AN UPPER BOUND ON THE INHOMOGENEOUS APPROXIMATION CONSTANTS

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Abstract

For an irrational real α and $\gamma \notin \mathbb{Z} + \mathbb{Z}\alpha$ it is well known that

$$\liminf_{|n|\to\infty} |n|||n\alpha - \gamma|| \leq \frac{1}{4}.$$

In the present paper we prove that, if the partial quotients, a_i, in the negative 'round-up' continued fraction expansion of α have $R := \liminf_{i\to\infty} a_i$ odd, then the bound 1/4 can be replaced by

$$\frac{1}{4}\left(1 - \frac{1}{R}\right)\left(1 - \frac{1}{R^2}\right),$$

which is optimal. The optimal bound for even $R \geq 4$ was already known.

1. Introduction

For a real irrational α and real γ in $[0, 1)$ we define the inhomogeneous approximation constant

$$M(\alpha, \gamma) := \liminf_{|n|\to\infty} |n|||n\alpha - \gamma||, \tag{1}$$

where $||x||$ is the distance from x to the nearest integer. Corresponding to the case of worst inhomogeneous approximation, we define

$$\rho(\alpha) := \sup_{\gamma\notin\mathbb{Z}+\mathbb{Z}\alpha} M(\alpha, \gamma).$$

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From a well known theorem of Minkowski, see for example [1, Ch. III] or [8, IV.9],

$$\rho(\alpha) \leq \frac{1}{4}, \tag{2}$$

Grace [3] giving examples with $\rho(\alpha) = \frac{1}{4}$. These examples, though, have continued fraction expansions with the partial quotients a_i satisfying $\liminf_{i\to\infty} a_i = \infty$. When the partial quotients of α are uniformly bounded, Khinchin [4] improved Inequality (2) to

$$\rho(\alpha) \leq \frac{1}{4}\sqrt{1 - 4M^2(\alpha, 0)}.$$

We are interested in improving Inequality (2) when the partial quotients have a bounded subsequence. Fukasawa [2] used the nearest integer continued fraction expansion to obtain bounds of this type. Here we shall use the *negative expansion*:

$$\alpha = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots}}} =: [0; a_1, a_2, a_3, \cdots]^-, \tag{3}$$

where the integers $a_i \geq 2$ are generated by always rounding up instead of down:

$$\alpha_0 := \{\alpha\} = \alpha, \quad a_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil, \quad \alpha_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil - \frac{1}{\alpha_n}.$$

We write α_i and $\bar{\alpha}_i$ for the forwards and backwards expansions from the ith point:

$$\alpha_i := [0; a_{i+1}, a_{i+2}, \ldots]^-, \qquad \bar{\alpha}_i := [0; a_i, a_{i-1}, \ldots, a_1]^-.$$

A method to evaluate $M(\alpha, \gamma)$ from Equation (3) and a sequence of integers t_i obtained from an appropriate α-expansion of γ,

$$\gamma = \sum_{i=1}^{\infty} \frac{1}{2}(a_i - 2 + t_i) D_{i-1}, \qquad D_{i-1} := \alpha_0 \alpha_1 \cdots \alpha_{i-1}, \tag{4}$$

was given in [7]. We review this in Section 2; see [5] for alternative algorithms to compute $M(\alpha, \gamma)$. Setting

$$R := \liminf_{i\to\infty} a_i,$$

it was also shown in [7] that if $R \geq 3$, then

$$\rho(\alpha) \leq \frac{1}{4}\left(1 - \frac{1}{R}\right). \tag{5}$$

The restriction $R \geq 3$ is needed here; the examples of Grace with $\rho(\alpha) = \frac{1}{4}$ will have long strings of 2's in their negative expansion.

For $R \geq 4$ even, Inequality (5) is best possible; for example, as shown in [7, Theorem 2], if $\alpha = [0; \overline{R, 2N}]^-$, then taking all the $t_i = 0$ in (4) gives

$$\rho(\alpha) = M\left(\alpha, \frac{1}{2}(1-\alpha)\right) = \frac{1}{4} \liminf_{k\to\infty} \frac{(1-\alpha_k)(1-\bar{\alpha}_k)}{1-\alpha_k\bar{\alpha}_k}.$$

As $N \to \infty$, the $\alpha_{2k-1}, \bar{\alpha}_{2k-1} \to 0, 1/R$, the $\alpha_{2k}, \bar{\alpha}_{2k} \to 1/R, 0$, and

$$\lim_{N\to\infty} \rho(\alpha) = \frac{1}{4}\left(1 - \frac{1}{R}\right).$$

When R is odd though, Inequality (5) can be improved.

Theorem 1. *If R is odd then*

$$\rho(\alpha) \leq C(R) := \frac{1}{4}\left(1-\frac{1}{R}\right)\left(1-\frac{1}{R^2}\right) = \frac{1}{4}\left(1 - \frac{1}{R} - \frac{1}{R^2} + \frac{1}{R^3}\right). \qquad (6)$$

For odd R, Inequality (6) is also best possible.

Theorem 2. *Suppose that $\alpha = [0; \overline{R, NR}]^-$ where N and R are positive integers with R odd. Define γ_* to have Equation (4) with $t_{2i-1} = -1$ and $t_{2i} = N$. Then*

$$\lim_{N\to\infty} M(\alpha, \gamma_*) = C(R).$$

One can also obtain lower bounds on $\rho(\alpha)$ in terms of R. When $R \geq 4$ is even, we showed in [6] the optimal bound

$$\rho(\alpha) \geq \frac{R-2}{4(\sqrt{R^2-4}+1)} = \frac{1}{4}\left(1 - \frac{3}{R} + \frac{5}{R^2} + O(R^{-3})\right),$$

and when $R \geq 3$ is odd, a bound which is at least asymptotically optimal:

$$\rho(\alpha) \geq \frac{2R-2-\sqrt{(R+1)^2-4}}{4(\sqrt{(R+1)^2-4}-1)} = \frac{1}{4}\left(1 - \frac{3}{R} + \frac{4}{R^2} + O(R^{-3})\right).$$

Here we have necessarily excluded the classical homogeneous case $\gamma \in \mathbb{Z} + \alpha\mathbb{Z}$. Of course $M(\alpha, 0)$ depends on the largest rather than the smallest partial quotients. Using the negative expansion, if $R \geq 3$ and r denotes $\limsup_{i\to\infty} a_i$, then

$$\frac{1}{\sqrt{r^2-4}} \leq M(\alpha, 0) = \liminf_{i\to\infty} \frac{1}{a_i - \alpha_i - \bar{\alpha}_{i-1}} \leq \frac{1}{r - R + \sqrt{R^2-4}},$$

with equality in the lower bound for $\alpha = [0; \bar{r}]^-$ and the upper bound when the expansion of α consists of blocks $r, \underbrace{R, \ldots, R}_{l \text{ times}}$ with length $l \to \infty$. Notice $M(\alpha, 0)$ can only exceed the inhomogeneous upper bound, that is, Inequality (5) if R is even and Inequality (6) if R is odd, for a few small r. Namely, when $R = 3$ and we have $3 \leq r \leq 7$, when $R = 4$ and $r = 4$ or $r = 5$, and when $R = 5$ with $r = 5$.

2. The Sequence of Best Inhomogeneous Approximations

In [7] it was shown how to use Equation (3) of α to expand $\gamma \in (0,1)$ in the form $\gamma = \sum_{i=1}^{\infty} b_i D_{i-1}$, by taking

$$\gamma_0 := \gamma, \quad b_{n+1} := \left\lfloor \frac{\gamma_n}{\alpha_n} \right\rfloor, \quad \gamma_{n+1} := \left\{ \frac{\gamma_n}{\alpha_n} \right\},$$

and obtaining from the b_i the sequence of best inhomogeneous approximations for α and γ. To obtain more symmetric looking functions, it proves convenient to write $b_i = \frac{1}{2}(a_i - 2 - t_i)$, leading to a unique α-expansion having the form of Equation (4), where the t_i are integers with

$$-(a_i - 2) \le t_i \le a_i, \quad t_i \equiv a_i \mod 2,$$

with no blocks of the form $t_i = a_i$ with $t_j = a_j - 2$ for all $j > i$ or $t_{i+\ell} = a_{i+\ell}$ with $t_j = a_j - 2$ for any $i < j < i + \ell$.

If the sequence of integers in Equation (4) has $t_k = a_k$ infinitely often, then [7, Lemma 1] gives

$$M(\alpha,\gamma) \le \liminf_{\substack{k\to\infty \\ t_k = a_k}} \frac{\bar{\alpha}_k}{4(1-\bar{\alpha}_k\alpha_k)}. \tag{7}$$

For $R \ge 3$, this will always be smaller than Inequality (6), and so these γ can be safely ignored (see Inequality (9)).

If the sequence has $t_k = a_k$ at most finitely often, then by using [7, Theorem 1] to evaluate Equation (1), it is enough to look at the sequence of n of the form

$$Q_k, \quad Q_k + q_{k-1}, \quad -(q_k - q_{k-1} - Q_k), \quad -(q_k - Q_k),$$

where the q_i are the convergent denominators:

$$p_i/q_i = [0; a_1, \ldots, a_i]^-, \qquad q_i = (\bar{\alpha}_1 \cdots \bar{\alpha}_i)^{-1},$$

and

$$Q_k := \sum_{i=1}^{k} \frac{1}{2}(a_i - 2 + t_i) q_{i-1}.$$

Writing

$$d_k^- := t_k\bar{\alpha}_k + t_{k-1}\bar{\alpha}_k\bar{\alpha}_{k-1} + t_{k-2}\bar{\alpha}_k\bar{\alpha}_{k-1}\bar{\alpha}_{k-2} + \cdots,$$
$$d_k^+ := t_{k+1}\alpha_k + t_{k+2}\alpha_k\alpha_{k+1} + t_{k+3}\alpha_k\alpha_{k+1}\alpha_{k+2} + \cdots,$$

this is readily expressed in the more symmetric form given in Theorem 1 of [7]. We note (see [7]) that $|d_k^-| \le 1 - \bar{\alpha}_k$ and $|d_k^+| \le 1 - \alpha_k$ once k has passed the last $t_j = a_j$ and any following (necessarily finite) string of $t_j = a_j - 2$.

Lemma 1. *If $\gamma \notin \mathbb{Z} + \alpha\mathbb{Z}$ and the α–expansion of γ has $t_i = a_i$ at most finitely many times, then*

$$M(\alpha, \gamma) = \liminf_{k\to\infty} \min\{s_1(k), s_2(k), s_3(k), s_4(k)\},$$

where

$$\begin{aligned}
s_1(k) &:= \frac{1}{4}(1 - \bar{\alpha}_k + d_k^-)(1 - \alpha_k + d_k^+)/(1 - \bar{\alpha}_k\alpha_k),\\
s_2(k) &:= \frac{1}{4}(1 + \bar{\alpha}_k + d_k^-)(1 + \alpha_k - d_k^+)/(1 - \bar{\alpha}_k\alpha_k),\\
s_3(k) &:= \frac{1}{4}(1 - \bar{\alpha}_k - d_k^-)(1 - \alpha_k - d_k^+)/(1 - \bar{\alpha}_k\alpha_k),\\
s_4(k) &:= \frac{1}{4}(1 + \bar{\alpha}_k - d_k^-)(1 + \alpha_k + d_k^+)/(1 - \bar{\alpha}_k\alpha_k).
\end{aligned}$$

Notice that

$$s_1(k)s_3(k) = \frac{\left((1-\bar{\alpha}_k)^2 - (d_k^-)^2\right)\left((1-\alpha_k)^2 - (d_k^+)^2\right)}{16(1-\bar{\alpha}_k\alpha_k)^2} \leq \frac{(1-\bar{\alpha}_k)^2(1-\alpha_k)^2}{16(1-\bar{\alpha}_k\alpha_k)^2}.$$

If $t_k = a_k$ at most finitely often, this plainly gives

$$M(\alpha, \gamma) \leq \liminf_{k\to\infty} \frac{(1-\bar{\alpha}_k)(1-\alpha_k)}{4(1-\bar{\alpha}_k\alpha_k)}, \tag{8}$$

and $\bar{\alpha}_k > 1/R$ when $a_k = R$ readily gives Inequality (5). When the a_i are all even we can take the t_i to be 0 in Equation (4) and have equality in Inequality (8), but when R is odd, $|d_k^-|$ will not be small for the $a_k = R$ and Inequality (5) can be improved.

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Suppose that α has $R \geq 3$ odd. Setting $\beta := [0; \overline{R}]^-$, we can assume that $\alpha_k \leq \beta$ and $\bar{\alpha}_k \leq \beta$ as $k \to \infty$. When $R = 3$ we have $\beta = \frac{1}{2}(3 - \sqrt{5})$. We need to show that $M(\alpha, \gamma) \leq C(R)$ for any $\gamma \notin \mathbb{Z} + \alpha\mathbb{Z}$.

From Inequality (7) we can assume that Equation (4) does not have $t_k = a_k$ infinitely often, for otherwise, as $k \to \infty$ we have

$$\frac{\bar{\alpha}_k}{4(1-\bar{\alpha}_k\alpha_k)} = \frac{1}{4(a_k - \bar{\alpha}_{k-1} - \alpha_k)} \leq \frac{1}{4(R-2\beta)} \leq \frac{1}{4\sqrt{5}} < C(3) \leq C(R). \tag{9}$$

Hence by Lemma 1 we just need to show that there are infinitely many k with

$$\min\{s_1(k), s_2(k), s_3(k), s_4(k)\} \leq C(R).$$

Notice that

$$s_3(k)s_4(k) = \frac{\left((1-d_k^-)^2 - \bar{\alpha}_k^2\right)\left(1-(\alpha_k+d_k^+)^2\right)}{16(1-\bar{\alpha}_k\alpha_k)^2} < \frac{(1-d_k^-)^2}{16(1-\bar{\alpha}_k\alpha_k)^2}.$$

Hence if $a_k = R$ and $t_k \geq 3$, then $d_k^- = (t_k + d_{k-1}^-)\bar{\alpha}_k \geq (3+d_{k-1}^-)\bar{\alpha}_k > 2\bar{\alpha}_k$ and $\alpha_k \leq \beta$ give

$$\min\{s_3(k), s_4(k)\} \leq \frac{1-d_k^-}{4(1-\bar{\alpha}_k\alpha_k)} \leq \frac{1-2\bar{\alpha}_k}{4(1-\bar{\alpha}_k\beta)}.$$

Since $f(x) = \frac{(1-2x)}{(1-\beta x)}$ has $f'(x) = -\frac{(2-\beta)}{(1-\beta x)^2} < 0$ and $\bar{\alpha}_k > 1/R$, we get

$$\min\{s_3(k), s_4(k)\} \leq \frac{1-\frac{2}{R}}{4\left(1-\frac{\beta}{R}\right)} < C(R),$$

the latter inequality since

$$\begin{aligned}\left(1-\frac{1}{R}\right)\left(1-\frac{1}{R^2}\right)\left(1-\frac{\beta}{R}\right) - \left(1-\frac{2}{R}\right)& \\ =\frac{1}{R}\left(1-\frac{1}{R}\left(1-\frac{1}{R}\right) - \beta\left(1-\frac{1}{R}\right)\left(1-\frac{1}{R^2}\right)\right) &> 0.\end{aligned}$$

Likewise, if $t_k \leq -3$ we get $d_k^- \leq (-3+d_{k-1}^-)\bar{\alpha}_k < -2\bar{\alpha}_k$ and

$$\min\{s_1(k), s_2(k)\} \leq \frac{1+d_k^-}{4(1-\bar{\alpha}_k\alpha_k)} \leq C(R).$$

So we can assume that the $a_k = R$ have $t_k = \pm 1$. Suppose now that $a_k = R$ with $t_k = -1$ so that

$$d_k^- = t_k\bar{\alpha}_k + d_{k-1}^-\bar{\alpha}_k = -\bar{\alpha}_k + d_{k-1}^-\bar{\alpha}_k.$$

If $t_k = +1$ then we simply switch the roles of $s_1(k)$ and $s_3(k)$ and replace d_{k-1}^-, d_k^+ by $-d_{k-1}^-, -d_k^+$, respectively, in what follows.

`Case 1.` Suppose that $d_{k-1}^-,\ d_k^+ \leq \frac{1}{R}$.

Then

$$s_1(k) = \frac{\left(1-2\bar{\alpha}_k + d_{k-1}^-\bar{\alpha}_k\right)\left(1-\alpha_k+d_k^+\right)}{4(1-\bar{\alpha}_k\alpha_k)} \leq \frac{\left(1-\bar{\alpha}_k(2-\frac{1}{R})\right)\left(1-\alpha_k+\frac{1}{R}\right)}{4(1-\bar{\alpha}_k\alpha_k)}.$$

Since $\bar{\alpha}_k(1+\frac{1}{R}) < 1$ and $\bar{\alpha}_k > \frac{1}{R}$ we get

$$\begin{aligned}s_1(k) &< \frac{\left(1-\bar{\alpha}_k\left(2-\frac{1}{R}\right)\right)\left(1-\alpha_k+\frac{1}{R}\right)}{4\left(1-\alpha_k/\left(1+\frac{1}{R}\right)\right)} = \frac{1}{4}\left(1-\bar{\alpha}_k\left(2-\frac{1}{R}\right)\right)\left(1+\frac{1}{R}\right) \\ &< \frac{1}{4}\left(1-\frac{1}{R}\left(2-\frac{1}{R}\right)\right)\left(1+\frac{1}{R}\right) = C(R).\end{aligned}$$

`Case 2.` Suppose that d_{k-1}^{-}, $d_k^{+} \geq \frac{1}{R}$.

Then

$$s_3(k) = \frac{\left(1 - d_{k-1}^{-}\bar{\alpha}_k\right)\left(1 - \alpha_k - d_k^{+}\right)}{4(1 - \bar{\alpha}_k\alpha_k)} \leq \frac{\left(1 - \frac{\bar{\alpha}_k}{R}\right)\left(1 - \alpha_k - \frac{1}{R}\right)}{4(1 - \bar{\alpha}_k\alpha_k)}.$$

This time, using $\bar{\alpha}_k(1 - \frac{1}{R}) < 1$ and $\bar{\alpha}_k > \frac{1}{R}$, we get

$$s_3(k) < \frac{\left(1 - \frac{\bar{\alpha}_k}{R}\right)\left(1 - \alpha_k - \frac{1}{R}\right)}{4\left(1 - \alpha_k/(1 - \frac{1}{R})\right)} = \frac{1}{4}\left(1 - \frac{\bar{\alpha}_k}{R}\right)\left(1 - \frac{1}{R}\right) < C(R).$$

`Case 3.` Suppose that $d_{k-1}^{-} \leq \frac{1}{R}$ and $d_k^{+} \geq \frac{1}{R}$.

We observe that

$$d_k^{-} = -\bar{\alpha}_k + d_{k-1}^{-}\bar{\alpha}_k \leq -\left(1 - \frac{1}{R}\right)\bar{\alpha}_k,$$

and

$$\min\{s_1(k), s_3(k)\} \leq \sqrt{s_1(k)s_3(k)} = \frac{1}{4}\sqrt{S},$$

with

$$\begin{aligned} S &= \frac{\left((1 - \bar{\alpha}_k)^2 - (d_k^{-})^2\right)\left((1 - \alpha_k)^2 - (d_k^{+})^2\right)}{(1 - \bar{\alpha}_k\alpha_k)^2} \\ &\leq \frac{\left((1 - \bar{\alpha}_k)^2 - \left(1 - \frac{1}{R}\right)^2\bar{\alpha}_k^2\right)\left((1 - \alpha_k)^2 - \frac{1}{R^2}\right)}{(1 - \bar{\alpha}_k\alpha_k)^2}. \end{aligned}$$

Hence we have

$$\begin{aligned} S &< \left((1 - \bar{\alpha}_k)^2 - \left(1 - \frac{1}{R}\right)^2\bar{\alpha}_k^2\right)\left(1 - \frac{1}{R^2}\right) \\ &< \left(\left(1 - \frac{1}{R}\right)^2 - \left(1 - \frac{1}{R}\right)^2\frac{1}{R^2}\right)\left(1 - \frac{1}{R^2}\right) = C(R)^2, \end{aligned} \tag{10}$$

the first inequality in (10) holding since

$$\begin{aligned} &(1 - \bar{\alpha}_k\alpha_k)^2\left(1 - \frac{1}{R^2}\right) - \left((1 - \alpha_k)^2 - \frac{1}{R^2}\right) \\ &\qquad = \alpha_k\left(2 - (2 - \bar{\alpha}_k\alpha_k)\bar{\alpha}_k\left(1 - \frac{1}{R^2}\right) - \alpha_k\right) > 0, \end{aligned}$$

and the second since $\bar{\alpha}_k > 1/R$ and

$$f(x) = (1 - x)^2 - \left(1 - \frac{1}{R}\right)^2 x^2 = 1 - 2x + \left(\frac{2}{R} - \frac{1}{R^2}\right)x^2$$

is plainly decreasing on $0 \le x \le \frac{1}{2}$.

`Case 4.` Suppose that $d_{k-1}^- \ge \frac{1}{R}$ and $d_k^+ \le \frac{1}{R}$.

This is almost the same as Case 3, except we observe that

$$d_{k-1}^+ = (t_k + d_k^+)\alpha_{k-1} \le -\left(1 - \frac{1}{R}\right)\alpha_{k-1},$$

and use

$$\min\{s_1(k-1), s_3(k-1)\} \le \sqrt{s_1(k-1)s_3(k-1)} = \frac{1}{4}\sqrt{S},$$

with

$$\begin{aligned} S &= \frac{\left((1-\bar{\alpha}_{k-1})^2 - (d_{k-1}^-)^2\right)\left((1-\alpha_{k-1})^2 - (d_{k-1}^+)^2\right)}{(1-\bar{\alpha}_{k-1}\alpha_{k-1})^2} \\ &\le \frac{\left((1-\bar{\alpha}_{k-1})^2 - \frac{1}{R^2}\right)\left((1-\alpha_{k-1})^2 - (1-\frac{1}{R})^2\alpha_{k-1}^2\right)}{(1-\bar{\alpha}_{k-1}\alpha_{k-1})^2}. \end{aligned}$$

The proof follows, using $\bar{\alpha}_{k-1}$ and α_{k-1} in place of α_k and $\bar{\alpha}_k$, respectively. □

Proof of Theorem 2. Suppose that α and γ_* have $a_{2k-1} = R$, $t_{2k-1} = -1$, $a_{2k} = NR$, and $t_{2k} = N$ for all k. Plainly, as $N \to \infty$,

$$\bar{\alpha}_{2k-1}, \alpha_{2k} = \frac{1}{R - \dfrac{1}{NR - O(1)}} \to \frac{1}{R}, \quad \alpha_{2k-1}, \bar{\alpha}_{2k} = \frac{1}{NR - O(1)} \to 0,$$

while for $k \ge 2$

$$\begin{aligned} d_{2k-1}^- &= -\bar{\alpha}_{2k-1} + (N + d_{2k-3}^-)\bar{\alpha}_{2k-2}\bar{\alpha}_{2k-1} \\ &= \bar{\alpha}_{2k-1}\left(-1 + \frac{N + O(1)}{NR - O(1)}\right) \to -\frac{1}{R} + \frac{1}{R^2}, \end{aligned}$$

and

$$d_{2k-1}^+ = (N + d_{2k}^+)\alpha_{2k-1} = \frac{N + O(1)}{NR - O(1)} \to \frac{1}{R}.$$

Likewise $d_{2k}^+ \to -\frac{1}{R} + \frac{1}{R^2}$ and $d_{2k}^- \to \frac{1}{R}$. So

$$\begin{aligned} s_1(2k-1), s_1(2k) &\to \frac{1}{4}\left(1 - \frac{2}{R} + \frac{1}{R^2}\right)\left(1 + \frac{1}{R}\right) = C(R), \\ s_3(2k-1), s_3(2k) &\to \frac{1}{4}\left(1 - \frac{1}{R^2}\right)\left(1 - \frac{1}{R}\right) = C(R), \end{aligned}$$

while $s_4(2k-1) > s_2(2k-1)$ and $s_2(2k) > s_4(2k)$ with

$$s_2(2k-1), s_4(2k) \to \frac{1}{4}\left(1+\frac{1}{R^2}\right)\left(1-\frac{1}{R}\right) > C(R).$$

Hence as $N \to \infty$

$$M(\alpha,\gamma_*) = \liminf_{k\to\infty} \min\{s_1(k), s_2(k), s_3(k), s_4(k)\} \to C(R). \qquad \square$$

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