

**RECURSIVELY SELF-CONJUGATE PARTITIONS****William J. Keith***Department of Mathematics, Drexel University, Philadelphia, PA 19104, USA*
wj26@math.drexel.edu*Received: 1/24/10, Revised: 7/30/10, Accepted: 8/26/10, Published: 3/9/11***Abstract**

A class of partitions that exhibit substantial symmetry, called recursively self-conjugate partitions, are defined and analyzed. They are found to have connections to non-squashing partitions and other combinatorial objects.

1. Introduction

In [3], Kathy Ji and Herb Wilf discuss “recursively palindromic” (RP) words, which are ordered sequences (a_1, \dots, a_k) in some alphabet such that the word is palindromic, i.e., $a_i = a_{k-i}$, and the left and right half-sequences $a_1, \dots, a_{\lfloor \frac{k}{2} \rfloor}$ and $a_{\lceil \frac{k}{2} \rceil + 1}, \dots, a_k$ are themselves recursively palindromic.

The study of RP words is intended to provide information about the parity of the set in which they appear. The set must be closed under the series of operations that reverse words, reverse the left and right half of words individually, reverse the left and right halves of each half in both halves, and so forth. The parity of the set of RP words in a set is then the parity of the set, for we may pair any nonpalindrome with its reverse, any palindrome with nonpalindromic halves with the word that has these halves reversed, and so forth. The fixed points of this series of involutions are the RP words.

In this article we will apply these ideas to partitions, using the familiar operation of conjugation as our involution and the Durfee square as the dividing point. A natural adaptation of the concepts leads us to the idea of a *recursively self-conjugate partition*. While the tool proves insufficient to attack the parity of $p(n)$, in enumerating these partitions and their properties we find the following main results:

Theorem 1. *The number of recursively self-conjugate partitions of largest part k is equal to the number of non-squashing partitions of k .*

While some numbers have no recursively self-conjugate partitions, this behavior is finite:

Theorem 2. *There exist recursively self-conjugate partitions of all $n > 545$.*

We produce some generating functions for these partitions, and while none of the results are really satisfactory (so this project is still open), some of the methods may be of interest and possibly bear more fruit for another investigator. Several other open questions, such as asymptotics, are listed in the conclusion.

1.1. Definitions

We start with the following definition.

Definition 3. A sequence a_1, \dots, a_k is *RP, recursively palindromic*, if it is palindromic and its left half $(a_1, \dots, a_{\lfloor \frac{k}{2} \rfloor})$ is RP, considering the empty sequence and singletons as RP.

Example 4. AABAACAABAA is RP; it is palindromic, its left half (the singleton C is in neither half) is the palindrome AABAA, the left half of this is AA, and the left half of this is the singleton A. MADAMIMADAM is a palindrome, and its left half MADAM is palindromic, but MA and AM are distinguishable reverses, so this is not RP. MADAMIMADAM is paired with the word AMDMAIAMDMA.

Ji and Wilf [3] found the following.

Theorem 5. (Ji, Wilf) *The number of RP words of length n in an alphabet of K letters is $K^{\alpha(n)}$, where $\alpha(n)$ is the sum of the binary digits of n .*

Here we will apply these ideas to partitions.

Definition 6. Let n be a nonnegative integer. A vector λ *partitions* n , which we denote $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \vdash n$, if $\lambda_i \in \mathbb{N}$, with $\sum \lambda_i = n$.

We do not distinguish between partitions with different orders of parts, so we typically write the parts in nonincreasing order, $\lambda_i \geq \lambda_{i+1}$.

We draw a partition with its *Ferrers diagram*, which is an array of squares occupying a quadrant at points (j, i) if $\lambda_j \geq i$, i.e., each row is of length equal to a part of the partition:

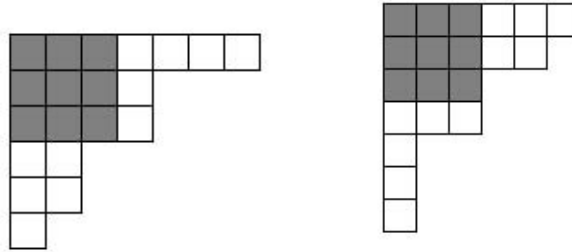


Figure 1: $\lambda = (7, 4, 4, 2, 2, 1)$; $\lambda' = (6, 4, 4, 2, 2, 1)$

The shaded squares indicate that the first three parts of λ are at least 3, while the fourth part is not at least 4. The i such that $\lambda_i \geq i$ but $\lambda_{i+1} < i + 1$ is the size of the *Durfee square* of λ . The name comes from the fact that visually, k is the side length of the largest square inscribable in the Ferrers diagram of the partition.

Partitions are unaffected by the palindrome involution, but the *conjugate* of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is the partition

$$\lambda' = (\mu_1, \dots, \mu_m) \text{ with } \mu_i = |\{\lambda_j : \lambda_j \leq i\}| .$$

The two partitions in Figure 1 are conjugate to each other. Partitions fixed under conjugation are *self-conjugate*: one such is given in Figure 2.

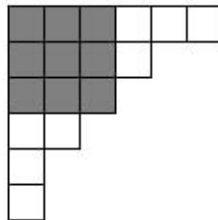


Figure 2: $\lambda = (6, 4, 3, 2, 1, 1)$

One can subdivide a partition into its Durfee square, say of size k , and two smaller partitions, the one to the right of the Durfee square, consisting of fewer than k parts, and the one below the Durfee square, consisting of parts of size less than k . Call the upper partition $\tau(\lambda) = (\tau_1, \dots)$ and the lower partition $\beta(\lambda) = (\beta_1, \dots)$. Since



conjugation fixes the Durfee square, we can take these as the analogs of the left and right half of the partition, and attempt to apply the RP construction.

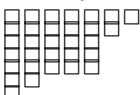
On most partitions, the recursion fails, since conjugation cannot be applied to each half alone with any guarantee that the result is a partition. In Figure 1, for example, with $\lambda = (7, 4, 4, 2, 2, 1)$, the upper partition is $(4, 1, 1)$, which has conjugate $(3, 1, 1, 1)$ of 4 parts thus not fitting above the square. We could ask, however, for what subset of partitions this line of investigation is valid.

If λ is self-conjugate, $\tau = \beta'$. Suppose that λ is self-conjugate, $\tau(\lambda)$ is self-conjugate, $\tau(\tau(\lambda))$ is self-conjugate, and so on indefinitely. (Eventually, $\tau(\dots\tau(\lambda))$ is the empty set, which we take to partition 0.) Call such a partition *recursively self-conjugate*, or formally,

Definition 7. A partition is *Durfee-conjugable*, *DC*, if the portions right of and below its Durfee square are themselves partitions with both number of parts and size of parts at most the size of the Durfee square. A partition is *recursively-conjugable*, *RC*, if it is Durfee-conjugable and the portions above and below its Durfee square are also RC.

Definition 8. A partition is *Durfee-self-conjugate* (*DSC*) if it is DC and conjugating the portions above and below the Durfee square fixes the partition; it is *recursively self-conjugate* (*RSC*) if it is RC and the bottom and right partitions are empty or RSC.

Example 9. The partition $(5, 3, 3, 1, 1)$ is Durfee conjugable: . It is also self-conjugate, but the portions above and below its Durfee square can be safely conjugated: 

Thus, it is not recursively self-conjugate. On the other hand, the partition $\lambda = (7, 6, 5, 5, 5, 2, 1)$ is: . Here we have $\tau(\lambda) = (2, 1)$, $\tau^2(\lambda) = 1$, $\tau^3(\lambda) = \emptyset$.

A table of the smallest examples of recursively self-conjugate partitions follows.



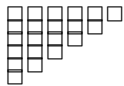

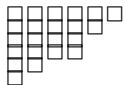


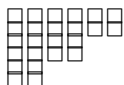





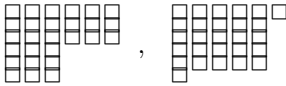

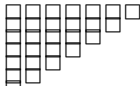


1		11		21	
2	None	12		22	
3		13	None	23	None
4		14	None	24	
5	None	15		25	
6		16		26	None
7	None	17		27	
8	None	18		28	
9		19	None	29	None
10		20	None	30	None

Table 1: Rsc partitions for first 30 n

We are thus interested in enumerative questions concerning recursively self-conjugate partitions, which for this paper shall be abbreviated rscps. We assign the number of recursively self-conjugate partitions of n to be $rscp(n)$. Among the questions of interest to us are:

- 1) Can we produce a generating function? Perhaps a recurrence?
- 2) Can we relate rscps to any previously-studied combinatorial objects?
- 3) Not all whole numbers have rscps, e.g. 2. Where are these 0s happening? In the audience at the original presentation of this work, James Sellers of Penn State asked: do they continue indefinitely? Does the function $rscp(n)$ take on all whole number values, and if so, does it take on any of them infinitely often?
- 4) What is the asymptotic behavior of the counting function?

2. Connection to Non-Squashing Partitions

One interesting feature of rscps connects them to *non-squashing partitions*:


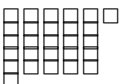
Definition 10. A *non-squashing partition* of n is a partition $(\lambda_r, \dots, \lambda_1)$ of n such that for all i , $\lambda_i \geq \sum_{j < i} \lambda_j$.

For instance, $8 + 5 + 1 + 1$ is a non-squashing partition of 15, but $6 + 4 + 3 + 2$ is not, since $4 < 3 + 2$.

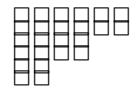
Mike Hirschhorn and James Sellers ([2]) showed that this is equal to the number of partitions of k into powers of 2, the binary partitions. This was showed in [2] by a generating function argument, but a simple bijection can also be constructed: the parts of a non-squashing partition λ can be built by adding corresponding parts of the vectors $(2^{r-2}, 2^{r-3}, \dots, 4, 2, 1, 1)$, $(2^{r-3}, 2^{r-4}, \dots, 2, 1, 1, 0)$, \dots , $(1, 0, \dots, 0, 0, 0, 0)$. Each of these is a partition of 2^i with i ranging over $1 \leq i \leq r - 1$.

The size of any Durfee square in a recursively self-conjugate partition must be at least as much as the sum of the sizes of all remaining Durfee squares in its sub-partitions. The recursivity of the definition means any rscp is entirely made up of its Durfee squares, the sum of which are its largest part. Thus we have

Theorem 11. *The number of recursively self-conjugate partitions of largest part k is equal to the number of non-squashing partitions of k .*

For example, the partitions  and  of 27 correspond to

the non-squashing partitions $3 + 3$ and $5 + 1$ of 6 respectively, while the partition

 of 24 corresponds to the partition $4 + 2$.

Interestingly, we can also construct a simple bijection between rscps and binary partitions, which is not the same map. This will arise more naturally later, when we discuss generating functions, and we first require a lemma on the shape of rscps.

3. Zeros and Growth

The set of numbers without recursively self-conjugate partitions is finite, and the last such number is $n = 545$. To prove this, we first need the following.

Lemma 12. *The i th smallest part of a recursively self-conjugate partition is never smaller than i , that is, an rscp of side length k dominates the partition $(k, \dots, 3, 2, 1)$.*

Proof. Since the bottom left element of the Ferrers diagram of an rscp is itself part of the last Durfee square in the recursive breakdown of the partition, the diagonal northwest of that square is filled to the size of that square. This square is either the only Durfee square in the partition, or it sits below one of the same size or larger.

If the Durfee square above is of the same size, then it is immediately followed by another square reflecting the first.

If the Durfee square is larger, then the diagonal continues in a chord across this square and repeats the diagonal of the original square on the opposite side.

We now encounter the next size of Durfee square, repeating until we have reached the largest square, which contains the main diagonal of the diagram. Everything then reflects. \square

Remark This can also be interpreted as pointing out that an rscp is a highly symmetric example of a Dyck path.

Now we can proceed with the verification of the claim. The strategy is to show that, once a long interval of numbers with nonzero $rscp(n)$ is found, it is possible to construct rscps of larger n by appending smaller rscps to either side of a sufficiently large Durfee square.

Theorem 13. *There exist recursively self-conjugate partitions of all $n > 545$.*

Proof. Suppose $rscp(n) > 0$ for $n \in [2y + 2, (y + 3)^2]$ for some y , with $y \geq 10$.

Above and below a Durfee square of size y^2 , i.e., of side y , add recursively self-conjugate partitions of all numbers from $2y + 2$ to $4y + 8$. This can be done since for $y \geq 10$, the triangular number of side y is of size $\frac{y^2+y}{2} > 4y + 12$, so rscps of these numbers fit below a Durfee square of side y .

Adding a pair of partitions of size $2y + 2$ yields a total of $y^2 + 4y + 4 = (y + 2)^2$, and $4y + 8$ yields $y^2 + 8y + 16 = (y + 4)^2$. This ensures that all numbers of the same parity as y from $(y + 2)^2$ to $(y + 4)^2$ have rscps.

Above and below a Durfee square of size $(y + 1)^2$, add rscps of all numbers from $2y + 4$ to $4y + 12$, thus ensuring that all numbers of the same parity as $y + 1$ from $(y + 3)^2$ to $(y + 5)^2$ have rscps.

We have constructed rscps of all numbers from $(y + 3)^2$ to $(y + 4)^2$. Set $y' = y + 1$; we have that all $n \in [2y' + 2, (y' + 3)^2]$ have rscps, and $y' \geq 10$. The process may now be repeated to construct rscps in the next interval between squares, and so forth indefinitely. Therefore all we need to find is the first acceptable interval.

Brute-force calculation by constructing rscps shows that at least one example exists from $n = 546$ ($2y + 2 = 546$, so $(y + 3)^2 = 75625$) to $n = 75625$. Thus the theorem holds. \square

Thus, in response to Prof. Sellers' questions:

There are 45 numbers with no rscps. The entire list of the missing is $n = 2, 5, 7, 8, 13, 14, 19, 20, 23, 26, 29, 30, 32, 35, 39, 41, 46, 50, 52, 53, 62, 63, 65, 74, 77, 92, 95, 104, 107, 109, 110, 116, 119, 128, 158, 159, 170, 173, 182, 185, 221, 248, 251, 317, 545$.

Now that we have shown that there are no zeroes after 545, a similar technique can prove that no value $rscp(n) = c$ is taken infinitely many times: let n be large enough that $(n - j^2)/2$ takes on c different values less than \sqrt{j} but bigger than 545. Each such $(n - j^2)/2$ will yield an rscp of different size Durfee square.

On the other hand, while $rscp(n)$ grows rather slowly and might hit all integer values at least once, a priori there seems to be no reason why it should not eventually be growing fast enough to miss some value. However, this question is entirely open.

No congruences have to date been observed in computationally accessible arithmetic progressions. However, there might be some not in linear progressions, but by size of contained triangle, square, or largest part.

4. Recurrence and Generating Functions

A generating function for $rscp(n)$ seems to be surprisingly difficult to produce cleanly, and the author would be quite interested in seeing one.

It is fairly easy to generate rscps by the recurrence suggested by the proof of Theorem 13:

$$rscp(n, k) = \sum_{\substack{j = \lceil \frac{k}{2} \rceil \\ j \equiv n \pmod{2}}}^{\lfloor \sqrt{n} \rfloor} rscp\left(\frac{n - j^2}{2}, k - j\right)$$

where $rscp(n, k)$ is the number of rscps of n of side length exactly k . This counts all side-partitions rather than simply confirming the existence of one. Unfortunately, this seems to lead to no nice generating function.

We showed earlier that the sizes of the Durfee squares form a non-squashing partition. The largest square appears once, the next largest appears twice, the next largest appears 4 times, and so forth. Thus we might write

$$\sum rscp(n)q^n = \sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_r) \\ \lambda \text{ non-squashing}}} q^{\sum \lambda_r + 2\lambda_{r-1} + \dots + 2^{r-1}\lambda_1}.$$

It is better, in a generating function, to allow indices to be chosen independently of each other, and we can do this to the Durfee-square breakdown by identifying

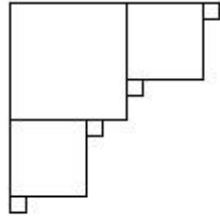


Figure 3: Durfee squares marked in an rscp.

the excess of the sides of each Durfee square over the sum of the sides of smaller squares, as marked on Figure 4.

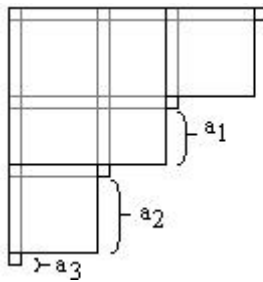


Figure 4: An rscp broken down by the Durfee square excesses.

With a bit of algebraic manipulation, this leads to a generating function for the number of rscps of weight n with k nonzero recursive Durfee squares:

$$\sum rscp_k(n)q^n = \sum_{a_i \geq 0, a_k > 0} q^{(a_1 + a_2 + 2a_3 + 4a_4 + \dots + 2^{k-2}a_k)^2 + \dots + 2(a_2 + a_3 + 2a_4 + \dots + 2^{k-3}a_k)^2 + \dots + 2^{k-3}(a_{k-1} + a_k)^2 + 2^{k-2}a_k^2}$$

We showed that every rscp contains the triangle built on its largest part. Each rscp can also be described as a Durfee square – formed by adding a reversed triangular “cap” to the middle of this triangle – with an rscp above and below the square. These in turn will be made in the same fashion, adding “caps” of equal sizes to the middle of their smaller triangles.

Recursively, then, we can describe an rscp as built on a triangle of side $k - 1$, with an additional boundary of length k made up of the bottom rows of a middle

triangular cap, a triangular cap to either side, four caps in between these, and so forth. This, then, is an explicit map between rscps and binary partitions.

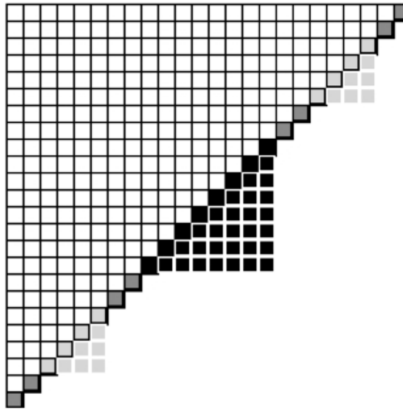


Figure 5: An rscp with triangular caps marked.

Figure 5 is an example of an rscp with side length 24. The outer diagonal of length 24 is partitioned into a middle triangle of size 8 (the black squares), two smaller triangles of size 4 (the light gray squares), and eight triangles of size 1. This corresponds to the partition $24 = 8^1 + 2^4 + 1^8$.

The multiplicities of parts need not be binary numbers, even though they are in this example. It is interesting to note that, if we take this map from rscps to binary partitions, and the map from rscps to non-squashing partitions followed by the natural map from there to binary partitions, not all rscps are fixed points. The example above is, but the partition $(6, 6, 4, 4, 2, 2)$ is not; counting its triangular caps leads to the binary partition 2^3 of 6, while it corresponds to the non-squashing partition $4 + 2$, which maps to $2^2 + 1^2$.

This association allows us to describe another (somewhat better?) representation of a generating function for $rscp(n, k)$. We need the operator CT_z , the “constant term” operator, defined by

$$CT_z \sum_{k=-\infty}^{\infty} f(k)z^k = f(0)$$

where $f(k)$ may be a function of any number of other variables. Then we can write

$$\sum rscp(n, k)q^n t^k = CT_z \sum_{k \geq 0} \frac{q \binom{k}{2} t^k z^{-k}}{\prod_i \Delta(z^{2^i}, q^{2^i})}$$

where $\Delta(z, q) = \sum_{j \geq 0} z^j q^{\binom{j}{2}}$.

Since $\Delta(z, q)$ is the series of nonnegative terms in the triangular version of Jacobi's triple product, it is possible that this approach will yield a cleaner generating function with more study.

5. Further Work

There are several outstanding questions on these objects which may merit investigation. A better generating function is surely to be desired, and is probably a prerequisite for answering the question of the asymptotics of the objects.

Can we construct a conjugation which meets the needs of the original attempt to refine self-conjugacy in investigating the parity of $p(n)$? We would need:

- a) An involution which is conjugation on non-self-conjugate partitions;
- b) which preserves the condition of self-conjugacy when operating on parts above and below the Durfee square;
- c) and can operate on partitions with a known maximum part size and number of parts,
- d) preserving any condition of self-mapping which held at shallower recursion.

Other questions are also under investigation. At the INTEGERS Conference in 2009 where this work was first presented, Sellers also suggests examining the behavior of partitions related to m -non-squashing partitions, which insist that $\lambda_i > m(\sum_{j < i} \lambda_j)$. Rishi Nath of CUNY-York suggested a number of interesting questions and conjectures regarding the properties of rscps relating to odd hooks and their status as t -cores (partitions without hooks of length t). Some of these have been proved and a followup paper in joint authorship is forthcoming.

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