.

Observe that by the definition of 1 we have $\mathrm{H} \vdash \mathrm{B}_5$.

Theorem 21. $\mathrm{H} \vdash \mathrm{B}_5'$.

Proof. By Proposition 13, $x \cap \overline{x} = \overline{\overline{x} \cup \overline{\overline{x}}} = \overline{\overline{x} \cup x} = 0$. ☐

Proposition 22. $\mathrm{H} \vdash x \cup 0 = x = x \cap 1$.

Proof. First, apply the Huntington equation to 0 and 0 to obtain:

$$\overline{1} = 0 = \overline{\overline{0} \cup \overline{0}} \cup \overline{\overline{0} \cup 0} = \overline{\overline{0} \cup \overline{0}} \cup 0 = \overline{1 \cup 1} \cup \overline{1}. \tag{1}$$

Second, use equation (1) to obtain:

$$1 = 1 \cup \overline{1} = 1 \cup \left(\overline{1 \cup 1} \cup \overline{1}\right) = \left(1 \cup \overline{1}\right) \cup \left(\overline{1 \cup 1}\right) = 1 \cup \overline{1 \cup 1}. \tag{2}$$

Third, use equation (2) to obtain:

$$1 = (1 \cup 1) \cup \overline{1 \cup 1} = 1 \cup \left(1 \cup \overline{1 \cup 1}\right) = 1 \cup 1. \tag{3}$$

Fourth, use equations 1 and 3 to obtain:

$$0 = \overline{1} = \overline{1 \cup 1} \cup \overline{1} = \overline{1} \cup \overline{1} = 0 \cup 0. \tag{4}$$

Fifth, use the Huntington equation applied to x and x and equation (4) to obtain,

$$x \cup 0 = \left(\overline{\overline{x} \cup \overline{x}} \cup \overline{\overline{x} \cup x}\right) \cup 0 = \overline{\overline{x} \cup \overline{x}} \cup 0 \cup 0 = \overline{\overline{x} \cup \overline{x}} \cup 0 = \overline{\overline{x} \cup \overline{x}} \cup \overline{\overline{x} \cup x} = x. \tag{5}$$

Finally, by equation (5) and 13, $x \cap 1 = \overline{\overline{x} \cup \overline{1}} = \overline{\overline{x} \cup 0} = \overline{\overline{x}} = x$. ☐

Proposition 23. $\mathrm{H} \vdash x \cup x = x = x \cap x$.

Proof. By the Huntington equation applied to x and x, and Proposition 22,

$$x = \overline{\overline{x} \cup \overline{x}} \cup \overline{\overline{x} \cup x} = \overline{\overline{x} \cup \overline{x}} \cup 0 = \overline{\overline{x} \cup \overline{x}} = x \cap x.$$

Therefore, by Propositions 16 and 13 we have $x \cup x = \overline{\overline{x} \cap \overline{x}} = \overline{\overline{x}} = x$. ☐

Proposition 24. $\mathrm{H} \vdash x \cap 0 = 0 \wedge x \cup 1 = 1$.

Proof. By Proposition 23, $x \cup 1 = x \cup x \cup \overline{x} = x \cup \overline{x} = 1$. Thus $x \cap 0 = \overline{\overline{x} \cup \overline{0}} = \overline{\overline{x} \cup 1} = \overline{1} = 0$. ☐

Theorem 25. $\mathrm{H} \vdash \mathrm{B}_3$.

Proof. Use the Huntington equation to expand x by y, then simplify with Proposition 23 and the Huntington equation applied to x and y:

$$x \cup (x \cap y) = \overline{\overline{x} \cup \overline{y}} \cup \overline{\overline{x} \cup y} \cup \overline{\overline{x} \cup \overline{y}} = \overline{\overline{x} \cup \overline{y}} \cup \overline{\overline{x} \cup y} = x.$$ ☐

Theorem 26. $\mathrm{H} \vdash \mathrm{B}_3'$.

Proof. By Proposition 15, B_3, and Proposition 13,

$$x \cap (x \cup y) = \overline{\overline{x} \cup \overline{x \cup y}} = \overline{\overline{x} \cup (\overline{x} \cap \overline{y})} = \overline{\overline{x}} = x.$$ ☐

Theorem 27. $\mathrm{H} \vdash \mathrm{B}_4$.

Proof. First, use the Proposition 17 to expand $x \cap (y \cup z)$ by y:

$$x \cap (y \cup z) = [x \cap (y \cup z) \cap y] \cup [x \cap (y \cup z) \cap \overline{y}],$$

which by B_3' applied to y and z simplifies to:

$$x \cap (y \cup z) = [x \cap y] \cup [x \cap (y \cup z) \cap \overline{y}].$$

Now use Proposition 17 to expand each square-bracketed term by z,

$$\begin{aligned}
x \cap (y \cup z) &= [(x \cap y \cap z) \cup (x \cap y \cap \overline{z})] \cup [(x \cap (y \cup z) \cap \overline{y} \cap z) \cup (x \cap (y \cup z) \cap \overline{y} \cap \overline{z})] \\
&= [(x \cap y \cap z) \cup (x \cap y \cap \overline{z})] \cup [(x \cap \overline{y} \cap z) \cup (x \cap (y \cup z) \cap \overline{y} \cap \overline{z})] && \text{(by } \mathrm{B}_3'\text{)} \\
&= [(x \cap y \cap z) \cup (x \cap y \cap \overline{z})] \cup [(x \cap \overline{y} \cap z) \cup (x \cap (y \cup z) \cap \overline{y \cup z})] && \text{(by 15)} \\
&= [(x \cap y \cap z) \cup (x \cap y \cap \overline{z})] \cup [(x \cap \overline{y} \cap z) \cup (x \cap 0)] && \text{(by } \mathrm{B}_5'\text{)} \\
&= [(x \cap y \cap z) \cup (x \cap y \cap \overline{z})] \cup [(x \cap \overline{y} \cap z) \cup 0] && \text{(by 24)} \\
&= (x \cap y \cap z) \cup (x \cap y \cap \overline{z}) \cup (x \cap \overline{y} \cap z) && \text{(by 22)} \\
&= (x \cap y \cap z) \cup (x \cap y \cap \overline{z}) \cup (x \cap y \cap z) \cup (x \cap \overline{y} \cap z) && \text{(by 23)} \\
&= (x \cap y) \cup (x \cap z). && \text{(by 17)}
\end{aligned}$$

☐

The proof of Theorem 27 was discovered by Don Monk.

Theorem 28. $\mathrm{H} \vdash \mathrm{B}_4'$.

Proof.

$$
\begin{aligned}
x \cup (y \cap z) &= \overline{\overline{\overline{\overline{x}} \cup \overline{\overline{y} \cup \overline{z}}}} && \text{(by 13)} \\
&= \overline{\overline{x} \cap (\overline{y} \cup \overline{z})} \\
&= \overline{(\overline{x} \cap \overline{y}) \cup (\overline{x} \cap \overline{z})} && \text{(by B}_4\text{)} \\
&= \overline{\overline{x \cup y} \cup \overline{x \cup z}} && \text{(by 15)} \\
&= (x \cup y) \cap (x \cup z).
\end{aligned}
$$

□

Theorem 29. H ⊢ B.

Proof. By the definition of 1 and Theorems 18, 19, 25, 26, 21, 27, and 28. □

5 $\mathrm{R} + \mathrm{W}_1 \vdash \mathrm{H}$

In 1992, Winker [5] proved that each of the following axioms is a sufficient condition for a Robbins algebra to satisfy the Huntington equation, and therefore to be a Boolean algebra.

(W_{-2})	$\overline{\overline{x}} = x$	(double negation)
(W_{-1})	$x \cup 0 = x$	(zero)
(W_0)	$a \cup a = a$	(idempotent)
(W_1)	$a \cup b = b$	(absorption)
(W_2)	$\overline{a \cup b} = \overline{b}$	(absorption within negation)

Note that in the above equations, x is a variable, while 0, a, and b are constants.

Theorem 30. $\mathrm{R} + \mathrm{W}_{-2} \vdash \mathrm{H}$.

Proof. Since H_1 and H_2 are identical to R_1 and R_2, all we need to prove is the Huntington equation. Apply the Robbins equation to $\overline{x}$ and y to get $\overline{\overline{\overline{x} \cup y} \cup \overline{\overline{x} \cup \overline{y}}} = \overline{x}$. Then by W_{-2},

$$\overline{\overline{x} \cup y} \cup \overline{\overline{x} \cup \overline{y}} = \overline{\overline{\overline{\overline{x} \cup y} \cup \overline{\overline{x} \cup \overline{y}}}} = \overline{\overline{x}} = x.$$

Commuting terms on the left yields the Huntington equation. □

Theorem 31. $\mathrm{R} + \mathrm{W}_{-1} \vdash \mathrm{W}_{-2}$.

Proof. Use the Robbins equation to expand 0 by x, then simplify with W_{-1} to obtain:

$$0 = \overline{\overline{0 \cup x} \cup \overline{0 \cup \overline{x}}} = \overline{\overline{x} \cup \overline{\overline{x}}}. \tag{6}$$

Use the Robbins equation to expand $\overline{x}$ by $\overline{\overline{x}}$, then simplify with equation (6) and W_{-1} to obtain:

$$\overline{x} = \overline{\overline{\overline{x} \cup \overline{\overline{x}}} \cup \overline{\overline{x} \cup \overline{\overline{\overline{x}}}}} = \overline{0 \cup \overline{\overline{x} \cup \overline{\overline{\overline{x}}}}} = \overline{\overline{\overline{x} \cup \overline{\overline{\overline{x}}}}}.$$

Use the Robbins equation to expand $\overline{\overline{\overline{x}}}$ by $\overline{x}$, then simplify with equation (6) applied to $\overline{x}$ to obtain:

$$\overline{\overline{\overline{x}}} = \overline{\overline{\overline{\overline{\overline{x}}} \cup \overline{x}} \cup \overline{\overline{\overline{\overline{x}}} \cup \overline{\overline{x}}}} = \overline{\overline{\overline{\overline{\overline{x}}} \cup \overline{x}} \cup 0} = \overline{\overline{\overline{\overline{\overline{x}}} \cup \overline{x}}}.$$

Therefore $\overline{x} = \overline{\overline{\overline{x}}}$. Substitute $\overline{x \cup y} \cup \overline{x \cup \overline{y}}$ in this equation, then apply the Robbins equation to both sides to get:

$$x = \overline{\overline{x \cup y} \cup \overline{x \cup \overline{y}}} = \overline{\overline{\overline{\overline{x \cup y} \cup \overline{x \cup \overline{y}}}}} = \overline{\overline{x}}.$$

□

Theorem 32. $\mathrm{R} + \mathrm{W}_0 \vdash \mathrm{W}_{-1}$.

Proof. Suppose $a \cup a = a$, and define $0 = \overline{a \cup \overline{a}}$. We will show that $x \cup 0 = x$.

Use the Robbins equation to expand a by a, then simplify with W_0 to get:

$$a = \overline{\overline{a \cup a} \cup \overline{a \cup \overline{a}}} = \overline{\overline{a} \cup 0}. \tag{7}$$

Use the Robbins equation to expand $a \cup x$ by a, then simplify with W_0 to get:

$$a \cup x = \overline{\overline{a \cup x \cup a} \cup \overline{a \cup x \cup \overline{a}}} = \overline{\overline{a \cup x} \cup \overline{a \cup x \cup \overline{a}}}. \tag{8}$$

Use the Robbins equation to expand x by $\overline{a} \cup 0$, then simplify with equation (7) to get:

$$x = \overline{\overline{x \cup \overline{a} \cup 0} \cup \overline{x \cup \overline{\overline{a} \cup 0}}} = \overline{\overline{x \cup \overline{a} \cup 0} \cup \overline{x \cup a}}. \tag{9}$$

Use the Robbins equation to expand $\overline{a}$ by $a \cup \overline{a}$, then simplify with equation (7) to get:

$$\overline{a} = \overline{\overline{\overline{a} \cup a \cup \overline{a}} \cup \overline{\overline{a} \cup \overline{a \cup \overline{a}}}} = \overline{\overline{a \cup \overline{a} \cup \overline{a}} \cup a}. \tag{10}$$

Apply equation (9) to a, then simplify with W_0 to get:

$$a = \overline{\overline{a \cup \overline{a} \cup 0} \cup \overline{a \cup a}} = \overline{\overline{a \cup \overline{a} \cup 0} \cup \overline{a}}. \tag{11}$$

Use the Robbins equation to expand a by $a \cup \overline{a} \cup \overline{a}$, then simplify with equation (10) and W_0 to get:

$$a = \overline{\overline{a \cup a \cup \overline{a} \cup \overline{a}} \cup \overline{a \cup \overline{a \cup \overline{a} \cup \overline{a}}}} = \overline{\overline{a \cup \overline{a} \cup \overline{a}} \cup \overline{a}}. \tag{12}$$

Use the Robbins equation to expand $\overline{a \cup \overline{a} \cup \overline{a}}$ by a, then simplify with equations (10) and (12) to get:

$$\overline{a \cup \overline{a} \cup \overline{a}} = \overline{\overline{\overline{a \cup \overline{a} \cup \overline{a}} \cup a} \cup \overline{\overline{a \cup \overline{a} \cup \overline{a}} \cup \overline{a}}} = \overline{\overline{a} \cup a} = 0. \tag{13}$$

By equations (10) and (13):

$$\overline{a} = \overline{a \cup \overline{a \cup \overline{a} \cup \overline{a}}} = \overline{a \cup 0}. \tag{14}$$

Apply equation (8) to 0, then simplify with equations (14) and (11) to get:

$$a \cup 0 = \overline{\overline{a \cup 0} \cup \overline{a \cup 0 \cup \overline{a}}} = \overline{\overline{a} \cup \overline{a \cup 0 \cup \overline{a}}} = a. \tag{15}$$

Use the Robbins equation to expand $x \cup 0$ by a, then simplify with equations (15) and (9) to get:

$$x \cup 0 = \overline{\overline{x \cup 0 \cup a} \cup \overline{x \cup 0 \cup \overline{a}}} = \overline{\overline{x \cup a} \cup \overline{x \cup 0 \cup \overline{a}}} = x.$$ □

The proof of Theorem 32 was found by EQP. It is shorter than Winker's original proof.

Lemma 33. $\mathrm{R} \vdash \overline{a \cup \overline{b \cup c}} = \overline{a \cup b \cup \overline{c}} \rightarrow a \cup b = a$.

Proof. Use the Robbins equation to expand $a \cup b$ by c, then apply the hypothesis and simplify with the Robbins equation applied to a and $b \cup c$:

$$a \cup b = \overline{\overline{a \cup b \cup c} \cup \overline{a \cup b \cup \overline{c}}} = \overline{\overline{a \cup b \cup c} \cup \overline{a \cup \overline{b \cup c}}} = a.$$ □

Lemma 34. $\mathrm{R} \vdash \overline{a \cup \overline{b \cup c}} = \overline{b \cup \overline{a \cup c}} \rightarrow a = b$.

Proof. Use the Robbins equation to expand a by $b \cup c$, then apply the hypothesis and simplify with the Robbins equation applied to b and $a \cup c$:

$$a = \overline{\overline{a \cup b \cup c} \cup \overline{a \cup \overline{b \cup c}}} = \overline{\overline{b \cup a \cup c} \cup \overline{b \cup \overline{a \cup c}}} = b. \qquad \square$$

Lemma 35. $\mathrm{R} \vdash \overline{a \cup \overline{b}} = c \to \overline{\overline{a \cup b} \cup c} = a.$

Proof. Use the Robbins equation to expand a by b, then apply the hypothesis:

$$a = \overline{\overline{a \cup b} \cup \overline{a \cup \overline{b}}} = \overline{\overline{a \cup b} \cup c}. \qquad \square$$

Lemma 36. *For every positive integer k,* $\mathrm{R} \vdash \overline{a \cup \overline{b}} = c \to \overline{a \cup \overline{b \cup k(a \cup c)}} = c.$

Proof. By induction on k. Let $b_0 = b$ and $b_k = b \cup k(a \cup c)$. By hypothesis, $\overline{a \cup \overline{b_0}} = c$. Now assume $\overline{a \cup \overline{b_k}} = c$. Then by Lemma 35, $a = \overline{\overline{a \cup b_k} \cup c}$, so we have

$$\overline{a \cup \overline{b_{k+1}}} = \overline{\overline{\overline{a \cup b_k} \cup c} \cup \overline{b_k \cup a \cup c}} = c,$$

by the Robbins equation applied to c and $a \cup b_k$. $\square$

Lemma 37. *For every positive integer k,* $\mathrm{R} \vdash \overline{\overline{a \cup \overline{b}} \cup \overline{b}} = a \to \overline{b \cup k(a \cup \overline{a \cup \overline{b}})} = \overline{b}.$

Proof. Let $\overline{a \cup \overline{b}} = c$ and $b_k = b \cup k(a \cup c)$. Then by Lemma 36,

$$\overline{a \cup \overline{b_k}} = \overline{a \cup \overline{b \cup k(a \cup c)}} = c.$$

By hypothesis, $\overline{c \cup \overline{b}} = a$, so by Lemma 36,

$$\overline{c \cup \overline{b_k}} = \overline{c \cup \overline{b \cup k(c \cup a)}} = a.$$

Therefore,

$$\overline{\overline{b_k} \cup \overline{\overline{b} \cup c}} = \overline{\overline{b_k} \cup a} = c = \overline{\overline{b} \cup a} = \overline{\overline{b} \cup \overline{\overline{b_k} \cup c}},$$

so by Lemma 34 applied to $\overline{b_k}$, $\overline{b}$, and c, we have $\overline{b_k} = \overline{b}$. $\square$

This proof of Lemma 37 was discovered by an automated theorem prover [6].

Lemma 38. *For every positive integer k,* $\mathrm{R} \vdash \overline{a \cup b} = \overline{b} \to \overline{b \cup k(a \cup \overline{a \cup \overline{b}})} = \overline{b}.$

Proof. Observe that by hypothesis, $\overline{\overline{a \cup \overline{b}} \cup \overline{b}} = \overline{\overline{a \cup b} \cup \overline{a \cup \overline{b}}} = a$, so the conclusion follows from Lemma 37. $\square$

Lemma 39. $\mathrm{R} \vdash \overline{2a \cup b} = \overline{b} = \overline{3a \cup b} \to 2a \cup b = 3a \cup b.$

Proof. Applying Lemma 38 to $2a$ and b with $k = 1$ yields:

$$\overline{2a \cup b \cup \overline{2a \cup \overline{b}}} = \overline{b \cup 2a \cup \overline{2a \cup \overline{b}}} = \overline{b}.$$

Applying Lemma 38 with $k = 1$ to a and $2a \cup b$ yields:

$$\overline{2a \cup b \cup a \cup \overline{a \cup \overline{b}}} = \overline{2a \cup b \cup a \cup \overline{a \cup \overline{2a \cup b}}} = \overline{2a \cup b} = \overline{b}.$$

Hence

$$\overline{2a \cup b \cup \overline{2a \cup \overline{b}}} = \overline{2a \cup b \cup a \cup \overline{a \cup \overline{b}}},$$

so by Lemma 33 we have $2a \cup b = 2a \cup b \cup a$. $\square$

Lemma 40. $\mathrm{R} \vdash (\overline{a \cup b} = \bar{b} \vee \overline{\overline{a \cup \bar{b}} \cup \bar{b}} = a) \rightarrow b \cup 2(a \cup \overline{a \cup \bar{b}}) = b \cup 3(a \cup \overline{a \cup \bar{b}})$.

Proof. Apply either Lemma 37 or 38 to obtain

$$\overline{b \cup 2(a \cup \overline{a \cup \bar{b}})} = \bar{b} = \overline{b \cup 3(a \cup \overline{a \cup \bar{b}})}.$$

Then use Lemma 39 applied to $a \cup \overline{a \cup \bar{b}}$ and b to conclude $b \cup 2(a \cup \overline{a \cup \bar{b}}) = b \cup 3(a \cup \overline{a \cup \bar{b}})$. □

Theorem 41. $\mathrm{R} + \mathrm{W}_1 \vdash \mathrm{W}_0$.

Proof. Let $a \cup b = b$. Define $c = b \cup 2(\overline{a \cup \bar{b}})$ and $d = c \cup \overline{c \cup \bar{c}}$. We will show that $3d \cup 3d = 3d$.

First observe that by W_1,

$$a \cup c = a \cup b \cup 2(\overline{a \cup \bar{b}}) = b \cup 2(\overline{a \cup \bar{b}}) = c. \tag{16}$$

By W_1 we have $\overline{a \cup b} = \bar{b}$, so by Lemma 38 and W_1,

$$\bar{b} = \overline{b \cup 2(a \cup \overline{a \cup \bar{b}})} = \overline{b \cup 2(\overline{a \cup \bar{b}})} = \bar{c}. \tag{17}$$

By equation (17), W_1, and Lemma 40,

$$\begin{aligned} c \cup \overline{a \cup \bar{c}} &= b \cup 2(\overline{a \cup \bar{b}}) \cup \overline{a \cup \bar{b}} \\ &= b \cup 3(\overline{a \cup \bar{b}}) \\ &= b \cup 3(a \cup \overline{a \cup \bar{b}}) \\ &= b \cup 2(a \cup \overline{a \cup \bar{b}}) \\ &= b \cup 2(\overline{a \cup \bar{b}}) \\ &= c \end{aligned} \tag{18}$$

By equations (16), (18), and the Robbins equation applied to c and $a \cup \bar{c}$ we have

$$\overline{\overline{c \cup \bar{c}} \cup \bar{c}} = \overline{\overline{c \cup a \cup \bar{c}} \cup \overline{c \cup \overline{a \cup \bar{c}}}} = c, \tag{19}$$

which satisfies the hypothesis of Lemma 40 applied to c and c. Hence,

$$c \cup 2d = c \cup 2(c \cup \overline{c \cup \bar{c}}) = c \cup 3(c \cup \overline{c \cup \bar{c}}) = c \cup 3d. \tag{20}$$

Therefore, $4d = 3d \cup c \cup \overline{c \cup \bar{c}} = 2d \cup c \cup \overline{c \cup \bar{c}} = 3d$. Repeat twice more to obtain $6d = 3d$. □

Theorem 42. $\mathrm{R} + \mathrm{W}_1 \vdash \mathrm{H}$

Proof. By Theorems 30, 32, and 41. □

6 $\mathrm{R} \vdash \mathrm{W}_1$

Lemma 43. $\mathrm{R} \vdash \overline{\overline{\bar{x} \cup y} \cup \overline{x \cup y}} = y$.

Proof. This is just a restatement of the Robbins equation. □

Lemma 44. $\mathrm{R} \vdash \overline{\overline{\overline{x \cup y} \cup \bar{x} \cup y} \cup y} = \overline{x \cup y}$.

Proof. Applying Lemma 43 to $\overline{x} \cup y$ and $\overline{x \cup y}$ yields:

$$\overline{\overline{\overline{\overline{x} \cup y} \cup \overline{x \cup y}} \cup \overline{\overline{x} \cup y \cup \overline{x \cup y}}} = \overline{x \cup y}.$$

Use Lemma 43 applied to x and y to simplify the left-hand side of the equation:

$$\overline{y \cup \overline{\overline{x} \cup y \cup \overline{x \cup y}}} = \overline{x \cup y}.$$

Commuting terms on the left-hand side yields the desired result. $\square$

Lemma 45. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{x} \cup y} \cup x \cup y} \cup y} = \overline{\overline{x} \cup y}$.

Proof. Applying Lemma 43 to $x \cup y$ and $\overline{\overline{x} \cup y}$ yields:

$$\overline{\overline{\overline{x \cup y} \cup \overline{\overline{x} \cup y}} \cup \overline{x \cup y \cup \overline{\overline{x} \cup y}}} = \overline{\overline{x} \cup y}.$$

Use Lemma 43 applied to x and y to simplify the left-hand side of the equation:

$$\overline{y \cup \overline{x \cup y \cup \overline{\overline{x} \cup y}}} = \overline{\overline{x} \cup y}.$$

Commuting terms on the left-hand side yields the desired result. $\square$

Lemma 46. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{\overline{x} \cup y} \cup x \cup y} \cup y} \cup \overline{\overline{x} \cup y}} = y$.

Proof. Applying Lemma 43 to $\overline{\overline{x} \cup y} \cup x \cup y$ and y yields:

$$\overline{\overline{\overline{\overline{\overline{x} \cup y} \cup x \cup y} \cup y} \cup \overline{\overline{\overline{x} \cup y} \cup x \cup y \cup y}} = y.$$

Use Lemma 45 applied to x and y to simplify the left-hand side of the equation:

$$\overline{\overline{\overline{x} \cup y} \cup \overline{\overline{\overline{x} \cup y} \cup x \cup y \cup y}} = y.$$

Commuting terms on the left-hand side yields the desired result. $\square$

Lemma 47. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{\overline{\overline{x} \cup y} \cup x \cup y} \cup y} \cup \overline{\overline{x} \cup y} \cup z} \cup \overline{y \cup z}} = z$.

Proof. Applying Lemma 43 to w and z yields:

$$\overline{\overline{\overline{w} \cup z} \cup \overline{w \cup z}} = z.$$

Let $w = \overline{\overline{\overline{\overline{x} \cup y} \cup x \cup y} \cup y} \cup \overline{\overline{x} \cup y}$. Then by Lemma 46, $\overline{w} = y$, so we have:

$$\overline{\overline{y \cup z} \cup \overline{w \cup z}} = z.$$

which expands to:

$$\overline{\overline{y \cup z} \cup \overline{\overline{\overline{\overline{\overline{x} \cup y} \cup x \cup y} \cup y} \cup \overline{\overline{x} \cup y} \cup z}} = z.$$

Commuting terms on the left-hand side yields the desired result. $\square$

Lemma 48. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{\overline{\overline{x} \cup y} \cup x \cup y} \cup y} \cup \overline{\overline{x} \cup y} \cup \overline{y \cup z} \cup z} \cup z} = \overline{y \cup z}$.

Proof. Applying Lemma 43 to $\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup z$ and $\overline{y\cup z}$ yields:

$$\overline{\overline{\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup z}\cup\overline{y\cup z}}\cup\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup z\cup\overline{y\cup z}}}=\overline{y\cup z}.$$

Use Lemma 47 applied to x, y, and z to simplify the left-hand side of the equation:

$$\overline{z\cup\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup\overline{y\cup z}\cup z}}=\overline{y\cup z}.$$

Commuting terms on the left-hand side yields the desired result. □

Lemma 49.

$$\mathrm{R}\vdash\overline{\overline{\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup\overline{y\cup z}\cup z}\cup z\cup u}\cup\overline{\overline{y\cup z}\cup u}}=u.$$

Proof. Applying Lemma 43 to w and u yields:

$$\overline{\overline{\overline{w}\cup u}\cup\overline{w\cup u}}=z.$$

Let $w=\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup\overline{y\cup z}\cup z}\cup z$. Then by Lemma 48, $\overline{w}=\overline{y\cup z}$, so we have:

$$\overline{\overline{\overline{y\cup z}\cup u}\cup\overline{w\cup u}}=u,$$

which expands to:

$$\overline{\overline{\overline{y\cup z}\cup u}\cup\overline{\overline{\overline{\overline{\overline{x}\cup y}\cup x\cup y\cup y}\cup\overline{\overline{x}\cup y}\cup\overline{y\cup z}\cup z}\cup z\cup u}}=u.$$

Commuting terms on the left-hand side yields the desired result. □

Lemma 50. $\mathrm{R}\vdash\overline{\overline{\overline{\overline{3x}\cup x}\cup\overline{3x}}\cup\overline{\overline{\overline{3x}\cup x}\cup 5x}}=\overline{\overline{3x}\cup x}.$

Proof. Applying Lemma 44 to $3x$ and $\overline{\overline{3x}\cup x}\cup 2x$ yields:

$$\overline{\overline{\overline{3x\cup\overline{\overline{3x}\cup x}\cup 2x}\cup\overline{3x}\cup\overline{\overline{3x}\cup x}\cup 2x}\cup\overline{\overline{3x}\cup x}\cup 2x}=\overline{3x\cup\overline{\overline{3x}\cup x}\cup 2x},$$

which simplifies to:

$$\overline{\overline{\overline{\overline{\overline{3x}\cup x}\cup 5x}\cup\overline{3x}\cup\overline{\overline{3x}\cup x}\cup 2x}\cup\overline{\overline{3x}\cup x}\cup 2x}=\overline{\overline{\overline{3x}\cup x}\cup 5x}. \qquad (21)$$

Applying Lemma 49 to $3x$, x, $2x$, and $\overline{\overline{3x}\cup x}$ yields:

$$\overline{\overline{\overline{\overline{\overline{\overline{3x}\cup x}\cup 3x\cup x\cup x}\cup\overline{\overline{3x}\cup x}\cup\overline{x\cup 2x}\cup 2x}\cup 2x\cup\overline{\overline{3x}\cup x}}\cup\overline{\overline{x\cup 2x}\cup\overline{\overline{3x}\cup x}}}=\overline{\overline{3x}\cup x},$$

which simplifies to:

$$\overline{\overline{\overline{\overline{\overline{\overline{3x}\cup x}\cup 5x}\cup\overline{\overline{3x}\cup x}\cup\overline{3x}\cup 2x}\cup 2x\cup\overline{\overline{3x}\cup x}}\cup\overline{\overline{3x}\cup\overline{\overline{3x}\cup x}}}=\overline{\overline{3x}\cup x}.$$

Now use equation (21) to further simplify, yielding:

$$\overline{\overline{\overline{\overline{3x}\cup x}\cup 5x}\cup\overline{\overline{3x}\cup\overline{\overline{3x}\cup x}}}=\overline{\overline{3x}\cup x}.$$

Commuting terms on the left-hand side yields the desired result. □

Lemma 51. $\mathrm{R} \vdash \overline{\overline{\overline{3x} \cup x} \cup 5x} = \overline{3x}$.

Proof. Applying Lemma 43 to $\overline{\overline{3x} \cup x} \cup \overline{3x}$ and $\overline{\overline{\overline{3x} \cup x} \cup 5x}$ yields:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x}} \cup \overline{\overline{\overline{3x} \cup x} \cup 5x}} \cup \overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup \overline{\overline{\overline{3x} \cup x} \cup 5x}}} = \overline{\overline{\overline{3x} \cup x} \cup 5x}.$$

Use Lemma 50 applied to x to simplify the left-hand side of the equation:

$$\overline{\overline{\overline{3x} \cup x} \cup \overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup \overline{\overline{\overline{3x} \cup x} \cup 5x}}} = \overline{\overline{\overline{3x} \cup x} \cup 5x}. \tag{22}$$

Applying Lemma 47 to $3x$, x, and $\overline{3x}$ yields:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup 3x \cup x \cup x} \cup \overline{\overline{3x} \cup x} \cup \overline{3x}} \cup \overline{x \cup \overline{3x}}} = \overline{3x}. \tag{23}$$

The left-hand sides of equations (22) and (23) are AC identical. Therefore, $\overline{\overline{\overline{3x} \cup x} \cup 5x} = \overline{3x}$. ☐

Lemma 52. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup 2x} \cup \overline{3x}} = \overline{\overline{3x} \cup x} \cup 2x$.

Proof. Applying Lemma 43 to $3x$ and $\overline{\overline{3x} \cup x} \cup 2x$ yields:

$$\overline{\overline{\overline{3x} \cup \overline{\overline{3x} \cup x} \cup 2x} \cup \overline{3x \cup \overline{\overline{3x} \cup x} \cup 2x}} = \overline{\overline{3x} \cup x} \cup 2x,$$

which simplifies to:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup 2x} \cup \overline{\overline{3x} \cup x} \cup 5x} = \overline{\overline{3x} \cup x} \cup 2x.$$

Use Lemma 51 to simplify the left-hand side of the equation:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup 2x} \cup \overline{3x}} = \overline{\overline{3x} \cup x} \cup 2x.$$

☐

Lemma 53. $\mathrm{R} \vdash \overline{\overline{\overline{3x} \cup x} \cup \overline{3x}} = x$.

Proof. Applying Lemma 43 to $\overline{\overline{3x} \cup x} \cup 4x$ and x yields:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup 4x} \cup x} \cup \overline{\overline{\overline{3x} \cup x} \cup 4x \cup x}} = x,$$

which simplifies to:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup 4x} \cup x} \cup \overline{\overline{\overline{3x} \cup x} \cup 5x}} = x.$$

Use Lemma 51 applied to x to simplify the left-hand side of the equation:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup 4x} \cup x} \cup \overline{3x}} = x. \tag{24}$$

Applying Lemma 45 to $3x$ and x yields:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup 3x \cup x} \cup x} = \overline{\overline{3x} \cup x},$$

which simplifies to:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup 4x} \cup x} = \overline{\overline{3x} \cup x}. \tag{25}$$

Use equation (25) to simplify the left-hand side of equation (24), yielding:

$$\overline{\overline{\overline{3x} \cup x} \cup \overline{3x}} = x.$$

☐

Lemma 54. $\mathrm{R} \vdash \overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup y} \cup \overline{x \cup y}} = y.$

Proof. Applying Lemma 43 to $\overline{\overline{3x} \cup x} \cup \overline{3x}$ and y yields:

$$\overline{\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x}} \cup y} \cup \overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup y}} = y.$$

Use Lemma 53 applied to x to simplify the left-hand side of the equation:

$$\overline{\overline{x \cup y} \cup \overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup y}} = y.$$

Commuting terms on the left-hand side yields the desired result. □

Theorem 55. $\mathrm{R} \vdash \mathrm{W}_1$

Proof. Applying Theorem 54 to x and $2x$ yields:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup 2x} \cup \overline{x \cup 2x}} = 2x,$$

which simplifies to:

$$\overline{\overline{\overline{\overline{3x} \cup x} \cup \overline{3x} \cup 2x} \cup \overline{3x}} = 2x.$$

Use this equation to simplify the left-hand side of Theorem 52, yielding:

$$2x = \overline{\overline{3x} \cup x} \cup 2x.$$ □

Theorem 56. $\mathrm{R} \vdash \mathrm{B}$.

Proof. By Theorems 29, 42, and 55. □

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