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REMARKS ON THE PÓLYA–VINOGRADOV INEQUALITY

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Abstract

We establish a numerically explicit version of the Pólya–Vinogradov inequality for the sum of values of a Dirichlet character on an interval. While the technique of proof is essentially that of Landau from 1918, the result we obtain has better constants than in other numerically explicit versions that have been found more recently.

– Dedicated to Mel Nathanson on his 65th birthday

1. Introduction

Let χ be a non-principal Dirichlet character to the modulus q and let

$$S(\chi) = \max_{M,N} \left| \sum_{a=M}^{N} \chi(a) \right|.$$

The Pólya–Vinogradov inequality (independently discovered by Pólya and Vinogradov in 1918) asserts that there is a universal constant c such that

$$S(\chi) \le cq^{1/2} \log q. \tag{1}$$

This inequality is remarkable for several reasons: it is relatively easy to prove, it is surprisingly strong, and it has many applications. To indicate its strength, it is known that $S(\chi) \gg q^{1/2}$ for χ primitive, a result of Montgomery and Vaughan [10]; this shows that apart from the log factor in (1), the inequality is best possible. Montgomery and Vaughan [9] have also shown on the Riemann Hypothesis for Dirichlet L-functions that $S(\chi) \ll q^{1/2} \log\log q$. This contrasts with a result of

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Paley from 1932 that $|S(\chi)| \gg q^{1/2} \log\log q$ for infinitely many quadratic characters χ.

There has been exciting recent work on improving the order of magnitude in (1). In [5], Granville and Soundararajan show that one can save a small power of $\log q$ in (1) when χ has odd order. This result has been improved by Goldmakher [4], who has shown that for each fixed odd number $g > 1$, for χ of order g,

$$S(\chi) \le q^{1/2} (\log q)^{\Delta_g + o(1)}, \quad \Delta_g = \frac{g}{\pi} \sin \frac{\pi}{g}, \quad q \to \infty.$$

Given the usefulness of the Pólya–Vinogradov inequality it is interesting to prove a version of it that is completely numerically explicit. Qiu [12] has shown that for χ primitive, $|S(\chi)| \le (4/\pi^2) q^{1/2} \log q + 0.38 q^{1/2}$ plus some lesser, but explicit terms. In a series of two papers, the later being [13], Simalarides obtained the leading coefficient $1/\pi$ in the case where the interval is $[0, N]$ and $\chi(-1) = -1$.

In [2], Bachman and Rachakonda show that

$$|S(\chi)| < \frac{1}{3 \log 3} q^{1/2} \log q + 6.5 q^{1/2},$$

the emphasis being on the method that was previously introduced by Dobrowolski and Williams. Working through the details of their proof several years ago (unpublished), I was able to make the small improvement to

$$\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \le \frac{2}{3 \log 3} q^{1/2} \log \frac{N}{q^{1/2}} + \frac{7}{3} q^{1/2}$$

for $q \ge 22{,}500$ and $N \ge q^{1/2}$. When $N/q^{1/2}$ is not too large, this inequality gives an explicit bound with very good constants. (I recently learned that an almost identical result appeared in Vinogradov [14].)

It is interesting that some early papers on the Pólya–Vinogradov inequality, though not advertised as such, had numerically explicit estimates, or estimates that could be easily made numerically explicit. For example, in [8], Landau recapitulates Pólya's argument, getting for χ primitive,

$$S(\chi) \le \frac{1}{\pi} q^{1/2} \log q + \frac{2}{\pi} q^{1/2} \log\log q + 2 q^{1/2}, \tag{2}$$

see page 83. (To achieve this, on page 81 one should *not* replace the denominator $(n+1)x$ with nx.) Landau goes on to improve the leading coefficient $1/\pi$ in (2) to $1/(2^{1/2}\pi)$ in the case of even characters and $1/(2\pi)$ in the case of odd characters. (A character is even or odd if it is an even or odd function, respectively, and this is determined by whether $\chi(-1) = 1$ or -1, respectively.) With a little effort, explicit secondary terms can be worked out here. And with this effort, one would have an explicit inequality that majorizes the others that are described above.

In [6], Hildebrand mentions that an unpublished observation of Bateman allows an improvement of the constant $1/(2^{1/2}\pi)$ in the case of even characters to $2/\pi^2$. Namely, Bateman replaces an estimate for $\sum_{m=1}^{n} |\sin m\theta|/m$ in Landau's paper with a better one found in the book of Pólya–Szego [11]. In this note we carefully work through the Landau–Bateman argument with an eye towards a numerically explicit final result. We prove the following theorem.

Theorem 1. *For χ a primitive character to the modulus $q > 1$, we have*

$$S(\chi) \leq \begin{cases} \dfrac{2}{\pi^2} q^{1/2} \log q + \dfrac{4}{\pi^2} q^{1/2} \log\log q + \dfrac{3}{2} q^{1/2}, & \chi \text{ even}, \\ \dfrac{1}{2\pi} q^{1/2} \log q + \dfrac{1}{\pi} q^{1/2} \log\log q + q^{1/2}, & \chi \text{ odd}. \end{cases}$$

Let

$$S_0(\chi) = \max_N \left| \sum_{a=0}^{N} \chi(a) \right|,$$

so that we always have $S(\chi) \leq 2S_0(\chi)$ via the triangle inequality. In the case of even characters, this may be reversed, that is, $S(\chi) = 2S_0(\chi)$. Thus, the above inequalities for $S(\chi)$ above may be divided by 2 for χ even when considering an initial interval.

2. Preliminaries

Let

$$C_n(x) = \sum_{j=1}^{n} \frac{\cos jx}{j}.$$

It is easy to see that for all x, $C_n(x) \leq C_n(0) = \sum_{j=1}^{n} 1/j$, which tends to infinity like $\log n$ as n tends to infinity. However, we are concerned with a lower bound for $C_n(x)$. As a function of x, $C_n(x)$ is periodic with period 2π and satisfies $C_n(2\pi - x) = C_n(x)$. Thus, it suffices to consider its minimum on the interval $[0, \pi]$. In 1913, Young [16] showed that $C_n(x) \geq -1$ for all n and x. This is best possible, since $C_1(\pi) = -1$. However, if $n > 1$, one can do slightly better, as shown by Brown and Koumandos [3]; they have $C_n(x) \geq -5/6$ in this case. This too is best possible, since $C_3(\pi) = -5/6$. We begin with the following simple result which is useful when n is large.

Lemma 2. *For all real numbers x and all positive integers n, we have*

$$C_n(x) > -\log 2 - \frac{2}{n}.$$

Proof. As noted above, we may assume that $0 \le x \le \pi$. In [16] it is shown that the minimum of $C_n(x)$ on this interval occurs at

$$x_n = \left\lfloor \frac{n+1}{2} \right\rfloor \frac{2\pi}{n+1}$$

(also see [11], Problem 27 in Part VI). First assume that n is odd. Then $x_n = \pi$ and

$$C_n(x_n) = \sum_{j=1}^{n} \frac{(-1)^j}{j}.$$

This alternating series converges to $-\log 2$. The last term is negative, so the finite sum is slightly below the limiting value by an amount less than the first of the terms from $n+1$ to infinity, which is $1/(n+1)$. Thus, $C_n(x) > -\log 2 - 1/(n+1)$.

Now assume that n is even. Then, $n-1$ is odd, so that

$$C_n(x) = C_{n-1}(x) + \frac{\cos nx}{n} > -\log 2 - \frac{1}{n} + \frac{\cos nx}{n} \ge -\log 2 - \frac{2}{n},$$

which proves the lemma. □

We remark that while it is not hard to improve the expression $-2/n$ in the lemma, the number $-\log 2$ is best possible since $C_n(\pi) \to -\log 2$ as $n \to \infty$.

We use Lemma 2 to get an upper bound for the series

$$S_n(x) = \sum_{j=1}^{n} \frac{|\sin jx|}{j},$$

following the outline in [11], Problems 34 and 38 in Part VI.

Lemma 3. *For all real numbers x and positive integers n, we have*

$$S_n(x) < \frac{2}{\pi} \log n + \frac{2}{\pi} \left(\gamma + \log 2 + \frac{3}{n} \right),$$

where γ is the Euler–Mascheroni constant.

Proof. We begin with the Fourier expansion for $|\sin\theta|$ as recorded in [11], Problem 34 in Part VI:

$$|\sin\theta| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2m\theta}{4m^2 - 1}.$$

Thus,

$$S_n(x) = \frac{2}{\pi} \sum_{j=1}^{n} \frac{1}{j} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{4m^2 - 1} \sum_{j=1}^{n} \frac{\cos 2mjx}{j}.$$

Using Lemma 2, we thus have

$$S_n(x) < \frac{2}{\pi}\sum_{j=1}^{n}\frac{1}{j} + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\log 2 + 2/n}{4m^2-1} = \frac{2}{\pi}\sum_{j=1}^{n}\frac{1}{j} + \frac{2}{\pi}\left(\log 2 + \frac{2}{n}\right).$$

It remains to note that the harmonic series satisfies

$$\sum_{j=1}^{n}\frac{1}{j} < \log n + \gamma + \frac{1}{n}. \tag{3}$$

□

For a real number x and positive integer n, let

$$R_n(x) = \sum_{j=n+1}^{\infty}\frac{\sin jx}{j}.$$

In [8, page 81], Landau follows Pólya's argument, getting an estimate for the sum of $|R_n(x)|$ at equally spaced points in $(0,\pi]$. Here we do slightly better.

Lemma 4. *If $q > 1$ is an integer, then*

$$\sum_{a=1}^{q-1}\left|R_n\left(\frac{2\pi a}{q}\right)\right| < \frac{2}{\pi}\frac{q}{n+1}\left(\log q + \log\left(\frac{4}{\pi}\right) + \frac{\pi^2}{12q^2}\right).$$

Proof. Following the argument in [8], we have for $0 < x \le \pi$ and every positive integer n,

$$|R_n(x)| \le \frac{1}{(n+1)\sin(x/2)}.$$

Since $R_n(2\pi - x) = -R_n(x)$, the same estimate holds when $\pi < x < 2\pi$. Thus,

$$\sum_{a=1}^{q-1}\left|R_n\left(\frac{2\pi a}{q}\right)\right| \le \frac{1}{n+1}\sum_{a=1}^{q-1}\frac{1}{\sin(\pi a/q)}. \tag{4}$$

On $(0,\pi)$ the function $\csc x$ is concave up. Thus, for a an integer with $1 \le a \le q-1$, we have

$$\csc(\pi a/q) < \frac{q}{\pi}\int_{\pi(a-1/2)/q}^{\pi(a+1/2)/q}\csc t\,\mathrm{d}t,$$

so that

$$\sum_{a=1}^{q-1}\frac{1}{\sin(\pi a/q)} < \frac{q}{\pi}\int_{\pi/(2q)}^{\pi-\pi/(2q)}\csc t\,\mathrm{d}t = \frac{2q}{\pi}\log\left(\frac{1+\cos(\pi/(2q))}{\sin(\pi/(2q))}\right).$$

Using $\cos t < 1$ and $\sin t > t - t^3/6$ for $t > 0$, we have

$$\sum_{a=1}^{q-1} \frac{1}{\sin(\pi a/q)} < \frac{2q}{\pi} \log\left(\frac{4q}{\pi} \frac{1}{1-(\pi/(2q))^2/6}\right).$$

And using $\log(1/(1-t)) < \log(1+2t) < 2t$ for $0 < t < 1/2$, we have

$$\sum_{a=1}^{q-1} \frac{1}{\sin(\pi a/q)} < \frac{2q}{\pi} \left(\log q + \log\left(\frac{4}{\pi}\right) + \frac{\pi^2}{12q^2}\right). \tag{5}$$

Thus, the result follows from (4). □

The lemma may be slightly improved using Alzer and Koumandos [1, Lemma 2]: the expression $\log(4/\pi) + \pi^2/(12q^2)$ $(= 0.24156\cdots + o(1))$ may be replaced with $\gamma - \log(\pi/2)$ $(= 0.12563\ldots)$. (An earlier result of Watson [15] had $\gamma - \log(\pi/2) + o(1)$.)

Let q be an integer with $q \geq 3$ and let M, N be numbers with $0 \leq M < N < q$. Let $\Phi_{q,M,N} = \Phi$ be the function on $[0, 2\pi)$ satisfying

$$\Phi(x) = \begin{cases} 1/2, & \text{if } x = 2\pi M/q \text{ or } x = 2\pi N/q \\ 1, & \text{if } 2\pi M/q < x < 2\pi N/q \\ 0, & \text{otherwise.} \end{cases}$$

Then, as in Landau [8], page 82, the real Fourier series for $\Phi(x)$ is

$$a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx),$$

where $a_0 = (N-M)/q$ and for $m \geq 1$,

$$\begin{aligned} a_m &= \frac{1}{\pi m}\left(\sin\frac{2\pi Nm}{q} - \sin\frac{2\pi Mm}{q}\right), \\ b_m &= \frac{-1}{\pi m}\left(\cos\frac{2\pi Nm}{q} - \cos\frac{2\pi Mm}{q}\right). \end{aligned} \tag{6}$$

Further, this series converges to $\Phi(x)$ uniformly on $[0, 2\pi)$.

Recall that if χ is a Dirichlet character to the modulus q, then the Gauss sum $\tau(\chi)$ is defined as

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e(a/q),$$

where $e(x) = e^{2\pi i x}$. If χ is a primitive character with modulus q, then

$$|\tau(\chi)| = q^{1/2}, \quad \overline{\chi}(m)\tau(\chi) = \sum_{a=1}^{q} \chi(a) e(am/q), \tag{7}$$

for every integer m, see [7, (3.12) and (3.14)].

Suppose that χ is primitive with modulus $q > 1$ (and so $q \geq 3$). Using that cos is even and sin is odd, we have for every integer m,

$$\sum_{a=1}^{q} \chi(a) \sin \frac{2\pi am}{q} = \sum_{-q/2<a<q/2} \chi(a) \sin \frac{2\pi am}{q} = 0 \tag{8}$$

when χ is even, and

$$\sum_{a=1}^{q} \chi(a) \cos \frac{2\pi am}{q} = \sum_{-q/2<a<q/2} \chi(a) \cos \frac{2\pi am}{q} = 0 \tag{9}$$

when χ is odd. Thus, from (7), (8), (9), and using $e(x) = \cos 2\pi x + i \sin 2\pi x$,

$$\overline{\chi}(m)\tau(\chi) = \begin{cases} \displaystyle\sum_{a=1}^{q} \chi(a) \cos \frac{2\pi am}{q}, & \text{if } \chi \text{ is even,} \\ \\ \displaystyle i \sum_{a=1}^{q} \chi(a) \sin \frac{2\pi am}{q}, & \text{if } \chi \text{ is odd.} \end{cases} \tag{10}$$

3. The Case of Even Characters

Suppose that χ is even and primitive mod q with $q > 1$ (and so $q \geq 5$). We treat the case when the interval $[M, N]$ is an initial interval; that is, $M = 0$. As remarked at the end of the Introduction, after multiplying by 2, our estimate will stand for $S(\chi)$.

We introduce the function $\Phi(x)$ discussed above. With a_m, b_m the coefficients in (6), we have for any positive integer n,

$$\begin{aligned} \sum_{a=0}^{N} \chi(a) &= \frac{1}{2}\chi(N) + \sum_{a=1}^{q} \chi(a) \Phi\left(\frac{2\pi a}{q}\right) \\ &= \frac{1}{2}\chi(N) + \sum_{a=1}^{q} \chi(a) \left(a_0 + \sum_{m=1}^{n} \left(a_m \cos \frac{2\pi am}{q} + b_m \sin \frac{2\pi am}{q} \right) \right) \\ &\quad + \sum_{a=1}^{q} \chi(a) \sum_{m=n+1}^{\infty} \left(a_m \cos \frac{2\pi am}{q} + b_m \sin \frac{2\pi am}{q} \right) \\ &= \frac{1}{2}\chi(N) + A_n + B_n, \text{ say.} \end{aligned}$$

Using (8) and (10),

$$A_n = \sum_{m=1}^{n} a_m \overline{\chi}(m)\tau(\chi).$$

Thus, by (7), (6), and Lemma 3,

$$|A_n| = q^{1/2}\left|\sum_{m=1}^{n} a_m\overline{\chi}(m)\right| \le q^{1/2}\sum_{m=1}^{n}\frac{1}{\pi m}\left|\sin\frac{2\pi Nm}{q}\right|$$
$$\le \frac{2}{\pi^2}q^{1/2}\left(\log n+\gamma+\log 2+\frac{3}{n}\right).$$

Entering the coefficients (6) in the expression for B_n we get

$$B_n = \frac{1}{\pi}\sum_{a=1}^{q}\chi(a)\sum_{m=n+1}^{\infty}\left(\frac{1}{m}\sin\frac{2\pi(N-a)m}{q}+\frac{1}{m}\sin\frac{2\pi am}{q}\right)$$
$$= \frac{1}{\pi}\sum_{a=1}^{q}\chi(a)\left(R_n\left(\frac{2\pi(N-a)}{q}\right)+R_n\left(\frac{2\pi a}{q}\right)\right).$$

Thus, by Lemma 4,

$$|B_n| \le \frac{1}{\pi}\sum_{a=1}^{q}\left(\left|R_n\left(\frac{2\pi(N-a)}{q}\right)\right|+\left|R_n\left(\frac{2\pi a}{q}\right)\right|\right) = \frac{2}{\pi}\sum_{a=1}^{q}\left|R_n\left(\frac{2\pi a}{q}\right)\right|$$
$$= \frac{2}{\pi}\sum_{a=1}^{q-1}\left|R_n\left(\frac{2\pi a}{q}\right)\right| < \frac{4}{\pi^2}\frac{q}{n+1}\left(\log q+\log\left(\frac{4}{\pi}\right)+\frac{\pi^2}{12q^2}\right).$$

Putting these estimates together, we have

$$\left|\sum_{a=0}^{N}\chi(a)\right| \le \frac{1}{2}+|A_n|+|B_n|$$
$$< \frac{1}{2}+\frac{2}{\pi^2}q^{1/2}\left(\log n+\gamma+\log 2+\frac{3}{n}\right)$$
$$+\frac{4}{\pi^2}\frac{q}{n+1}\left(\log q+\log\left(\frac{4}{\pi}\right)+\frac{\pi^2}{12q^2}\right).$$

We now choose $n = \lfloor 2q^{1/2}\log q\rfloor$, so that

$$\left|\sum_{a=0}^{N}\chi(a)\right| < \frac{2}{\pi^2}q^{1/2}\left(\frac{1}{2}\log q+\log\log q\right)$$
$$+\frac{2}{\pi^2}q^{1/2}\left(\log 4+\gamma+1+\frac{3}{n}+\frac{\log(4/\pi)}{\log q}+\frac{\pi^2}{12q^2\log q}\right)+\frac{1}{2}$$
$$< \frac{1}{\pi^2}q^{1/2}\log q+\frac{2}{\pi^2}q^{1/2}\log\log q+\frac{3}{4}q^{1/2},$$

for $q \ge 21$. An inspection of the cases of even primitive characters with conductor $q < 21$ shows that the estimate holds here as well. (In fact, each such $S_0(\chi)$ is smaller than the least significant term $\frac{3}{4}q^{1/2}$ according to calculations kindly supplied by Jonathan Bober and Leo Goldmakher.) This proves Theorem 1 in the case of even characters.

4. Odd Characters

Now we consider the case of an odd primitive character χ. We consider a general interval $[M, N]$ in $[0, q)$. As with the even case, we have

$$\begin{aligned}
&\sum_{a=M}^{N} \chi(a) \\
&\quad = \frac{1}{2}\chi(M) + \frac{1}{2}\chi(N) + \sum_{m=1}^{n}\sum_{a=1}^{q} \chi(a)\left(a_m \cos\frac{2\pi am}{q} + b_m \sin\frac{2\pi am}{q}\right) \\
&\qquad + \frac{1}{\pi}\sum_{a=1}^{q}\chi(a)\left(R_n\left(\frac{2\pi(N-a)}{q}\right) - R_n\left(\frac{2\pi(M-a)}{q}\right)\right) \\
&\quad = \frac{1}{2}\chi(M) + \frac{1}{2}\chi(N) + C_n + D_n, \text{ say.}
\end{aligned}$$

Using (9), (10), and (6) we have

$$C_n = \frac{i\tau(\chi)}{\pi}\sum_{m=1}^{n}\frac{\overline{\chi}(m)}{m}\left(\cos\frac{2\pi Nm}{q} - \cos\frac{2\pi Mm}{q}\right),$$

so that by (7), we have

$$|C_n| \le \frac{q^{1/2}}{\pi}\sum_{m=1}^{n}\frac{1}{m}|\cos m\alpha - \cos m\beta|,$$

where $\alpha = 2\pi N/q$ and $\beta = 2\pi M/q$. Now

$$\begin{aligned}
\sum_{m=1}^{n}\frac{1}{m}|\cos m\alpha - \cos m\beta| &= 2\sum_{m=1}^{n}\frac{1}{m}\left|\sin\frac{m(\alpha+\beta)}{2}\sin\frac{m(\alpha-\beta)}{2}\right| \\
&\le 2\left(\sum_{m=1}^{n}\frac{1}{m}\sin^2\frac{m(\alpha+\beta)}{2}\right)^{1/2}\left(\sum_{m=1}^{n}\frac{1}{m}\sin^2\frac{m(\alpha-\beta)}{2}\right)^{1/2} \\
&= 2\left(\sum_{m=1}^{n}\frac{1}{2m}(1-\cos m(\alpha+\beta))\right)^{1/2}\left(\sum_{m=1}^{n}\frac{1}{2m}(1-\cos m(\alpha-\beta))\right)^{1/2} \\
&\le \log n + \gamma + \log 2 + \frac{3}{n},
\end{aligned}$$

using Lemma 2 and (3) for the last step. Thus,

$$|C_n| \le \frac{q^{1/2}}{\pi}\left(\log n + \gamma + \log 2 + \frac{3}{n}\right).$$

The expression for D_n is estimated as with B_n, getting

$$|D_n| \le \frac{4}{\pi^2}\frac{q}{n+1}\left(\log q + \log\left(\frac{4}{\pi}\right) + \frac{\pi^2}{12q^2}\right).$$

We now take $n = \lfloor (4/\pi) q^{1/2} \log q \rfloor$. Putting the estimates together, we have for an odd primitive character χ,

$$\left| \sum_{a=M}^{N} \chi(a) \right| \le \frac{1}{2\pi} q^{1/2} \log q + \frac{1}{\pi} q^{1/2} \left(\log\log q + \log\left(\frac{4}{\pi}\right) + \gamma + \log 2 + \frac{3}{n} \right) + \frac{1}{\pi} q^{1/2} \left(1 + \frac{\log(4/\pi)}{\log q} + \frac{\pi^2}{12 q^2 \log q} \right) + 1.$$

Collecting the significant terms, we have

$$\left| \sum_{a=M}^{N} \chi(a) \right| \le \frac{1}{2\pi} q^{1/2} \log q + \frac{1}{\pi} q^{1/2} \log\log q + q^{1/2}, \tag{11}$$

whenever $q \ge 45$. A check over smaller cases shows that this inequality holds there as well. (According to the calculations supplied by Jonathan Bober and Leo Goldmakher, $S(\chi) < q^{1/2}$ for primitive odd χ with $q \le 12$; the value of the right side of (11) at $q = 13$ exceeds 6 yet $S(\chi) < 6$ for $q \le 28$; and the value of the right side exceeds 10 at $q = 29$ yet $S(\chi) < 10$ for $q \le 44$.)

This completes the proof of Theorem 1 in the case of odd characters.

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