ON A GENERALIZATION OF A THEOREM OF IBUKIYAMA TO EVALUATE THREE IMPRIMITIVE CHARACTER SUMS

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Abstract
In a previous paper, we expressed three families of character sums by certain generalized Bernoulli functions which in turn were expressed by generalized Bernoulli numbers via a complicated and indirect process. In this paper, we generalize a theorem of Ibukiyama to directly express these generalized Bernoulli functions by generalized Bernoulli numbers. As a result, we can express the three families of character sums by generalized Bernoulli numbers in a more elegant fashion than was done before.

1. Introduction

Let χ be a Dirichlet character modulo m, and h be any positive integer prime to m. We put $\zeta = exp(2\pi i/m)$. Let $\tau(n,\chi)$ denote the Gaussian sum $\tau(n,\chi) = \sum_{j=0}^{m-1} \chi(j)\zeta^{jn}$. Let a, b be nonnegative integers. In [16], we obtained formulas expressing the closely related character sums

$$M_{a,b}(h,\chi) = \sum_{j=1}^{m-1} \frac{\chi(j)}{(\zeta^{hj}-1)^a(\zeta^j-1)^b},$$

$$S_{a,b}(h,\chi;e_1,\ldots,e_{a+b}) = \sum_{j_1,\cdots,j_{m+n}=1}^{m-1} \tau\left(h\sum_{k=1}^{a} j_k + \sum_{l=1}^{b} j_{a+l}, \chi\right) j_1^{e_1}\cdots j_{a+b}^{e_{a+b}}, \quad (1)$$

$$c_{a,b}(h,\chi) = \sum_{j=1}^{m-1} \cot^a\left(\frac{h\pi j}{m}\right)\cot^b\left(\frac{\pi j}{m}\right)\chi(j)$$

by generalized Bernoulli functions $\overline{A}_{k,\chi}(x)$ defined by (2), which were then expressed by generalized Bernoulli numbers through an indirect and complicated process. In this paper, by generalizing a theorem of Ibukiyama (Theorem 3), we directly express

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generalized Bernoulli functions $\overline{A}_{k,\chi}(x)$ by generalized Bernoulli numbers, thereby enabling a faster, more elegant evaluation of these character sums. Before stating our results, we need more notation.

For integers r, s with s prime to m, we define the Gaussian sum

$$\tau\left(r/s, \chi\right) = \sum_{j \,(\mathrm{mod}\, m)} \chi(j)\zeta^{jrs^{-1}},$$

where s^{-1} is regarded as an element of $\mathbb{Z}/m\mathbb{Z}$ such that $ss^{-1} \equiv 1 \,(\mathrm{mod}\ m)$, and j runs over a complete residue system modulo m. We will write $\tau(\chi)$ for $\tau(1, \chi)$. We also extend the definition of χ by multiplicativity by defining $\chi(r/s) = \chi(rs^{-1})$.

For $x \in \mathbb{Q}$ with denominator prime to m, we define two types of generalized Bernoulli functions $A_{k,\chi}(x), B_{k,\chi}(x)$ by the generating functions

$$\begin{aligned} \sum_{j=0}^{m-1} \frac{\tau(j+x,\chi)te^{(j+x)t}}{e^{mt}-1} &= \sum_{k=0}^{\infty} A_{k,\chi}(x)\frac{t^k}{k!}, \\ \sum_{j=0}^{m-1} \frac{\chi(j+x)te^{(j+x)t}}{e^{mt}-1} &= \sum_{k=0}^{\infty} B_{k,\chi}(x)\frac{t^k}{k!}. \end{aligned} \tag{2}$$

Note that if χ is primitive, then $A_{k,\chi}(x) = \tau(\chi)B_{k,\overline{\chi}}(x)$, and if χ is trivial (i.e., $m = 1$), then $A_{k,\chi}(x), B_{k,\chi}(x)$ both reduce to the ordinary Bernoulli polynomials $B_k(x)$ defined by

$$\frac{te^{tx}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}.$$

We define the generalized Bernoulli numbers $A_{k,\chi}$, $B_{k,\chi}$ and ordinary Bernoulli numbers B_k by $A_{k,\chi} = A_{k,\chi}(0)$, $B_{k,\chi} = B_{k,\chi}(0)$, and $B_k = B_k(0)$. We note that $A_{k,\chi}$, $B_{k,\chi}$ vanish whenever $\chi(-1) \neq (-1)^k$ unless $k = 1$ and χ is principal or trivial, respectively, in which case $A_{1,\chi} = \phi(m)B_1$ if χ is principal (see Lemma 5) and $B_{1,\chi} = B_1$ if χ is trivial, where ϕ denotes the Euler phi function.

Remark. The generalized Bernoulli functions $B_{k,\chi}(x)$ were introduced by Snyder [19] in connection with p-adic Dedekind sums and the generalized Bernoulli functions $A_{k,\chi}(x)$ were introduced by the present author in [16] to evaluate the character sums given by (1).

Corresponding to the generalized Bernoulli functions $A_{k,\chi}(x), B_{k,\chi}(x)$, we have the periodic generalized Bernoulli functions $\overline{A}_{k,\chi}(x), \overline{B}_{k,\chi}(x)$ given by $\overline{A}_{k,\chi}(x) = A_{k,\chi}(x - [x])$, $\overline{B}_{k,\chi}(x) = B_{k,\chi}(x - [x])$, where $[x]$ denotes the greatest integer not exceeding x.

For any natural number l and any integer u with $u \mid l$, denote by l_u the u-primary part of l, that is, the maximum integer which divides l and is prime to u. For any natural number n, we denote by $Y(n)$ the set of primitive Dirichlet characters modulo n. We denote by ϕ Euler's phi function. The following is Ibukiyama's Theorem 2 in [9].

Theorem 1 ([9, Theorem 2])**.** *Let χ be a nontrivial, primitive Dirichlet character modulo m. Let l be a natural number prime to m and c be a natural number prime to l with $1 \le c \le l-1$. If χ is nontrivial and primitive, we get*

$$\sum_{a=0}^{m-1} \chi(la+c)a = \frac{m}{\phi(l)} \sum_{u|l} \sum_{\delta \in Y(u)} \left(\delta(c^{-1}) B_{1,\delta\chi} \prod_{\substack{q|l_u \\ q \text{ prime}}} (1-\chi(q)\delta(q)) \right).$$

In [13], we generalized this theorem to express $\overline{B}_{k,\chi}(x)$ by generalized Bernoulli numbers $B_{k,\delta\chi}$.

Theorem 2 ([13, Theorem 3.1])**.** *Let χ be a Dirichlet character modulo m. Let l be a natural number prime to m and c be any integer prime to l. We get*

$$\overline{B}_{k,\chi}\left(\frac{c}{l}\right) = \frac{\overline{\chi}(l)}{\phi(l)l^{k-1}} \sum_{u|l} \sum_{\delta \in Y(u)} \left(\delta(c^{-1}) B_{k,\delta\chi} \prod_{\substack{q|l_u \\ q \text{ prime}}} (1-q^{k-1}\chi(q)\delta(q)) \right).$$

Remark. To see how Theorem 1 follows from Theorem 2, observe that for $1 \le c \le l-1$ and χ nontrivial and primitive, we have $\sum_{a=0}^{m-1} \chi(la+c)a = \chi(l)\overline{B}_{1,\chi}(c/l)$.

In this paper, we generalize Ibukiyama's theorem to express $\overline{A}_{k,\chi}(x)$ by generalized Bernoulli numbers $B_{k,\delta\overline{\psi}}$, where ψ is the primitive Dirichlet character which induces χ.

Theorem 3. *Let χ be a Dirichlet character modulo m. Let ψ be the primitive Dirichlet character modulo f which induces χ. Let l be a natural number prime to m and c be any integer prime to l. Setting*

$$J_{k,\chi} = \left(\frac{m}{f}\right)^k \prod_{\substack{p|m \\ p\nmid f \\ p \text{ prime}}} \left(1 - \frac{\psi(p)}{p^k}\right),$$

we have

$$\overline{A}_{k,\chi}\left(\frac{c}{l}\right) = \frac{\tau(\psi)\psi(l)}{\phi(l)l^{k-1}} \sum_{u|l} \sum_{\delta \in Y(u)} \left(\delta(c^{-1}m/f) J_{k,\bar{\delta}\chi} B_{k,\delta\overline{\psi}} \prod_{\substack{q|l_u \\ q \text{ prime}}} (1-q^{k-1}\overline{\psi}(q)\delta(q)) \right).$$

Remark. For primitive character χ, we have $\overline{A}_{k,\chi}(c/l) = \tau(\chi)\overline{B}_{k,\overline{\chi}}(c/l)$ and it is clear how Theorem 3 reduces to Theorem 2 in this case.

Expressing $\overline{A}_{k,\chi}(c/l)$ by generalized Bernoulli numbers directly by Theorem 3 is a significant improvement over the indirect method given in [16]. To illustrate the significance of this, we describe how $\overline{A}_{k,\chi}(c/l)$ was expressed by generalized Bernoulli numbers $B_{k,\delta\overline{\psi}}$ in [16]. We first expressed $\overline{A}_{k,\chi}(c/l)$ as a linear combination of generalized Bernoulli functions $\overline{B}_{k,\overline{\psi}}(x)$ as follows (see [16, Theorem 3.2]):

$$\overline{A}_{k,\chi}\left(\frac{c}{l}\right) = R^k q^{k-1}\mu(q)\tau(\psi) \sum_{e \pmod q} \mu((le+c_0,q))\phi((le+c_0,q))\overline{B}_{k,\overline{\psi}}\left(\frac{le+c_0}{lq}\right), \tag{3}$$

where μ is the Möbius function and

$$q = \prod_{\substack{p|m \\ p\nmid f \\ p \text{ prime}}} p, \quad R = \frac{m}{fq},$$

and then applied Theorem 2 to each of the generalized Bernoulli functions on the right-hand side of (3). Therefore, we have reduced the problem in [16] of evaluating a complicated sum of generalized Bernoulli functions $\overline{B}_{k,\overline{\psi}}(x)$ with denominator lq to that of evaluating a single generalized Bernoulli function $\overline{A}_{k,\chi}(x)$ with denominator l. This is a substantial improvement because the expressions given by Theorems 2 and 3 get increasingly more complicated as the denominator gets larger. Consequently, Theorem 3 yields a faster, more elegant evaluation of the character sums given by (1), examples of which are given in Section 4.

The layout of this paper is as follows. In Section 2, we review some properties of the generalized Bernoulli functions $A_{k,\chi}(x)$, $B_{k,\chi}(x)$ such as their finite sum representations and multiplication formulas. In Section 3, we prove the main theorem of this paper (Theorem 3) which directly expresses $\overline{A}_{k,\chi}(c/l)$ by generalized Bernoulli numbers. In Section 4, we give examples using Theorem 3 to express the character sums $M_{a,b}(h,\chi)$, $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$, and $c_{a,b}(h,\chi)$ by generalized Bernoulli numbers.

2. Properties of Generalized Bernoulli Functions $A_{k,\chi}(x)$, $B_{k,\chi}(x)$

We keep the notation used previously. Recall that χ is a Dirichlet character modulo m and ψ is the Dirichlet character modulo f which induces χ. In this section, we review some basic properties of the generalized Bernoulli functions $A_{k,\chi}(x)$, $B_{k,\chi}(x)$ such as their finite sum representations and multiplication formulas. With the exception of Corollary 6, we omit the proofs as they already appear in [16, 17].

From the definitions of $A_{k,\chi}(x)$, $B_{k,\chi}(x)$ given by (2), we get the following finite sum representations

$$A_{k,\chi}(x) = m^{k-1} \sum_{j=0}^{m-1} B_k\left(\frac{j+x}{m}\right) \tau(j+x,\chi),$$
$$B_{k,\chi}(x) = m^{k-1} \sum_{j=0}^{m-1} B_k\left(\frac{j+x}{m}\right) \chi(j+x),$$

or equivalently, for any natural number s prime to m and any integer r, we have

$$A_{k,\chi}\left(\frac{r}{s}\right) = \chi(s) m^{k-1} \sum_{j=0}^{m-1} B_k\left(\frac{sj+r}{sm}\right) \tau(sj+r,\chi),$$
$$B_{k,\chi}\left(\frac{r}{s}\right) = \overline{\chi}(s) m^{k-1} \sum_{j=0}^{m-1} B_k\left(\frac{sj+r}{hm}\right) \chi(sj+r).$$

Similarly, we have the following useful formulations for $\overline{A}_{k,\chi}(r/s)$, $\overline{B}_{k,\chi}(r/s)$:

$$\overline{A}_{k,\chi}\left(\frac{r}{s}\right) = \chi(s) m^{k-1} \sum_{j \pmod{m}} \overline{B}_k\left(\frac{sj+r}{sm}\right) \tau(sj+r,\chi),$$
$$\overline{B}_{k,\chi}\left(\frac{r}{s}\right) = \overline{\chi}(s) m^{k-1} \sum_{j \pmod{m}} \overline{B}_k\left(\frac{sj+r}{sm}\right) \chi(sj+r).$$

We state the multiplication formula for periodic Bernoulli functions which follows from Raabe's multiplication formula for ordinary Bernoulli polynomials:

$$\overline{B}_k(nx) = n^{k-1} \sum_{j \pmod{n}} \overline{B}_k\left(x + \frac{j}{n}\right) \quad (n \in \mathbb{N}). \tag{4}$$

As a natural generalization of Equation (4), we have the multiplication formula for periodic generalized Bernoulli functions.

Lemma 4 ([16, Lemma 2.1]). *Let χ, m be as above. Let $n \in \mathbb{N}$ with $(n,m) = 1$ and $x \in \mathbb{Q}$ with denominator relatively prime to m. We have*

$$\overline{A}_{k,\chi}(nx) = n^{k-1} \overline{\chi}(n) \sum_{j \pmod{n}} \overline{A}_{k,\chi}\left(x + \frac{j}{n}\right),$$
$$\overline{B}_{k,\chi}(nx) = n^{k-1} \chi(n) \sum_{j \pmod{n}} \overline{B}_{k,\chi}\left(x + \frac{j}{n}\right).$$

We note that when χ is trivial, Lemma 4 reduces to Equation (4).

We now describe the relationship between the two types of generalized Bernoulli numbers $A_{k,\chi}$ and $B_{k,\overline{\psi}}$.

Lemma 5 ([17, Lemma 2.1]). *Let χ, ψ, m, f be as above. We have*

$$A_{k,\chi} = B_{k,\overline{\psi}}\,\tau(\psi)\left(\frac{m}{f}\right)^k \prod_{\substack{p\mid m\\ p\nmid f\\ p\text{ prime}}} \left(1 - \frac{\psi(p)}{p^k}\right).$$

Corollary 6. *Let χ, ψ, m, f be as above. Let δ be a primitive Dirichlet character modulo u with $(u, m) = 1$. We have*

$$A_{k,\delta\chi} = B_{k,\overline{\delta\psi}}\,\delta(f)\chi(u)\tau(\delta)\tau(\psi)\left(\frac{m}{f}\right)^k \prod_{\substack{p\mid m\\ p\nmid f\\ p\text{ prime}}} \left(1 - \frac{\delta(p)\psi(p)}{p^k}\right).$$

Proof. Since $\delta\psi$ is the primitive Dirichlet character inducing $\delta\chi$ and $\tau(\delta\psi) = \delta(f)\chi(u)\tau(\delta)\tau(\psi)$, the corollary follows immediately from Lemma 5. □

3. Proof of Theorem 3

Fix a Dirichlet character χ modulo m. For any natural number l prime to m and any integer c, we have

$$\overline{A}_{k,\chi}\left(\frac{c}{l}\right) = \chi(l)m^{k-1} \sum_{j \,(\mathrm{mod}\, m)} \overline{B}_k\left(\frac{lj+c}{lm}\right)\tau(lj+c,\chi).$$

The aim of this section is to obtain a formula expressing $\overline{A}_{k,\chi}(c/l)$ by generalized Bernoulli numbers using only elementary methods from algebra and number theory.

For any natural number l and any integer u with $u \mid l$, denote by l_u the u-primary part of l, that is, the maximum integer which divides l and is prime to u. For any Dirichlet character δ, we denote by f_δ the conductor of δ. For any natural number n, we denote by $X(n)$ the set of primitive Dirichlet characters δ such that n is divisible by f_δ, and by $Y(n)$ the set of primitive Dirichlet characters with conductor n. Let ϕ denote Euler's phi function.

To prove Theorem 3, we prepare several lemmas which are generalizations of Lemmas 1-3 in [9].

Lemma 7. *For a natural number l prime to m and any $\delta \in X(l)$, we get*

$$\delta(m)\overline{\chi}(l/f_\delta)l^{k-1} \sum_{c \,(\mathrm{mod}\, l)} \overline{A}_{k,\chi}\left(\frac{c}{l}\right)\tau(c,\delta) = A_{k,\delta\chi}.$$

Proof. Since $\tau(c,\delta)=\tau(lj+c,\delta)$ and $\tau(n,\chi)\tau(n,\delta)=\overline{\chi}(f_\delta)\overline{\delta}(m)\tau(n,\delta\chi)$ $(n\in\mathbb{Z})$, we have

$$\begin{aligned}
&\delta(m)\overline{\chi}(l/f_\delta)l^{k-1}\sum_{c\,(\mathrm{mod}\,l)}\overline{A}_{k,\chi}\left(\frac{c}{l}\right)\tau(c,\delta)\\
&\quad=\delta(m)\chi(f_\delta)(lm)^{k-1}\sum_{c\,(\mathrm{mod}\,l)}\sum_{j\,(\mathrm{mod}\,m)}\overline{B}_k\left(\frac{lj+c}{lm}\right)\tau(lj+c,\chi)\tau(lj+c,\delta)\\
&\quad=(lm)^{k-1}\sum_{c\,(\mathrm{mod}\,l)}\sum_{j\,(\mathrm{mod}\,m)}\overline{B}_k\left(\frac{lj+c}{lm}\right)\tau(lj+c,\delta\chi)\\
&\quad=(lm)^{k-1}\sum_{n\,(\mathrm{mod}\,lm)}\overline{B}_k\left(\frac{n}{lm}\right)\tau(n,\delta\chi).
\end{aligned}$$

Since the Gaussian sum $\tau(n,\delta\chi)$ is periodic modulo $f_\delta m$, we get

$$\begin{aligned}
&\delta(m)\overline{\chi}(l/f_\delta)l^{k-1}\sum_{c\,(\mathrm{mod}\,l)}\overline{A}_{k,\chi}\left(\frac{c}{l}\right)\tau(c,\delta)\\
&\quad=(lm)^{k-1}\sum_{a\,(\mathrm{mod}\,f_\delta m)}\sum_{b\,(\mathrm{mod}\,l/f_\delta)}\overline{B}_k\left(\frac{f_\delta mb+a}{lm}\right)\tau(f_\delta mb+a,\delta\chi)\\
&\quad=(lm)^{k-1}\sum_{a\,(\mathrm{mod}\,f_\delta m)}\left(\sum_{b\,(\mathrm{mod}\,l/f_\delta)}\overline{B}_k\left(\frac{b}{l/f_\delta}+\frac{a}{lm}\right)\right)\tau(a,\delta\chi).
\end{aligned}$$

By the multiplication formula (4), we obtain

$$\begin{aligned}
&\delta(m)\overline{\chi}(l/f_\delta)l^{k-1}\sum_{c\,(\mathrm{mod}\,l)}\overline{A}_{k,\chi}\left(\frac{c}{l}\right)\tau(c,\delta)\\
&\quad=(lm)^{k-1}\left(\frac{f_\delta}{l}\right)^{k-1}\sum_{a\,(\mathrm{mod}\,f_\delta m)}\overline{B}_k\left(\frac{a}{f_\delta m}\right)\tau(a,\delta\chi)\\
&\quad=A_{k,\delta\chi}.
\end{aligned}$$

□

Following the method of Ibukiyama in [9], we obtain an inversion formula. We fix a natural number l which is prime to m and put $L=\prod_{q\mid l}q$, where q runs over primes. For any $n\mid l$, denote by l_n the n-primary part of l.

Lemma 8. *For any fixed number $d\in(\mathbb{Z}/l\mathbb{Z})^*$, we get*

$$\sum_{n\mid L}\phi(l_n)\overline{\chi}(l_nn)(l_nn)^{k-1}\overline{A}_{k,\chi}\left(\frac{e}{l_n}\right)=\sum_{\delta\in X(l)}\frac{\overline{\chi}(f_\delta)\delta(m^{-1}d)}{\tau(\delta)}A_{k,\delta\chi},$$

where e is an integer determined by $ne\equiv d \bmod l_n$.

Proof. We prove this lemma by taking the sum over $\delta \in X(l)$ of both sides of the formula in Lemma 7. For any integer c with $0 \le c \le l-1$, there exists a unique $n \mid L$ such that $n \mid c$ and $(c, L/n) = 1$. For such c, we have $\sum_{\delta \in X(l)} \delta(c) = \sum_{\delta \in X(l_n)} \delta(c)$, since $\delta(c) = 0$ whenever $(f_\delta, n) > 1$. Denote by $A(n)$ the following set of integers:

$$A(n) = \{c \in \mathbb{Z} : 0 \le c \le l-1,\, n \mid c,\, (c, L/n) = 1\}.$$

Then, applying Lemma 7, noting that $\tau(c, \delta) = \overline{\delta}(c)\tau(\delta)$, we get

$$\begin{aligned} \sum_{\delta \in X(l)} \frac{\overline{\chi}(f_\delta)\delta(m^{-1}d)}{\tau(\delta)} A_{k,\delta\chi} &= \overline{\chi}(l) l^{k-1} \sum_{\delta \in X(l)} \sum_{c \,(\mathrm{mod}\, l)} \overline{\delta}(d^{-1}c) \overline{A}_{k,\chi}\left(\frac{c}{l}\right) \\ &= \overline{\chi}(l) l^{k-1} \sum_{n \mid L} \sum_{c \in A(n)} \sum_{\delta \in X(l_n)} \delta(d^{-1}c) \overline{A}_{k,\chi}\left(\frac{c}{l}\right). \end{aligned} \tag{5}$$

Observe that

$$\sum_{\delta \in X(l_n)} \delta(d^{-1}c) = \begin{cases} \phi(l_n), & \text{if } d \equiv c \bmod l_n, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $C(n)$ the following set of integers:

$$C(n) = \{c \in \mathbb{Z} : 0 \le c \le l-1,\, n \mid c,\, (c, L/n) = 1, \text{ and } c \equiv d \bmod l_n\}.$$

Then, from (5), we get

$$\sum_{\delta \in X(l)} \frac{\overline{\chi}(f_\delta)\delta(m^{-1}d)}{\tau(\delta)} A_{k,\delta\chi} = \overline{\chi}(l) l^{k-1} \sum_{n \mid L} \phi(l_n) \sum_{c \in C(n)} \overline{A}_{k,\chi}\left(\frac{c}{l}\right). \tag{6}$$

Let e be an integer such that $ne \equiv d \bmod l_n$. Then $(e, l_n) = 1$ since $(d, l) = 1$. Hence, it follows that

$$C(n) = \{n(l_n a + e) : a \in \mathbb{Z},\, 0 \le a \le l/(l_n n) - 1\}.$$

By the multiplication formula Lemma 4, we obtain

$$\begin{aligned} \overline{\chi}(l) l^{k-1} \sum_{c \in C(n)} \overline{A}_{k,\chi}\left(\frac{c}{l}\right) &= \overline{\chi}(l) l^{k-1} \sum_{a \,(\mathrm{mod}\, l/l_n n)} \overline{A}_{k,\chi}\left(\frac{n(l_n a + e)}{l}\right) \\ &= \overline{\chi}(l_n n)(l_n n)^{k-1} \overline{A}_{k,\chi}\left(\frac{e}{l_n}\right). \end{aligned}$$

Thus, from (6), we get

$$\sum_{\delta \in X(l)} \frac{\overline{\chi}(f_\delta)\delta(m^{-1}d)}{\tau(\delta)} A_{k,\delta\chi} = \sum_{n \mid L} \phi(l_n) \overline{\chi}(l_n n)(l_n n)^{k-1} \overline{A}_{k,\chi}\left(\frac{e}{l_n}\right).$$

□

Lemma 9. *We fix a natural number l prime to m and an integer c prime to l. We define L and l_n for $n \mid l$ in the same way as in Lemma 8. Then, we get*

$$\phi(l)\overline{\chi}(l)l^{k-1}\overline{A}_{k,\chi}\left(\frac{c}{l}\right) = \sum_{n|L}\mu(n)\overline{\chi}(n)n^{k-1}\sum_{\delta\in X(l_n)}\frac{\overline{\chi}(f_\delta)\delta(m^{-1}n^{-1}c)}{\tau(\delta)}A_{k,\delta\chi},$$

where μ is the Möbius function.

Proof. For $u \mid v \mid L$ and any $d \in (\mathbb{Z}/l\mathbb{Z})^*$, we put

$$g(u,v,d) = \phi\left(l/l_u\right)\overline{\chi}\left(lv/(l_u u)\right)\left(lv/(l_u u)\right)^{k-1}\overline{A}_{k,\chi}\left(\frac{w}{l/l_u}\right),$$

where w is determined by $(v/u)w \equiv d \bmod (l/l_u)$. Also, we put

$$f(v,d) = \sum_{\delta\in X(l/l_v)}\frac{\overline{\chi}(f_\delta)\delta(m^{-1}d)}{\tau(\delta)}A_{k,\delta\chi}.$$

Next, we apply Lemma 8 for $(v, l/l_v)$ instead of (L, l). Noting that $(l/l_v)_m = l_m/l_v$ for any $m \mid v$, we get

$$\sum_{n|v}\phi(l_n/l_v)\overline{\chi}((l_n/l_v)n)((l_n/l_v)n)^{k-1}\overline{A}_{k,\chi}\left(\frac{e}{l_n/l_v}\right) = \sum_{\delta\in X(l/l_v)}\frac{\overline{\chi}(f_\delta)\delta(m^{-1}d)}{\tau(\delta)}A_{k,\delta\chi},$$

where e is determined by $ne \equiv d \bmod (l_n/l_v)$. For each $n \mid v$, we define u by $nu = v$. Then $l_n/l_v = l/l_u$, and we get

$$\sum_{u|v} g(u,v,d) = f(v,d).$$

For any $u \mid v \mid L$, we put $G(u) = g(u,L,c)$ and $F(u) = (L/u)^{k-1}\,\overline{\chi}(u^{-1}L)f(v,L^{-1}uc)$. Observe that $g(u,L,c) = (L/v)^{k-1}\,\overline{\chi}(v^{-1}L)g(u,v,L^{-1}vc)$, where $L^{-1}vc$ is regarded as an element of $(\mathbb{Z}/l_v\mathbb{Z})^*$. Hence, we get

$$\begin{aligned}\sum_{u|v} G(u) &= (L/v)^{k-1}\,\overline{\chi}(v^{-1}L)\sum_{u|v} g(u,v,L^{-1}vc)\\ &= (L/v)^{k-1}\,\overline{\chi}(v^{-1}L)f(v,L^{-1}vc)\\ &= F(v).\end{aligned}$$

Applying the Möbius inversion formula for $v = L$, we get

$$G(L) = \sum_{m|L}\mu(m)F\left(\frac{L}{m}\right).$$

Thus, we obtain

$$\phi(l)\overline{\chi}(l)l^{k-1}\overline{A}_{k,\chi}\left(\frac{c}{l}\right)=\sum_{n|L}\mu(n)\overline{\chi}(n)n^{k-1}\sum_{\delta\in X(l_n)}\frac{\overline{\chi}(f_\delta)\delta(m^{-1}n^{-1}c)}{\tau(\delta)}A_{k,\delta\chi}.$$

□

We are now in position to prove Theorem 3.

Proof of Theorem 3. We define L and l_n for $n \mid l$ in the same way as in Lemma 8. From Lemma 9, we get

$$\begin{aligned}\phi(l)\overline{\chi}(l)l^{k-1}\overline{A}_{k,\chi}\left(\frac{c}{l}\right)&=\sum_{n|L}\sum_{\delta\in X(l_n)}\frac{\mu(n)n^{k-1}\overline{\chi}(nf_\delta)\delta(m^{-1}n^{-1}c)}{\tau(\delta)}A_{k,\delta\chi}\\&=\sum_{n|l}\sum_{u|l_n}\sum_{\delta\in Y(u)}\frac{\mu(n)n^{k-1}\overline{\chi}(nu)\delta(m^{-1}n^{-1}c)}{\tau(\delta)}A_{k,\delta\chi}.\end{aligned}$$

Observe that $u \mid l_n$ for $n \mid l$ is equivalent to $n \mid l_u$ for $u \mid l$. Thus, we have

$$\begin{aligned}\phi(l)\overline{\chi}(l)l^{k-1}\overline{A}_{k,\chi}\left(\frac{c}{l}\right)&=\sum_{u|l}\sum_{n|l_u}\sum_{\delta\in Y(u)}\frac{\mu(n)n^{k-1}\overline{\chi}(nu)\delta(m^{-1}n^{-1}c)}{\tau(\delta)}A_{k,\delta\chi}\\&=\sum_{u|l}\sum_{\delta\in Y(u)}\frac{\delta(m^{-1}c)\overline{\chi}(u)}{\tau(\delta)}A_{k,\delta\chi}\prod_{\substack{q|l_u\\ q\ prime}}(1-q^{k-1}\overline{\chi}(q)\overline{\delta}(q)).\end{aligned}$$

Applying Corollary 6 to express $A_{k,\delta\chi}$ by generalized Bernoulli numbers $B_{k,\overline{\delta\chi}}$, we obtain Theorem 3. □

4. The Evaluation of $M_{a,b}(h,\chi)$, $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$, and $c_{a,b}(h,\chi)$

We remind the reader of the notation. Let χ be a Dirichlet character modulo m, and h be any positive integer prime to m. We put $\zeta = exp(2\pi i/m)$. Let $\tau(n,\chi)$ denote the Gaussian sum $\tau(n,\chi)=\sum_{j=0}^{m-1}\chi(j)\zeta^{jn}$. Let a, b be nonnegative integers. We assume that $m > 1$ and $a + b \geq 1$ to exclude the trivial cases. Without a loss of generality, we further assume that $a \geq 1$. In this section, we give examples expressing the closely related character sums

$$M_{a,b}(h,\chi)=\sum_{j=1}^{m-1}\frac{\chi(j)}{(\zeta^{hj}-1)^a(\zeta^j-1)^b}\ ,$$

$$S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})=\sum_{j_1,\cdots,j_{m+n}=1}^{m-1}\tau\left(h\sum_{k=1}^{a}j_k+\sum_{l=1}^{b}j_{a+l},\chi\right)j_1^{e_1}\cdots j_{a+b}^{e_{a+b}},$$

$$c_{a,b}(h,\chi)=\sum_{j=1}^{m-1}\cot^a\left(\frac{h\pi j}{m}\right)\cot^b\left(\frac{\pi j}{m}\right)\chi(j)$$

by generalized Bernoulli numbers. We first review the results in [16] expressing these sums by periodic generalized Bernoulli functions $\overline{A}_{k,\chi}(x)$.

Remark. The sums $M_{a,b}(h,\chi)$ and $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$ are natural generalizations of sums introduced and studied by Berndt [6] and Arakawa–Ibukiyama–Kaneko [3] in the context of the theory of modular forms. For examples, we refer the interested reader to [2, 3, 8, 9, 10, 11, 12, 13, 14, 15, 18, 20, 21]. The sum $c_{a,b}(h,\chi)$ is a variation of trigonometric sums first investigated by Ramanujan in connection to certain theta function identities. For examples, we refer the interested reader to [4, 5, 7].

For positive integers n and k, we denote by $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ the Stirling numbers of the first and second kind, respectively. That is, Stirling's cycle numbers $\left[{n \atop k}\right]$ denote the number of permutations of n letters (elements of the symmetric group of degree n) that consist of k disjoint cycles, and Stirling's subset numbers $\left\{{n \atop k}\right\}$ denote the number of ways to divide a set of n elements into k nonempty sets.

Theorem 10 ([16, Theorem 4.1])**.** *Let χ, m, h, a, b be as above. We have*

$$M_{a,b}(h,\chi)=\frac{(-1)^{a+b-1}\overline{\chi}(h)}{(a+b-1)!}$$
$$\times\sum_{j=1}^{a+b}\sum_{c=0}^{b(h-1)}C_h(b,c)\left(\sum_{r=0}^{a+b-j}\frac{(-1)^r}{r+j}\left[{a+b \atop r+j}\right]\binom{r+j}{j}\left(\frac{c}{h}\right)^r\right)\overline{A}_{j,\chi}\left(\frac{c}{h}\right),$$

where

$$C_h(b,c)=\begin{cases}1 & \text{if } b=0,\\ \displaystyle\sum_{k=0}^{[c/h]}(-1)^k\binom{b}{k}\binom{b-1+c-hk}{b-1} & \text{if } b\geq 1.\end{cases}$$

Theorem 11 ([16, Theorem 4.2])**.** *Let χ, m, h, a, b be as above. We have*

$$S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})=(-1)^{e_1+\cdots+e_{a+b}}$$
$$\times\sum_{\substack{1\leq k_j\leq e_j\\ 1\leq l_j\leq e_j-k_j+1\\ 1\leq j\leq a+b}}\left(\prod_{i=1}^{a+b}(-m)^{l_i}\binom{e_i}{l_i}\left\{{e_i-l_i+1 \atop k_i}\right\}(k_i-1)!\right)$$
$$\times M_{k_1+\cdots+k_a,k_{a+1}+\cdots+k_{a+b}}(h,\chi).$$

Remark. In the case where $e_j = 1$ for $j = 1, \ldots, a+b$, we get

$$S_{a,b}(h, \chi; 1, \ldots, 1) = m^{a+b} M_{a,b}(h, \chi).$$

Theorem 12 ([16, Theorem 4.3])**.** *Let χ, m, h, a, b be as above. We have*

$$c_{a,b}(h, \chi) = i^{a+b} \sum_{j=0}^{a} \sum_{k=0}^{b} 2^{j+k} \binom{a}{j} \binom{b}{k} M_{j,k}(h, \chi).$$

As a corollary, we obtained a formula for cotangent power sums considered by Apostol [1], Berndt [6] and others.

Corollary 13 ([16, Corollary 4.4])**.** *Let χ, m, a be as above. We have*

$$\begin{aligned} c_{a,0}(1, \chi) &= \sum_{j=1}^{m-1} \cot^a \left(\frac{\pi j}{m} \right) \chi(j) \\ &= -i^a \sum_{k=1}^{a} \frac{1}{k} \left(\sum_{j=k}^{a} \frac{(-2)^j \binom{a}{j} \left[{j \atop k} \right]}{(j-1)!} \right) A_{k,\chi} + \begin{cases} i^a \phi(m) & \text{if } \chi \text{ is principal,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since the sums $M_{a,b}(h, \chi)$, $S_{a,b}(h, \chi; e_1, \ldots, e_{a+b})$, and $c_{a,b}(h, \chi)$ can be expressed by periodic generalized Bernoulli functions $\overline{A}_{k,\chi}(x)$, and $\overline{A}_{k,\chi}(x)$ are expressible by generalized Bernoulli numbers by virtue of Theorem 3, so are the sums $M_{a,b}(h, \chi)$, $S_{a,b}(h, \chi; e_1, \ldots, e_{a+b})$, and $c_{a,b}(h, \chi)$. We now give examples expressing $M_{a,b}(h, \chi)$, $S_{a,b}(h, \chi; e_1, \ldots, e_{a+b})$, and $c_{a,b}(h, \chi)$ by generalized Bernoulli numbers.

We note the following useful fact that follows from the multiplication formula Lemma 4:

$$\sum_{c \pmod h} \overline{A}_{k,\chi}(c/h) = \frac{\chi(h)}{h^{k-1}} A_{k,\chi}. \tag{7}$$

Proposition 14. *Let $m > 1$ be a natural number prime to 4 and χ be a Dirichlet character modulo m. Let ψ be the primitive Dirichlet character modulo f which induces χ. We denote by δ the unique primitive Dirichlet character modulo 4. Let $J_{k,\chi}$ be defined as in Theorem 3. Let a be a nonnegative integer. We have*

$$(i)\quad M_{1,1}(4,\chi)=\sum_{j=1}^{m-1}\frac{\chi(j)}{(\zeta^{4j}-1)(\zeta^{j}-1)}$$
$$=-\tau(\psi)\left(\frac{1+\overline{\psi}(4)}{2}J_{1,\chi}B_{1,\overline{\psi}}+\frac{\delta(m/f)}{4}J_{1,\delta\chi}B_{1,\delta\overline{\psi}}+\frac{1}{8}J_{2,\chi}B_{2,\overline{\psi}}\right),$$

$$(ii)\quad M_{2,1}(4,\chi)=\sum_{j=1}^{m-1}\frac{\chi(j)}{(\zeta^{4j}-1)^2(\zeta^{j}-1)}$$
$$=\tau(\psi)\left(\frac{13-\overline{\psi}(2)+20\overline{\psi}(4)}{32}J_{1,\chi}B_{1,\overline{\psi}}+\frac{\delta(m/f)}{4}J_{1,\delta\chi}B_{1,\delta\overline{\psi}}+\frac{1+2\overline{\psi}(4)}{8}J_{2,\chi}B_{2,\overline{\psi}}\right.$$
$$\left.+\frac{\delta(m/f)}{32}J_{2,\delta\chi}B_{2,\delta\overline{\psi}}+\frac{1}{96}J_{3,\chi}B_{3,\overline{\psi}}\right),$$

$$(iii)\quad S_{1,1}(4,\chi;1,1)=\sum_{j,k=1}^{m-1}\tau(4j+k,\chi)jk$$
$$=-m^2\tau(\psi)\left(\frac{1+\overline{\psi}(4)}{2}J_{1,\chi}B_{1,\overline{\psi}}+\frac{\delta(m/f)}{4}J_{1,\delta\chi}B_{1,\delta\overline{\psi}}+\frac{1}{8}J_{2,\chi}B_{2,\overline{\psi}}\right),$$

$$(iv)\quad S_{1,1}(4,\chi;2,1)=\sum_{j,k=1}^{m-1}\tau(4j+k,\chi)j^2k$$
$$=-m^2\tau(\psi)\left(\frac{8m-3-\overline{\psi}(2)+4(2m+1)\overline{\psi}(4)}{16}J_{1,\chi}B_{1,\overline{\psi}}+\frac{m\delta(m/f)}{4}J_{1,\delta\chi}B_{1,\delta\overline{\psi}}\right.$$
$$\left.+\frac{m+4\overline{\psi}(4)}{8}J_{2,\chi}B_{2,\overline{\psi}}+\frac{\delta(m/f)}{16}J_{2,\delta\chi}B_{2,\delta\overline{\psi}}+\frac{1}{48}J_{3,\chi}B_{3,\overline{\psi}}\right),$$

$$(v)\quad c_{1,1}(4,\chi)=\sum_{j=1}^{m-1}\cot\left(\frac{4\pi j}{m}\right)\cot\left(\frac{\pi j}{m}\right)\chi(j)$$
$$=\tau(\psi)\left(\delta(m/f)J_{1,\delta\chi}B_{1,\delta\overline{\psi}}+\frac{1}{2}J_{2,\chi}B_{2,\overline{\psi}}\right)-\begin{cases}\phi(m) & \text{if } \chi \text{ is principal}\\ 0 & \text{otherwise}\end{cases},$$

$$(vi)\quad c_{2,1}(4,\chi)=\sum_{j=1}^{m-1}\cot^2\left(\frac{4\pi j}{m}\right)\cot\left(\frac{\pi j}{m}\right)\chi(j)$$
$$=-i\tau(\psi)\left(\frac{5-\overline{\psi}(2)+4\overline{\psi}(4)}{4}J_{1,\chi}B_{1,\overline{\psi}}+\frac{\delta(m/f)}{4}J_{2,\delta\chi}B_{2,\delta\overline{\psi}}+\frac{1}{12}J_{3,\chi}B_{3,\overline{\psi}}\right)$$
$$-\begin{cases}i\phi(m) & \text{if } \chi \text{ is principal}\\ 0 & \text{otherwise}\end{cases},$$

$$(vii)\quad c_{a,0}(1,\chi)=\sum_{j=1}^{m-1}\cot^a\left(\frac{\pi j}{m}\right)\chi(j)$$
$$=-i^a\tau(\psi)\sum_{k=1}^{a}\frac{1}{k}\left(\sum_{j=k}^{a}\frac{(-2)^j\binom{a}{j}\left[{j \atop k}\right]}{(j-1)!}\right)J_{k,\chi}B_{k,\overline{\psi}}+\begin{cases}i^a\phi(m) & \text{if } \chi \text{ is principal}\\ 0 & \text{otherwise}\end{cases}.$$

Proof. By Theorem 10 together with the help of (7), we get

$$\begin{aligned}
M_{1,1}(4,\chi) &= -A_{1,\chi} - \frac{1}{8}A_{2,\chi} + \frac{\overline{\psi}(4)}{4}\sum_{c=1}^{3} c\,\overline{A}_{1,\chi}(c/4),\\
M_{2,1}(4,\chi) &= A_{1,\chi} + \frac{3}{16}A_{2,\chi} + \frac{1}{96}A_{3,\chi}\\
&\quad + \overline{\psi}(4)\sum_{c=1}^{3}\left(\frac{-3c}{8} + \frac{c^2}{32}\right)\overline{A}_{1,\chi}(c/4) - \frac{\overline{\psi}(4)}{8}\sum_{c=1}^{3} c\,\overline{A}_{2,\chi}(c/4).
\end{aligned}$$

Since $A_{k,\chi} = \tau(\psi)J_{k,\chi}B_{k,\overline{\psi}}$ by Lemma 5, this becomes

$$\begin{aligned}
M_{1,1}(4,\chi) &= -\tau(\psi)\left(J_{1,\chi}B_{1,\overline{\psi}} + \frac{1}{8}J_{2,\chi}B_{2,\overline{\psi}}\right) + \frac{\overline{\psi}(4)}{4}\sum_{c=1}^{3} c\,\overline{A}_{1,\chi}(c/4),\\
M_{2,1}(4,\chi) &= \tau(\psi)\left(J_{1,\chi}B_{1,\overline{\psi}} + \frac{3}{16}J_{2,\chi}B_{2,\overline{\psi}} + \frac{1}{96}J_{3,\chi}B_{3,\overline{\psi}}\right) \qquad (8)\\
&\quad + \overline{\psi}(4)\sum_{c=1}^{3}\left(\frac{-3c}{8} + \frac{c^2}{32}\right)\overline{A}_{1,\chi}(c/4) - \frac{\overline{\psi}(4)}{8}\sum_{c=1}^{3} c\,\overline{A}_{2,\chi}(c/4).
\end{aligned}$$

By Theorem 3, noting that $\delta = \overline{\delta}$ so that $J_{k,\overline{\delta}\chi} = J_{k,\delta\chi}$, we have

$$\begin{aligned}
\overline{A}_{k,\chi}(1/4) &= \frac{\tau(\psi)\psi(4)}{2\cdot 4^{k-1}}\Big((1-2^{k-1}\overline{\psi}(2))J_{k,\chi}B_{k,\overline{\psi}} + \delta(m/f)J_{k,\delta\chi}B_{k,\delta\overline{\psi}}\Big),\\
\overline{A}_{k,\chi}(2/4) &= \overline{A}_{k,\chi}(1/2) = \frac{\tau(\psi)\psi(2)}{2^{k-1}}(1-2^{k-1}\overline{\psi}(2))J_{k,\chi}B_{k,\overline{\psi}},\\
\overline{A}_{k,\chi}(3/4) &= \frac{\tau(\psi)\psi(4)}{2\cdot 4^{k-1}}\Big((1-2^{k-1}\overline{\psi}(2))J_{k,\chi}B_{k,\overline{\psi}} - \delta(m/f)J_{k,\delta\chi}B_{k,\delta\overline{\psi}}\Big) \quad (k\in\mathbb{N}).
\end{aligned}$$

Thus, for $j,k\in\mathbb{N}$, we have

$$\begin{aligned}
&\overline{\psi}(4)\sum_{c=1}^{3} c^j\,\overline{A}_{k,\chi}(c/4)\\
&\quad = \frac{\tau(\psi)}{2\cdot 4^{k-1}}\Big((1+2^{j+k}\overline{\psi}(2)+3^j)(1-2^{k-1}\overline{\psi}(2))J_{k,\chi}B_{k,\overline{\psi}} + (1-3^j)\delta(m/f)J_{k,\delta\chi}B_{k,\delta\overline{\psi}}\Big).
\end{aligned}$$

Using this to evaluate the summations on the right-hand side of (8), we obtain the assertions (i) and (ii).

By Theorem 11, we have

$$\begin{aligned}
S_{1,1}(4,\chi;1,1) &= m^2 M_{1,1}(4,\chi),\\
S_{1,1}(4,\chi;2,1) &= m^2\Big((m-2)M_{1,1}(4,\chi) - 2M_{2,1}(4,\chi)\Big).
\end{aligned}$$

Thus the assertions (iii), (iv) follow from (i), (ii).

By Theorem 12, we have

$$
\begin{aligned}
c_{1,1}(4,\chi) &= (-1)\Big(M_{0,0}(4,\chi) + 2M_{0,1}(4,\chi) + 2M_{1,0}(4,\chi) + 4M_{1,1}(4,\chi)\Big),\\
c_{2,1}(4,\chi) &= -i\Big(M_{0,0}(4,\chi) + 2M_{0,1}(4,\chi) + 4M_{1,0}(4,\chi) + 8M_{1,1}(4,\chi)\\
&\qquad + 4M_{2,0}(4,\chi) + 8M_{2,1}(4,\chi)\Big).
\end{aligned} \tag{9}
$$

Observe that $M_{0,0}(4,\chi) = \phi(m)$ if χ is principal, and 0 otherwise, and by Theorem 10 and Lemma 5, we have

$$
\begin{aligned}
M_{1,0}(4,\chi) &= \overline{\psi}(4)A_{1,\chi} = \overline{\psi}(4)\tau(\psi)J_{1,\chi}B_{1,\overline{\psi}},\\
M_{0,1}(4,\chi) &= \psi(4)M_{1,0}(4,\chi) = A_{1,\chi} = \tau(\psi)J_{1,\chi}B_{1,\overline{\psi}},\\
M_{2,0}(4,\chi) &= -\overline{\psi}(4)\left(A_{1,\chi} + \frac{1}{2}A_{2,\chi}\right) = -\overline{\psi}(4)\tau(\psi)\left(J_{1,\chi}B_{1,\overline{\psi}} + \frac{1}{2}J_{2,\chi}B_{2,\overline{\psi}}\right).
\end{aligned}
$$

Plugging these values back into (9), the assertions (v), (vi) follow from (i), (ii).

Since $A_{k,\chi} = \tau(\psi)J_{k,\chi}B_{k,\overline{\psi}}$ by Lemma 5, the assertion (vii) follows from Corollary 13. □

Remark. In [16, Proposition 5.1], we considered the special case of χ being an imprimitive Dirichlet character modulo $m = f \cdot 3^n$ for some positive integer n with $(f, 12) = 1$, and ψ being the primitive Dirichlet character modulo f which induces χ. Since

$$
\begin{aligned}
J_{k,\chi} &= 3^{k(n-1)}(3^k - \psi(3)),\\
J_{k,\delta\chi} &= 3^{k(n-1)}(3^k + \psi(3)) \quad (k \in \mathbb{N}),
\end{aligned}
$$

we see that Proposition 14 implies [16, Proposition 5.1].

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