PERMUTATION PATTERNS OF THE ITERATED SYRACUSE FUNCTION

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Abstract

Let Ω be the set of odd positive integers and let $S : \Omega \to \Omega$ be the Syracuse function. It is proved that, for every permutation σ of $(1, 2, 3)$, the set of triples of the form $(m, S(m), S^2(m))$ with permutation pattern σ has positive density, and these densities are computed. However, there exist permutations τ of $(1, 2, 3, 4)$ such that no quadruple $(m, S(m), S^2(m), S^3(m))$ has permutation pattern τ. This implies the nonexistence of certain permutation patterns of n-tuples $(m, S(m), \ldots, S^{n-1}(m))$ for all $n \geq 4$.

1. Permutation Patterns

Let Ω be the set of odd positive integers. The *Syracuse function* is the arithmetic function $S : \Omega \to \Omega$ defined by

$$S(m) = \frac{3m+1}{2^e}$$

where e is the largest integer such that 2^e divides $3m + 1$. Equivalently, $S(m)$ is the *odd part*, that is, the largest odd divisor, of the even integer $3m + 1$. Note that $S(m) = 1$ if and only if $m = \sum_{i=0}^{k} 4^i = (4^{k+1} - 1)/3$ for some nonnegative integer k. The notorious *Collatz conjecture* asserts that for every positive integer m there exists an integer r such that $S^r(m) = 1$ (Lagarias [3], Tao [9], Wikipedia [11]). By supercomputer calculation, Barina [2] has verified the conjecture for all $m < 2^{68}$.

An arithmetic function is any function whose domain is a nonempty subset Ω of the set positive integers. Let $S : \Omega \to \Omega$ be an arithmetic function and, for $j \in \mathbf{N}$, let $S^j : \Omega \to \Omega$ be the jth iterate of S. Let $V = (v_i)_{i=1}^{n}$ be a finite sequence of positive integers. We say that an integer m in Ω has *increasing-decreasing pattern* V with respect to S if

$$m < S(m) < S^2(m) < \cdots < S^{v_1}(m)$$

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$$S^{v_1}(m) > S^{v_1+1}(m) > \cdots > S^{v_1+v_2}(m)$$
$$S^{v_1+v_2}(m) < S^{v_1+v_2+1}(m) < \cdots < S^{v_1+v_2+v_3}(m)$$

and, in general, if i is odd, then

$$S^{v_1+\cdots+v_{i-1}}(m) < S^{v_1+\cdots+v_{i-1}+1}(m) < \cdots < S^{v_1+\cdots+v_{i-1}+v_i}(m) \quad (1)$$

and if i is even, then

$$S^{v_1+\cdots+v_{i-1}}(m) > S^{v_1+\cdots+v_{i-1}+1}(m) > \cdots > S^{v_1+\cdots+v_{i-1}+v_i}(m). \quad (2)$$

The arithmetic function S is *wildly increasing-decreasing* if, for every finite sequence V of positive integers, there exists an integer $m \in \Omega$ such that m has increasing-decreasing pattern V with respect to S.

Nathanson [4] proved that the Syracuse function is wildly increasing-decreasing. In this paper we consider more subtle variations in successive iterates of the Syracuse function.

Let Σ_n be the group of permutations of $\{1, 2, 3, \ldots, n\}$. Let $X = (x_1, x_2, \ldots, x_n)$ be an n-tuple of distinct real numbers. We rearrange the coordinates of X to obtain an n-tuple $(y_1, y_2, \ldots, y_n)$ such that

$$y_1 < y_2 < \cdots < y_n.$$

There is a unique permutation $\sigma \in \Sigma_n$ such that

$$(x_1, x_2, \ldots, x_n) = (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}).$$

We call σ the *permutation pattern* of the n-tuple X and denote it by $(\sigma(1), \sigma(2), \ldots, \sigma(n))$. In standard form, this is the permutation $\left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ \sigma(a) & \sigma(2) & \cdots & \sigma(n) \end{smallmatrix}\right)$.

For example, if

$$X = (x_1, x_2, x_3, x_4) = (7, 13, 18, 11)$$

then

$$7 < 11 < 13 < 18.$$

We obtain

$$(y_1, y_2, y_3, y_4) = (7, 11, 13, 18).$$

and

$$(x_1, x_2, x_3, x_4) = (y_1, y_3, y_4, y_2).$$

The permutation pattern of the quadruple $(7, 13, 18, 11)$ is $\sigma = (1, 3, 4, 2)$.

It is an open problem to determine, for every positive integer n, the possible permutation patterns of the initial segments $\left(m, S(m), S^2(m), \ldots, S^{n-1}(m)\right)$ of the iterated Syracuse function for integers $m \in \Omega$ such that $S^i(m) \neq S^j(m)$ for $0 \leq i < j \leq n-1$. For every permutation $\sigma \in \Sigma_n$, let $\Gamma_\sigma(M)$ count the number of odd positive integers $m \leq M$ such that the n-tuple $\left(m, S(m), S^2(m), \ldots, S^{n-1}(m)\right)$ has distinct coordinates and has permutation pattern σ. The *permutation density* of $\sigma \in \Sigma_n$ is

$$d_n(\sigma) = \lim_{M \to \infty} \frac{\Gamma_\sigma(M)}{M/2}$$

(if the limit exists).

In this paper we prove that every permutation $\sigma \in \Sigma_3$ occurs with positive density. We also prove that there are permutations $\tau \in \Sigma_4$ such that τ not only has zero permutation density, but there exists no positive integer m with permutation pattern τ. If no n-tuple $\left(m, S(m), S^2(m), \ldots, S^{n-1}(m)\right)$ has permutation pattern $(a_1, \ldots, a_n) \in \Sigma_n$, then no $(n+1)$-tuple $\left(m, S(m), S^2(m), \ldots, S^{n-1}(m), S^n(m)\right)$ has permutation pattern $(a_1, \ldots, a_n, n+1) \in \Sigma_{n+1}$. It follows that, for all $n \geq 4$, there exist permutations τ in Σ_n such that no n-tuple $\left(m, S(m), S^2(m), \ldots, S^{n-1}(m)\right)$ has permutation pattern τ.

Theorem 1. *The following table gives the density of permutation patterns of triples* $\left(m, S(m), S^2(m)\right)$.

permutation pattern $\sigma \in \Sigma_3$	*permutation density* $d_3(\sigma)$
$(1,2,3)$	$1/4$
$(1,3,2)$	$1/8$
$(2,1,3)$	$1/8$
$(2,3,1)$	$1/8$
$(3,1,2)$	$1/8$
$(3,2,1)$	$1/4$

The proof of this result will follow immediately from Theorems 3 and 4 below.

Note: In the study of the Collatz conjecture, instead of the Syracuse function, investigators often use the Collatz functions

$$C(m) = \begin{cases} 3m+1 & \text{if } m \text{ is odd} \\ \frac{m}{2} & \text{if } m \text{ is even} \end{cases}$$

and

$$C_1(m) = \begin{cases} \frac{3m+1}{2} & \text{if } m \text{ is odd} \\ \frac{m}{2} & \text{if } m \text{ is even.} \end{cases}$$

The equivalent Collatz conjectures state that for every odd integer m there exist positive integers n and n_1 such that $C^n(m) = 1$ and $C_1^{n_1}(m) = 1$. Simons and de Weger [8] and Simons [5, 6, 7] have studied a different kind of increasing-decreasing behavior for the Collatz function. Albert, Gudmundsson, and Ulfarsson [1] considered permutations generated by the iterated function $C(m)$.

2. Permutation Patterns for Pairs

We begin with a simple but important calculation of permutation densities for pairs.

Theorem 2. *Let $m \in \Omega$. If $m \equiv 3 \pmod 4$, then $(m, S(m))$ has permutation pattern $(1,2)$. If $m \equiv 1 \pmod 4$ and $m > 1$, then $(m, S(m))$ has permutation pattern $(2,1)$.*

congruence class of m	*permutation pattern* $\sigma \in \Sigma_2$	*permutation density* $d_3(\sigma)$
$3 \pmod 4$	$(1,2)$	$1/2$
$1 \pmod 4$	$(2,1)$	$1/2$

Proof. If $m = 3 + 4x$ for some nonnegative integer x, then $3m + 1 = 10 + 12x$ and

$$S(m) = \frac{10 + 12x}{2} = 5 + 6x > 3 + 4x = m.$$

Thus, $(m, S(m))$ has permutation pattern $(1,2)$.

If $m = 1 + 4x$ for some positive integer x, then $3m + 1 = 4 + 12x$ and there is an integer $e \geq 2$ such that

$$S(m) = \frac{4 + 12x}{2^e} = \frac{1 + 3x}{2^{e-2}} \leq 1 + 3x < 1 + 4x = m$$

and $(m, S(m))$ has permutation pattern $(2,1)$. This completes the proof. □

Corollary 1. *Let A be the set of odd positive integers m for which the permutation pattern of the triple $(m, S(m), S^2(m))$ is*

$$(1,2,3), \qquad (1,3,2), \quad \text{or} \quad (2,3,1).$$

Let B be the set of odd positive integers m for which the permutation pattern of the triple $(m, S(m), S^2(m))$ is

$$(2,1,3), \qquad (3,1,2), \quad \text{or} \quad (3,2,1).$$

The set A has density $1/2$ and the set B has density $1/2$.

Proof. The pair $(m, S(m))$ has permutation pattern $(1,2)$ if and only if the triple $(m, S(m), S^2(m))$ has permutation pattern $(1,2,3)$, $(1,3,2)$, or $(2,3,1)$. Similarly, the pair $(m, S(m))$ has permutation pattern $(2,1)$ if and only if the triple $(m, S(m), S^2(m))$ has permutation pattern $(2,1,3)$, $(3,1,2)$, or $(3,2,1)$. This completes the proof. □

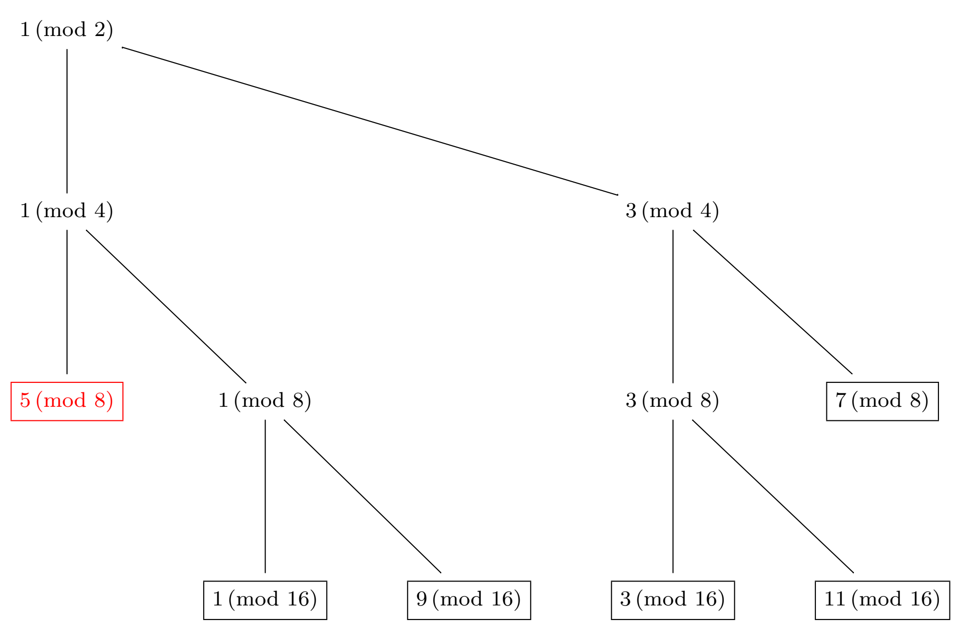
3. Permutation Patterns for Triples for $m \not\equiv 5 \pmod 8$

The calculation of permutation densities for triples $\big(m, S(m), S^2(m)\big)$ is divided into two cases. In the first case we consider odd positive integers $m \not\equiv 5 \pmod 8$ and in the second case we consider odd positive integers $m \equiv 5 \pmod 8$.

Theorem 3. *Every odd positive integer m such that $m \not\equiv 5 \pmod 8$ belongs to exactly one of the five congruence classes in the table below. Each congruence class uniquely determines the permutation pattern of the triple $(m, S(m), S^2(m))$ for all integers $m > 1$ in the congruence class.*

congruence class of m	*permutation pattern* $\sigma \in \Sigma_3$	*permutation density* $d_3(\sigma)$
$7 \pmod 8$	$(1, 2, 3)$	$1/4$
$9 \pmod{16}$	$(2, 1, 3)$	$1/8$
$11 \pmod{16}$	$(1, 3, 2)$	$1/8$
$3 \pmod{16}$	$(2, 3, 1)$	$1/8$
$1 \pmod{16}$	$(3, 2, 1)$	$1/8$

Proof. The following diagram shows the five congruence classes (in black boxes) that partition the integers $m \not\equiv 5 \pmod 8$ and the "missing" congruence class $5 \pmod 8$ (in the red box).



If $m \equiv 7 \pmod 8$, then for some nonnegative integer x we have

$$\begin{aligned} m &= 7 + 8x \\ S(m) &= \frac{22 + 24x}{2} = 11 + 12x \\ S^2(m) &= \frac{34 + 36x}{2} = 17 + 18x \end{aligned}$$

and

$$7 + 8x < 11 + 12x < 17 + 18x.$$

Therefore, $(m, S(m), S^2(m))$ has permutation pattern $(1, 2, 3)$.

If $m \equiv 9 \pmod{16}$, then for some nonnegative integer x we have

$$\begin{aligned} m &= 9 + 16x \\ S(m) &= \frac{28 + 48x}{4} = 7 + 12x \\ S^2(m) &= \frac{22 + 36x}{2} = 11 + 18x \end{aligned}$$

and

$$7 + 12x < 9 + 16x < 11 + 18x.$$

Therefore, $(m, S(m), S^2(m))$ has permutation pattern $(2, 1, 3)$.

If $m \equiv 11 \pmod{16}$, then for some nonnegative integer x we have

$$\begin{aligned} m &= 11 + 16x \\ S(m) &= \frac{34 + 48x}{2} = 17 + 24x \\ S^2(m) &= \frac{52 + 72x}{4} = 13 + 18x \end{aligned}$$

and

$$11 + 16x < 13 + 18x < 17 + 24x.$$

Therefore, $(m, S(m), S^2(m))$ has permutation pattern $(1, 3, 2)$.

If $m \equiv 3 \pmod{16}$, then for some positive integer x and nonnegative integer e we have

$$\begin{aligned} m &= 3 + 16x \\ S(m) &= \frac{10 + 48x}{2} = 5 + 24x \\ S^2(m) &= \frac{16 + 72x}{8} = \frac{2 + 9x}{2^e}. \end{aligned}$$

and

$$\frac{2 + 9x}{2^e} \leq 2 + 9x < 3 + 16x < 5 + 24x.$$

Therefore, $(m, S(m), S^2(m))$ has permutation pattern $(2, 3, 1)$.

If $m > 1$ and $m \equiv 1 \pmod{16}$, then for some positive integer x and nonnegative integer e we have

$$\begin{aligned} m &= 1 + 16x \\ S(m) &= \frac{4 + 48x}{4} = 1 + 12x \\ S^2(m) &= \frac{4 + 36x}{4} = \frac{1 + 9x}{2^e}. \end{aligned}$$

and

$$\frac{1 + 9x}{2^e} \leq 1 + 9x < 1 + 12x < 1 + 16x.$$

Therefore, $(m, S(m), S^2(m))$ has permutation pattern $(3, 2, 1)$. This completes the proof. □

4. Permutation Patterns for Triples for $m \equiv 5 \pmod{8}$

We shall prove that half of the odd positive integers congruent to $5 \pmod{8}$ have the 3-term permutation pattern $(3, 2, 1)$ and half have the 3-term permutation pattern $(3, 1, 2)$.

In the following proofs, e denotes a nonnegative integer.

Lemma 1. *Let $k \geq 1$. If m is a positive integer and*

$$m \equiv \sum_{i=0}^{k} 4^i + 2 \cdot 4^k \pmod{2 \cdot 4^{\mathrm{k+1}}}$$

then $m \equiv 5 \pmod{8}$ and the permutation pattern of $(m, S(m), S^2(m))$ is $(3, 2, 1)$.

Proof. For some nonnegative integer x we have

$$m = \frac{4^{k+1} - 1}{3} + 2 \cdot 4^k + 2 \cdot 4^{k+1}x \geq 13 + 32x$$

and

$$3m + 1 = 4^{k+1} + 3 \cdot 2 \cdot 4^k + 3 \cdot 2 \cdot 4^{k+1}x.$$

It follows that

$$S(m) = \frac{3m + 1}{2 \cdot 4^k} = 5 + 12x < 13 + 32x \leq m.$$

We have

$$3S(m) + 1 = 16 + 36x = 4(4 + 9x)$$

and so

$$S^2(m) = \frac{4 + 9x}{2^e} \leq 4 + 9x < 5 + 12x = S(m).$$

Therefore,

$$S^2(m) < S(m) < m$$

and the permutation pattern of $(m, S(m), S^2(m))$ is $(3, 2, 1)$. This completes the proof. □

Lemma 2. *Let $k \geq 1$. If m is a positive integer and*

$$m \equiv \sum_{i=0}^{k} 4^i \pmod{4^{\mathrm{k+2}}}$$

then $m \equiv 5 \pmod 8$ and either $m = \sum_{i=0}^{k} 4^i$ and $S(m) = S^2(m) = 1$ or the permutation pattern of $(m, S(m), S^2(m))$ is $(3, 2, 1)$.

Proof. For some nonnegative integer x we have

$$m = \frac{4^{k+1} - 1}{3} + 4^{k+2}x \geq 5 + 64x$$

and

$$3m + 1 = 4^{k+1} + 3 \cdot 4^{k+2}x.$$

If $x = 0$, then $3m + 1 = 4^{k+1}$ and $S(m) = S^2(m) = 1$.

If $x \geq 1$, then

$$S(m) = \frac{3m + 1}{4^{k+1}} = 1 + 12x < 5 + 64x \leq m$$

and

$$S^2(m) = \frac{1 + 9x}{2^e} \leq 1 + 9x < 1 + 12x = S(m).$$

Therefore,

$$S^2(m) < S(m) < m$$

and the permutation pattern of $(m, S(m), S^2(m))$ is $(3, 2, 1)$. This completes the proof. □

Lemma 3. *Let $k \geq 1$. If m is a positive integer and*

$$m \equiv \sum_{i=0}^{k+1} 4^i + 2 \cdot 4^k \pmod{2 \cdot 4^{\mathrm{k+1}}}$$

then $m \equiv 5 \pmod 8$ and the permutation pattern of $(m, S(m), S^2(m))$ is $(3, 1, 2)$.

Proof. For some nonnegative integer x we have

$$\begin{aligned} m &= \frac{4^{k+2}-1}{3} + 2 \cdot 4^k + 2 \cdot 4^{k+1} x \\ &\geq 29 + 32x \end{aligned}$$

and

$$3m+1 = 4^{k+2} + 3 \cdot 2 \cdot 4^k + 3 \cdot 2 \cdot 4^{k+1} x.$$

Therefore,

$$S(m) = \frac{3m+1}{2 \cdot 4^k} = 11 + 12x$$

and

$$S^2(m) = 17 + 18x.$$

We have

$$S(m) = 11 + 12x < S^2(m) = 17 + 18x < 29 + 32x \leq m$$

and the permutation pattern of $(m, S(m), S^2(m))$ is $(3, 1, 2)$. This completes the proof. □

Lemma 4. *Let $k \geq 1$. If m is a positive integer and*

$$m \equiv \sum_{i=0}^{k+1} 4^i + 4^{k+1} \pmod{4^{k+2}}$$

then $m \equiv 5 \pmod 8$ and the permutation pattern of $(m, S(m), S^2(m))$ is $(3, 1, 2)$.

Proof. For some nonnegative integer x we have

$$\begin{aligned} m &= \frac{4^{k+2}-1}{3} + 4^{k+1} + 4^{k+2} x \\ &\geq 37 + 64x \end{aligned}$$

and

$$3m+1 = 4^{k+2} + 3 \cdot 4^{k+1} + 3 \cdot 4^{k+2} x$$

It follows that

$$S(m) = \frac{3m+1}{4^{k+1}} = 7 + 12x$$

and

$$S^2(m) = 11 + 18x.$$

Therefore,

$$S(m) = 7 + 12 < S^2(m) = 11 + 18x < 37 + 64x \leq m$$

and the permutation pattern of $(m, S(m), S^2(m))$ is $(3, 1, 2)$. This completes the proof. □

Notation. Let $F(M)$ be a positive function of M. We denote by $o(F(M))$ a function $G(M)$ such that $\lim_{M\to\infty} \frac{G(M)}{F(M)} = 0$. Thus, $o(1)$ denotes a function $G(M)$ such that $\lim_{M\to\infty} G(M) = 0$. Note that $-o(1) = o(1)$ and $o(1) + o(1) = o(1)$. Also, $o(X(M)) = X(M)o(1)$.

The *counting function* of a set A of positive integers is

$$A(M) = \sum_{\substack{a\in A \\ a\leq M}} 1.$$

A subset A of a set X of positive integers has density α with respect to X if the limit

$$\lim_{M\to\infty} \frac{A(M)}{X(M)}$$

exists and equals α. Equivalently, A has density α with respect to X, denoted $d_X(A) = \alpha$, if

$$A(M) = \alpha X(M) + o(X(M)).$$

If $d_X(A) = 0$, then $A(M) = o(X(M))$.

Lemma 5. *Let X be a set of positive integers, let W be a subset of X with density $d_X(W) = \omega > 0$, and let*

$$W = R_0 \cup R_1 \cup \cdots \cup R_t$$

be a partition of W. Let $\alpha_1, \ldots, \alpha_t$ be positive real numbers such that $\sum_{i=1}^t \alpha_i = \omega$. If $d_X(R_0) = 0$ and if, for all ε with

$$0 < \varepsilon < \min(\alpha_1, \ldots, \alpha_t)$$

there is a subset $R_{i,\varepsilon}$ of R_i such that $d_X(R_{i,\varepsilon}) = \alpha_i - \varepsilon$, then $d_X(R_j) = \alpha_j$ for all $j \in \{1, \ldots, t\}$.

Proof. Let $M \geq 1$. Because the sets $R_0, R_1, \ldots, R_t$ partition the set W, we have the counting function equation

$$W(M) = R_0(M) + R_1(M) + \cdots + R_t(M).$$

The density condition $d_X(R_0) = 0$ implies

$$R_0(M) = o(X(M)).$$

Let $0 < \varepsilon < \min(\alpha_1, \ldots, \alpha_t)$. For all $i \in \{1, \ldots, t\}$, the subset condition $R_{i,\varepsilon} \subseteq R_i$ implies

$$0 \leq R_{i,\varepsilon}(M) \leq R_i(M).$$

The density condition

$$d_X(R_{i,\varepsilon}) = \lim_{M\to\infty} \frac{R_{i,\varepsilon}(M)}{X(M)} = \alpha_i - \varepsilon$$

implies

$$R_{i,\varepsilon}(M) = (\alpha_i - \varepsilon + o(1))X(M).$$

Also, $d_X(W) = \omega > 0$ implies

$$W(M) = (\omega + o(1))X(M).$$

For all $j \in \{1, \ldots, t\}$, we have

$$\begin{aligned}
(\alpha_j - \varepsilon + o(1))\, X(M) &= R_{j,\varepsilon}(M) \\
&\leq R_j(M) \\
&= W(M) - \sum_{\substack{i=0 \\ i \neq j}}^{t} R_i(M) \\
&\leq W(M) - R_0(M) - \sum_{\substack{i=1 \\ i \neq j}}^{t} R_{i,\varepsilon}(M) \\
&= (\omega + o(1))X(M) - o(1)X(M) - \sum_{\substack{i=1 \\ i \neq j}}^{t} ((\alpha_i - \varepsilon + o(1))X(M)) \\
&= \omega X(M) - \sum_{\substack{i=1 \\ i \neq j}}^{t} (\alpha_i - \varepsilon)X(M) + o(1)X(M) \\
&= X(M) \left(\omega - \sum_{\substack{i=1 \\ i \neq j}}^{t} \alpha_i + (t-1)\varepsilon + o(1) \right) \\
&= X(M)\, (\alpha_j + (t-1)\varepsilon + o(1)) \,.
\end{aligned}$$

Therefore,

$$\alpha_j - \varepsilon + o(1) \leq \frac{R_j(M)}{X(M)} \leq \alpha_j + (t-1)\varepsilon + o(1)$$

for all $\varepsilon > 0$ and so

$$\alpha_j + o(1) \leq \frac{R_j(M)}{X(M)} \leq \alpha_j + o(1).$$

Thus,

$$d_X(R_j) = \lim_{M \to \infty} \frac{R_j(X(M))}{X(M)} = \alpha_j$$

for all $j \in \{1, \ldots, t\}$. This completes the proof. □

Theorem 4. *Let m be a positive integer such that $m \equiv 5 \pmod 8$. If $m = \sum_{i=0}^{k} 4^i$ for some $k \geq 1$, then $\big(m, S(m), S^2(m)\big) = (m, 1, 1)$. If $m \neq \sum_{i=0}^{k} 4^i$ for some $k \geq 1$, then the triple $\big(m, S(m), S^2(m)\big)$ has permutation pattern $(3, 2, 1)$ or $(3, 1, 2)$. In the congruence class $m \equiv 5 \pmod 8$, the permutation densities are as follows:*

permutation pattern $\sigma \in \Sigma_3$	*permutation density* $d_3(\sigma)$
$(3,2,1)$	$1/8$
$(3,1,2)$	$1/8$

Proof. For every positive integer k, let

$$r_k = \sum_{i=0}^{k} 4^i = \frac{4^{k+1}-1}{3}$$

and let

$$\mathcal{R}_0 = \{r_k : k = 1,2,3\ldots\} = \{5, 21, 85, 341, 1365, \ldots\}.$$

The set $\mathcal{R}_0$ has density zero. We have $S(m) = 1$ if and only if $m = 1$ or $m \in \mathcal{R}_0$. If $m \in \mathcal{R}_0$, then $m \equiv 5 \pmod 8$.

Let

$$\mathcal{R}_1 = \left\{ m \equiv 5 \pmod 8 : \left(m, S(m), S^2(m)\right) \text{ has permutation pattern } (3,1,2) \right\}$$

and

$$\mathcal{R}_2 = \left\{ m \equiv 5 \pmod 8 : \left(m, S(m), S^2(m)\right) \text{ has permutation pattern } (3,2,1) \right\}.$$

The sets $\mathcal{R}_0, \mathcal{R}_1$, and $\mathcal{R}_2$ are pairwise disjoint.

Let

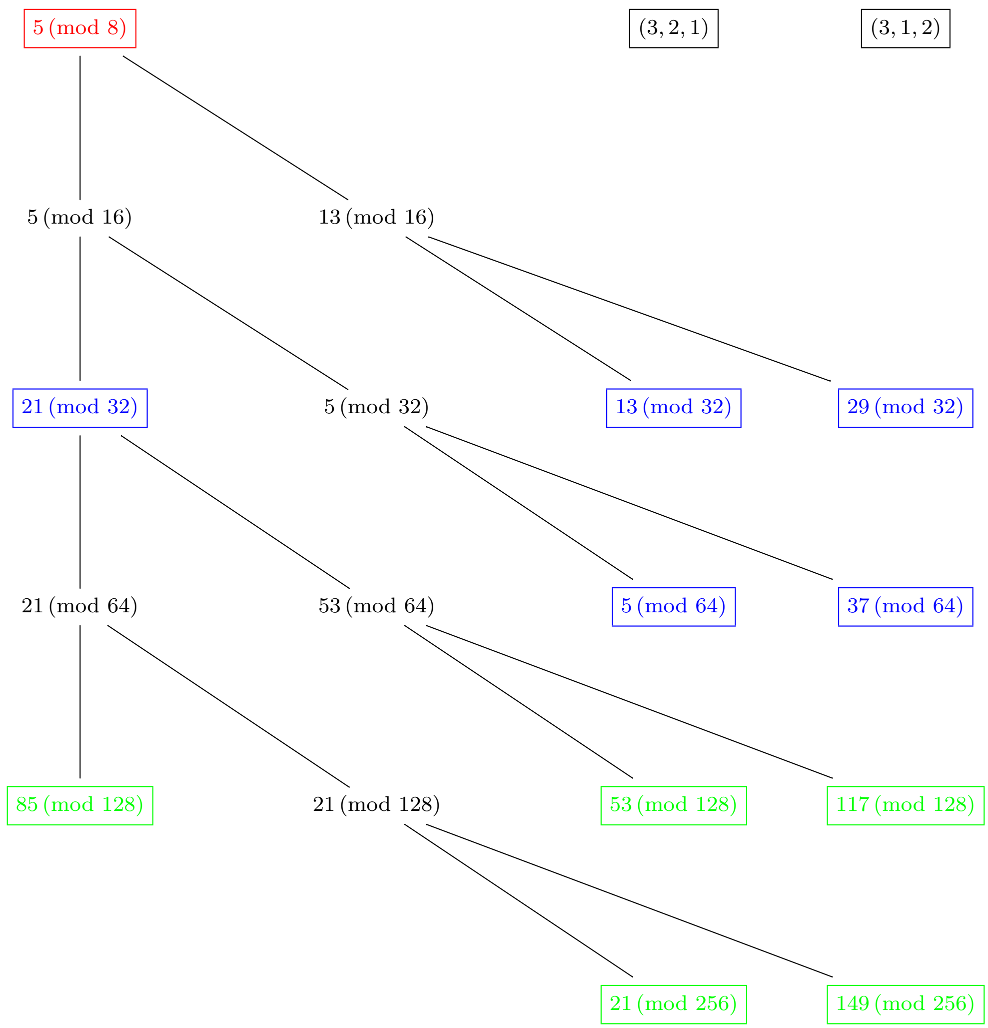
$$5 \pmod 8 \setminus \left(\mathcal{R}_0 \cup r_{k+1} \pmod{2 \cdot 4^{\mathrm{k}+1}}\right)$$
$$(5 \pmod 8 \setminus \mathcal{R}_0) \setminus \left(r_{k+1} \pmod{2 \cdot 4^{\mathrm{k}+1}}\right)$$

denote the set of positive integers $m \notin \mathcal{R}_0$ such that $m \equiv 5 \pmod 8$ but $m \not\equiv r_{k+1} \pmod{2 \cdot 4^{\mathrm{k}+1}}$. We shall prove by induction on k that this set is partitioned into two sets, one with permutation pattern $(3,2,1)$ and the other with permutation pattern $(3,1,2)$, and that each of these sets has permutation density

$$\frac{1}{8}\left(1 - \frac{1}{4^k}\right).$$

We begin with the cases $k = 1$ and $k = 2$.

Here is a picture of a partition of the congruence class $5 \pmod 8$ into disjoint unions of congruence classes:



The congruence class $5 \pmod 8 = r_1 \pmod 8$ is the disjoint union of the following five congruence classes (in the blue boxes):

$$\begin{aligned} 13 &\pmod{32} \\ 5 &\pmod{64} \\ 29 &\pmod{32} \\ 37 &\pmod{64} \\ 21 &\pmod{32} = r_2 \pmod{32}. \end{aligned}$$

Applying Lemmas 1 and 2 with $k = 1$, we see that every positive integer (except integers $m \in \mathcal{R}_0$) in the congruence classes $13 \pmod{32}$ and $5 \pmod{64}$ has permutation pattern $(3, 2, 1)$. Applying Lemmas 3 and 4 with $k = 1$, we see that every positive integer in the congruence classes $29 \pmod{32}$ and $37 \pmod{64}$ has permutation pattern $(3, 1, 2)$. Thus, the set of positive integers $m \notin \mathcal{R}_0$ in

$$5 \pmod{8} \setminus 21 \pmod{32} = 5 \pmod{8} \setminus r_2 \pmod{2 \cdot 4^2}$$

is partitioned into two sets, one with permutation pattern $(3, 2, 1)$ and the other with permutation pattern $(3, 1, 2)$. Each of these sets has density

$$\sum_{i=4}^{5} \frac{1}{2^i} = \frac{1}{8}\left(1 - \frac{1}{4}\right).$$

The congruence class $21 \mod 32 = r_2 \pmod{32}$ is the disjoint union of the following five congruence classes (in the green boxes):

$$\begin{aligned} 53 &\pmod{128} \\ 21 &\pmod{256} \\ 117 &\pmod{128} \\ 149 &\pmod{256} \\ 85 &\pmod{128} = r_3 \pmod{128}. \end{aligned}$$

Applying Lemmas 1 and 2 with $k = 2$, we see that every positive integer $m \notin \mathcal{R}_0$ in the congruence classes $53 \pmod{128}$ and $21 \pmod{256}$ has permutation pattern $(3, 2, 1)$. Applying Lemmas 3 and 4 with $k = 2$, we see that every positive integer in the congruence classes $117 \pmod{128}$ and $149 \pmod{256}$ has permutation pattern $(3, 1, 2)$. Thus, the set of positive integers $m \notin \mathcal{R}_0$ in

$$21 \pmod{32} \setminus 85 \pmod{128} = r_2 \pmod{2 \cdot 4^2} \setminus r_3 \pmod{2 \cdot 4^3}$$

is partitioned into two sets, one with permutation pattern $(3, 2, 1)$ and the other with permutation pattern $(3, 1, 2)$. Each of these sets has density

$$\sum_{i=6}^{7} \frac{1}{2^i} = \frac{1}{8}\left(\frac{1}{4} - \frac{1}{4^2}\right).$$

It follows that the set of positive integers $m \notin \mathcal{R}_0$ in

$$5 \pmod 8 \setminus 85 \pmod{128} = 5 \pmod 8 \setminus r_3 \pmod{2 \cdot 4^3}$$

is partitioned into two sets, one with permutation pattern $(3, 2, 1)$ and the other with permutation pattern $(3, 1, 2)$. Each of these sets has density

$$\frac{1}{8}\left(1 - \frac{1}{4^2}\right).$$

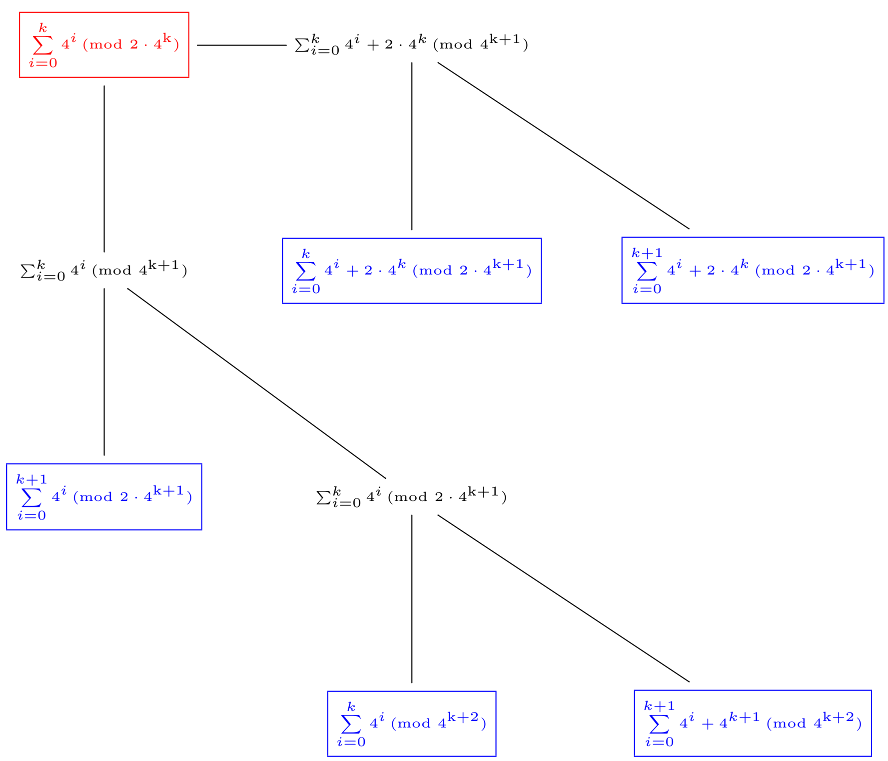
Let $k \geq 3$ and assume that the set of positive integers $m \notin \mathcal{R}_0$ in

$$5 \pmod 8 \setminus r_k \pmod{2 \cdot 4^{\mathrm{k}}}$$

is partitioned into two sets, one with permutation pattern $(3, 2, 1)$ and the other with permutation pattern $(3, 1, 2)$, and that each of these sets has density

$$\frac{1}{8}\left(1 - \frac{1}{4^{k-1}}\right).$$

The following diagram displays the partition of the "red" congruence class $r_k \pmod{2 \cdot 4^{\mathrm{k}}}$ into three "blue" congruence classes modulo $2 \cdot 4^{k+1}$ and two "blue" congruence classes modulo 4^{k+2}.



The congruence class $r_k \pmod{2 \cdot 4^{\mathrm{k}}}$ is the disjoint union of the following five congruence classes:

$$\sum_{i=0}^{k} 4^i + 2 \cdot 4^k \pmod{2 \cdot 4^{\mathrm{k+1}}}$$

$$\sum_{i=0}^{k} 4^i \pmod{4^{\mathrm{k+2}}}$$

$$\sum_{i=0}^{k+1} 4^i + 2 \cdot 4^k \pmod{2 \cdot 4^{\mathrm{k+1}}}$$

$$\sum_{i=0}^{k+1} 4^i + 4^{k+1} \pmod{4^{\mathrm{k+2}}}$$

$$\sum_{i=0}^{k+1} 4^i \pmod{2 \cdot 4^{\mathrm{k+1}}}.$$

Applying Lemmas 1 and 2, we see that every positive integer in the congruence classes $\sum_{i=0}^{k} 4^i + 2 \cdot 4^k \pmod{2 \cdot 4^{\mathrm{k+1}}}$ and $\sum_{i=0}^{k} 4^i \pmod{4^{\mathrm{k+2}}}$ has permutation pattern $(3, 2, 1)$. Applying Lemmas 3 and 4, we see that every positive integer in the congruence classes $\sum_{i=0}^{k+1} 4^i + 2 \cdot 4^k \pmod{2 \cdot 4^{\mathrm{k+1}}}$ and $\sum_{i=0}^{k+1} 4^i + 4^{k+1} \pmod{4^{\mathrm{k+2}}}$ has permutation pattern $(3, 1, 2)$. Thus, the positive integers in the set

$$r_k \pmod{2 \cdot 4^{\mathrm{k}}} \setminus r_{k+1} \pmod{2 \cdot 4^{\mathrm{k+1}}}$$

are partitioned into two sets, one with permutation pattern $(3, 2, 1)$ and the other with permutation pattern $(3, 1, 2)$. Each of these sets has density

$$\frac{1}{4^{k+1}} + \frac{2}{4^{k+2}} = \frac{1}{8}\left(\frac{1}{4^{k-1}} - \frac{1}{4^k}\right).$$

Thus, the positive integers in

$$\begin{aligned} &5 \pmod 8 \setminus r_{k+1} \pmod{2 \cdot 4^{\mathrm{k+1}}} \\ &\quad = \big(5 \pmod 8 \setminus r_k \pmod{2 \cdot 4^{\mathrm{k}}}\big) \bigcup \big(r_k \pmod{2 \cdot 4^{\mathrm{k}}} \setminus r_{k+1} \pmod{2 \cdot 4^{\mathrm{k+1}}}\big) \end{aligned}$$

are partitioned into two sets, one with permutation pattern $(3, 2, 1)$ and the other with permutation pattern $(3, 1, 2)$. Each of these sets has density

$$\frac{1}{8}\left(1 - \frac{1}{4^{k-1}}\right) + \frac{1}{8}\left(\frac{1}{4^{k-1}} - \frac{1}{4^k}\right) = \frac{1}{8}\left(1 - \frac{1}{4^k}\right).$$

This completes the induction.

For every $\varepsilon > 0$ there is an integer k such that $1/(8 \cdot 4^k) < \varepsilon$ and so both sets $\mathcal{R}_1$ and $\mathcal{R}_2$ contain subsets of density greater than $1/8 - \varepsilon$. The set $\mathcal{R}_0$ has density 0 and the congruence class $5 \pmod 8$ has density $1/4$ with respect to Ω. Applying Lemma 5 with $X = \Omega$, $W = \{m > 1 : m \equiv 5 \pmod 8\}$, $t = 2$, and $\alpha_1 = \alpha_2 = 1/8$ to the partition $5 \pmod 8 = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2$, we see that $\mathcal{R}_1$ has density $1/8$ and $\mathcal{R}_2$ has density $1/8$. This completes the proof. □

5. Some Impossible Permutation Patterns for Quadruples

By Theorem 1, every triple permutation pattern is the permutation pattern of triples $\big(m, S(m), S^2(m)\big)$ of the iterated Syracuse function for a set of integers m of positive density. The story for quadruple permutation patterns is different. In this section we prove that there are quadruple permutation patterns that never occur as permutation patterns of quadruples $\big(m, S(m), S^2(m), S^3(m)\big)$ of the iterated Syracuse function.

Theorem 5. *Consider quadruples*

$$\big(m, S(m), S^2(m), S^3(m)\big)$$

such that $S^i(m) \neq S^j(m)$ *for* $i \neq j$ *and the triple* $\big(m, S(m), S^2(m)\big)$ *has permutation pattern* $(1, 2, 3)$, *that is,*

$$m < S(m) < S^2(m).$$

For these quadruples there are four possible quadruple permutation patterns:

$$(1, 2, 3, 4), \qquad (1, 2, 4, 3), \qquad (2, 3, 4, 1), \qquad (1, 3, 4, 2).$$

The density of each of these permutation patterns is as follows:

permutation pattern $\sigma \in \Sigma_4$	*permutation density* $d_4(\sigma)$
$(1, 2, 3, 4)$	$1/8$
$(1, 2, 4, 3)$	$1/16$
$(2, 3, 4, 1)$	$1/16$
$(1, 3, 4, 2)$	0

Moreover, the permutation pattern $(1, 3, 4, 2)$ *never occurs.*

Note that $1/4 = 1/8 + 1/16 + 1/16$ is the Syracuse function permutation pattern density of the triple $(1, 2, 3)$.

Proof. By Theorems 3 and 4, we have $m < S(m) < S^2(m)$ if and only if

$$m \equiv 7 \pmod 8.$$

Then $m = 7 + 8x$ for some nonnegative integer x and

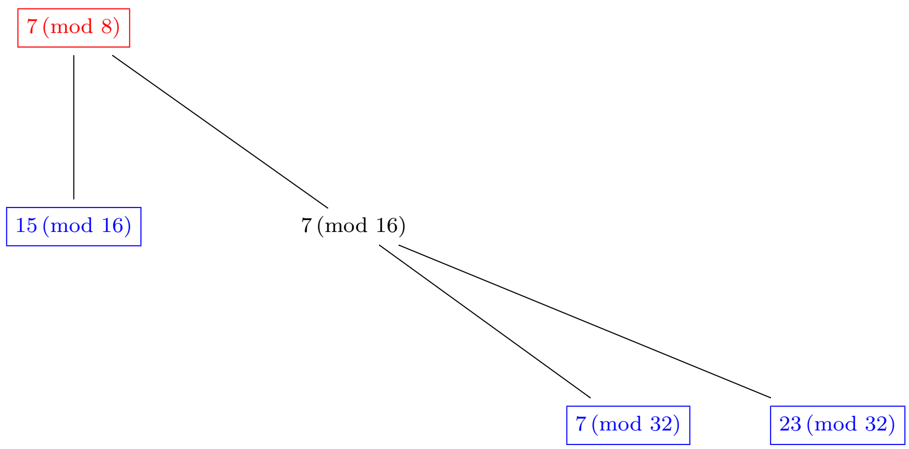
$$m = 7 + 8x < S(m) = 11 + 12x < S^2(m) = 17 + 18x.$$

It follows that

$$S^3(m) = \frac{26 + 27x}{2^e}$$

for some nonnegative integer e.

The congruence class $7 \pmod 8$ is the disjoint union of the congruence classes $15 \pmod{16}$, $7 \pmod{32}$, and $23 \pmod{32}$.



If $m \equiv 15 \pmod{16}$, then the integers x and $26 + 27x$ are odd and so $e = 0$. We have

$$m < S(m) < S^2(m) = 17 + 18x < 26 + 27x = S^3(x)$$

and so the quadruple $(m, S(m), S^2(m), S^3(m))$ has permutation pattern $(1, 2, 3, 4)$.

If $m \equiv 7 \pmod{32}$, then $x = 4y$ and $e = 1$. We obtain

$$S^3(m) = \frac{26 + 27x}{2} = 13 + 54y = 13 + \left(\frac{27}{2}\right) x.$$

The inequality

$$7 + 8x < 11 + 12x < 13 + \left(\frac{27}{2}\right) x < 17 + 18x$$

implies

$$m < S(m) < S^3(m) < S^2(m).$$

The quadruple $(m, S(m), S^2(m), S^3(m))$ has permutation pattern $(1, 2, 4, 3)$.

If $m \equiv 23 \pmod{32}$, then $x = 2 + 4y$ and $e \geq 2$.

$$\begin{aligned} S^3(m) &= \frac{26+27x}{2^e} = \frac{80+27\cdot 4y}{2^e} = \frac{20+27y}{2^{e-2}} \\ &\leq 20+27y = 20+27\left(\frac{x-2}{4}\right) \\ &= \frac{13}{2}+\left(\frac{27}{4}\right)x < 7+8x = m. \end{aligned}$$

The quadruple $(m, S(m), S^2(m), S^3(m))$ has permutation pattern $(2,3,4,1)$.

We see that the permutation pattern $(1,3,4,2)$ never occurs, and that the permutation patterns $(1,2,3,4)$, $(1,2,4,3)$, and $(2,3,4,1)$ have permutation densities $1/8$, $1/16$, and $1/16$, respectively. This completes the proof. □

Theorem 6. *Consider quadruples*

$$\big(m, S(m), S^2(m), S^3(m)\big)$$

such that $S^i(m) \neq S^j(m)$ *for* $i \neq j$ *and the triple* $\big(m, S(m), S^2(m)\big)$ *has permutation pattern* $(1,3,2)$, *that is,*

$$m < S^2(m) < S(m).$$

For these quadruples there are four possible quadruple permutation patterns:

$$(1,3,2,4), \qquad (2,4,3,1), \qquad (1,4,2,3), \qquad (1,4,3,2).$$

The density of each of these permutation patterns is as follows:

permutation pattern $\sigma \in \Sigma_4$	*permutation density* $d_4(\sigma)$
$(1,3,2,4)$	$1/16$
$(2,4,3,1)$	$1/16$
$(1,4,2,3)$	0
$(1,4,3,2)$	0

Moreover, the permutation patterns $(1,4,3,2)$ *and* $(1,4,2,3)$ *never occur.*

Note that $1/8 = 1/16+1/16$ is the Syracuse function permutation pattern density of the triple $(1,3,2)$.

Proof. By Theorems 3 and 4, we have $m < S^2(m) < S(m)$ if and only if

$$m \equiv 11 \pmod{16}.$$

Then $m = 11 + 16x$ for some nonnegative integer x and

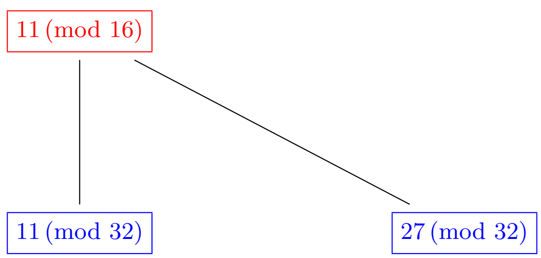
$$m = 11 + 16x < S^2(m) = 13 + 18x < S(m) = 17 + 24x.$$

It follows that

$$S^3(m) = \frac{20 + 27x}{2^e}$$

for some nonnegative integer e.

The congruence class $11 \pmod{16}$ is the disjoint union of the congruence classes $11 \pmod{32}$ and $27 \pmod{32}$.



If $m \equiv 27 \pmod{32}$, then x and $20 + 27x$ are odd and so $e = 0$. We have

$$S(m) = 17 + 24x < 20 + 27x = S^3(m)$$

and the quadruple $(m, S(m), S^2(m), S^3(m))$ has permutation pattern $(1, 3, 2, 4)$.

If $m \equiv 11 \pmod{32}$, then x is even and $e \geq 1$. We obtain

$$S^3(m) = \frac{20 + 27x}{2^e} \leq 10 + \left(\frac{27}{2}\right) x < 11 + 16x = m.$$

The quadruple $(m, S(m), S^2(m), S^3(m))$ has permutation pattern $(2, 4, 3, 1)$.

We see that the permutation patterns $(1, 4, 3, 2)$ and $(1, 4, 2, 3)$ never occur and that each of the permutation patterns $(1, 3, 2, 4)$ and $(2, 4, 3, 1)$ has density $1/16$. This completes the proof. $\square$

Theorem 7. *Consider quadruples*

$$\big(m, S(m), S^2(m), S^3(m)\big)$$

such that $S^i(m) \neq S^j(m)$ *for* $i \neq j$ *and the triple* $\big(m, S(m), S^2(m)\big)$ *has permutation pattern* $(2, 1, 3)$, *that is,*

$$S(m) < m < S^2(m).$$

For these quadruples there are four possible quadruple permutation patterns:

$$(2, 1, 3, 4), \qquad (2, 1, 4, 3), \qquad (3, 1, 4, 2), \qquad (3, 2, 4, 1).$$

The density of each of these permutation patterns is as follows:

permutation pattern $\sigma \in \Sigma_4$	*permutation density* $d_4(\sigma)$
$(2, 1, 3, 4)$	$1/16$
$(3, 1, 4, 2)$	$1/32$
$(3, 2, 4, 1)$	$1/32$
$(2, 1, 4, 3))$	0

Moreover, the permutation pattern $(2, 1, 4, 3)$ *never occurs.*

Note that $1/8 = 1/16+1/32+1/32$ is the Syracuse function permutation pattern density of the triple $(2, 1, 3)$.

Proof. By Theorems 3 and 4, we have $S(m) < m < S^2(m)$ if and only if

$$m \equiv 9 \pmod{16}.$$

Then $m = 9 + 16x$ for some nonnegative integer x and

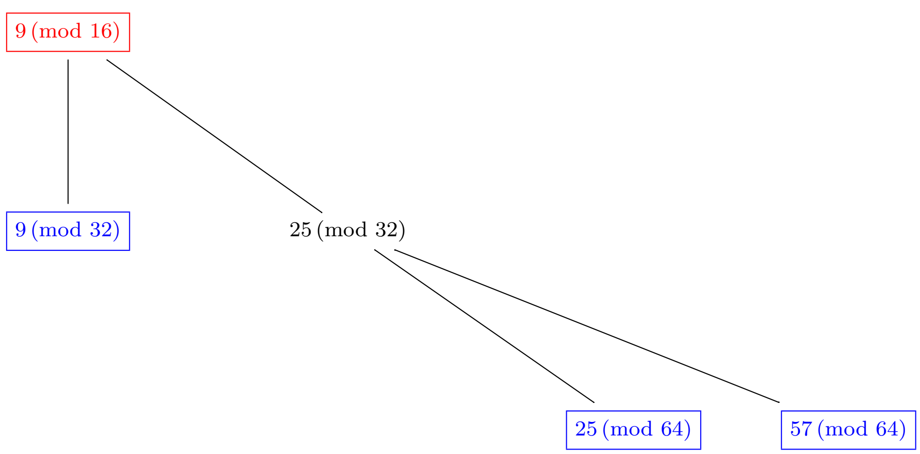
$$S(m) = 7 + 12x < m = 9 + 16x < S^2(m) = 11 + 18x.$$

It follows that

$$S^3(m) = \frac{17 + 27x}{2^e}$$

for some nonnegative integer e.

The congruence class $9 \pmod{16}$ is the disjoint union of the congruence classes $9 \pmod{32}$, $25 \pmod{64}$, and $57 \pmod{64}$.



If $m \equiv 9 \pmod{32}$, then x is even, $17 + 27x$ is odd, and so $e = 0$. We have

$$S^2(m) = 11 + 18x < S^3(m) = 17 + 27x$$

and the quadruple $(m, S(m), S^2(m), S^3(m))$ has permutation pattern $(2, 1, 3, 4)$.

If $m \equiv 57 \pmod{64}$, then $x = 3 + 4y$ and $e = 1$. We obtain

$$\begin{aligned} S^3(m) &= \frac{17 + 27(3 + 4y)}{2} = 49 + 54y \\ &= 49 + 54\left(\frac{x - 3}{4}\right) = \frac{17}{2} + \left(\frac{27}{2}\right)x. \end{aligned}$$

The inequality

$$7 + 12x < \frac{17}{2} + \left(\frac{27}{2}\right) x < 9 + 16x$$

implies

$$S(m) < S^3(m) < m < S^2(m).$$

The quadruple $(m, S(m), S^2(m), S^3(m))$ has permutation pattern $(3, 1, 4, 2)$.

If $m \equiv 25 \pmod{64}$, then $x = 1 + 4y$ and $e \geq 2$. We obtain

$$\begin{aligned} S^3(m) &= \frac{17 + 27(1 + 4y)}{2^e} = \frac{11 + 27y}{2^{e-2}} \\ &\leq 11 + 27y = 11 + 27\left(\frac{x-1}{4}\right) = \frac{17}{4} + \left(\frac{27}{4}\right) x \\ &< 7 + 12x = S(m). \end{aligned}$$

The quadruple $(m, S(m), S^2(m), S^3(m))$ has permutation pattern $(3, 2, 4, 1)$.

We see that the permutation pattern $(2, 1, 4, 3)$ never occurs, and that the permutation patterns $(2, 1, 3, 4)$, $(3, 1, 4, 2)$, and $(3, 2, 4, 1)$ have densities $1/16$, $1/32$, and $1/32$, respectively. This completes the proof. □

6. Dropping Time

An odd integer $m > 1$ has *dropping time* $D(m) = k$ if k is the smallest positive integer such that $S^k(m) < m$, and dropping time $D(m) = \infty$ if $S^k(m) > m$ for all positive integers k. The Collatz conjecture is equivalent to the statement that $D(m) < \infty$ for all odd integers $m > 1$. We have $D(m) > k$ if and only if $S^i(m) \neq 1$ for all $i \in \{1, 2, \ldots, k\}$ and the permutation pattern of the $(k+1)$-tuple $\left(m, S(m), S^2(m), \ldots, S^k(m)\right)$ is not of the form $(1, a_2, a_3, \ldots, a_{k+1})$, where $(a_2, a_3, \ldots, a_{k+1})$ is any permutation of $(2, 3, \ldots, k+1)$.

Theorem 8. *Let $N_k(x)$ count the number of odd integers $m \leq x$ such that $D(m) \leq k$. Then*

$$\overline{N}_1 = \lim_{x\to\infty} \frac{N_1(x)}{x/2} = \frac{1}{2}$$

$$\overline{N}_2 = \lim_{x\to\infty} \frac{N_2(x)}{x/2} = \frac{5}{8}$$

$$\overline{N}_3 = \lim_{x\to\infty} \frac{N_3(x)}{x/2} = \frac{3}{4}$$

Proof. Permutation pattern densities give explicit values for $N_k(x)$ for $k = 1, 2$, and 3. An odd integer $m > 1$ has dropping time $D(m) = 1$ if and only if $S(m) < m$

if and only if the pair $(m, S(m))$ has permutation pattern $(2,1)$, and so $\overline{N}_1$ is the density of the permutation pattern $(2,1)$. We have $\overline{N}_1 = 1/2$ by Theorem 2.

The odd integer $m > 1$ has dropping time $D(m) \leq 2$ if and only if the permutation pattern of the triple $(m, S(m), S^2(m))$ is not $(1,2,3)$ or $(1,3,2)$. By Theorem 1, the permutation pattern $(1,2,3)$ has density $1/4$ and the permutation pattern $(1,3,2)$ has density $1/8$. Therefore, $\overline{N}_2 = 1 - 1/4 - 1/8 = 5/8$.

The odd integer $m > 1$ has dropping time $D(m) \leq 3$ if and only if the quadruple $(m, S(m), S^2(m), S^3(m))$ has permutation pattern not equal to $(1, a_2, a_3, a_4)$, where (a_2, a_3, a_4) is any of the six permutations of $2, 3$, and 4. By Theorems 5 and 6, the sum of the densities of these six permutation patterns is $1/4$ and so $\overline{N}_3 = 1 - 1/4 = 3/4$. □

The dropping time function is the Syracuse function analogue of the stopping time function of Riho Terras [10]. It would be of interest to prove (similar to results of Terras) that the limit

$$\overline{N}_k = \lim_{x \to \infty} \frac{N_k(x)}{x/2}$$

exists for all positive integers k and that

$$\lim_{k \to \infty} N_k = 1.$$

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References

[1] M. Albert, B. Gudmundsson, and H. Ulfarsson, Collatz meets Fibonacci, *Math. Mag.* **95** (2022), 130–136.

[2] D. Barina, Convergence verification of the Syracuse problem, *Journal of Supercomputing* **77** (2021), 2681–2688.

[3] J. C. Lagarias, The $3x + 1$ problem and its generalizations, *Amer. Math. Monthly* **92** (1985), 3-23.

[4] M. B. Nathanson, Patterns in the iteration of an arithmetic function, 2022, arXiv:2208.02242.

[5] J. L. Simons, A simple (inductive) proof for the non-existence of 2-cycles of the $3x+1$ problem, *J. Number Theory* **123** (2007), 10–17.

[6] J. L. Simons, On the (non-)existence of m-cycles for generalized Syracuse sequences, *Acta Arith.* **131** (2008), 217–254.

[7] J. L. Simons, Exotic Collatz cycles, *Acta Arith.* **134** (2008), 201–209.

[8] J. Simons and B.de Weger, Theoretical and computational bounds for m-cycles of the $3n+1$-problem, *Acta Arith.* **117** (2005), 51–70.

[9] T. Tao, Almost all orbits of the Collatz map attain almost bounded values, *Forum Math., Pi* **10** (2022), Paper No. E12.

[10] R. Terras, A stopping time problem on the positive integers, *Acta Arith.* **30** (1976), 241–252.

[11] Wikipedia, Collatz conjecture