A COMBINATORIAL PROOF OF A PARTITION PERIMETER INEQUALITY

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Abstract

The partition perimeter is a statistic defined to be one less than the sum of the number of parts and the largest part. Recently, Amdeberhan, Andrews, and Ballantine proved the following analog of Glaisher's theorem: for all $m \geq 2$ and $n \geq 1$, there are at least as many partitions with perimeter n and parts repeating fewer than m times as there are partitions with perimeter n with parts not divisible by m. In this work, we provide a combinatorial proof of their theorem by relating the combinatorics of the partition perimeter to that of compositions. Using this technique, we also show that a composition theorem of Huang implies a refinement of another perimeter theorem of Fu and Tang.

1. Introduction

The *partitions* of an integer $n \geq 1$, which are finite sequences of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ where the *parts* λ_i sum to n and satisfy the inequalities $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$, have long been studied for their wealth of enumerative properties. Many seemingly unrelated families of partitions have been shown to be equinumerous. Euler's theorem, which is stated here, is a classic example of this kind of result.

Theorem 1 ([2], Corollary 1.2)**.** *For all $n \geq 1$, there are as many partitions of n with odd parts as there are partitions of n with distinct parts.*

The *perimeter* of a partition is a statistic defined to be the value $\lambda_1 + \ell(\lambda) - 1$ where $\ell(\lambda)$ is the number of parts in λ. This is Straub's definition [8], which is the most convenient for the purposes of this work. Many other partition statistics like the *rank* $\lambda_1 - \ell(\lambda)$ allow infinitely many partitions to take on each value, and so must be studied in addition to partition *size* $n = |\lambda|$. There are only 2^{k-1} partitions

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with perimeter $k \geq 1$, which makes viable the study of partitions by their perimeter alone.

Interest in enumeration by perimeter (see [1, 3, 5]) spawned from the following result of Straub, which is remarkably similar in presentation to Euler's theorem.

Theorem 2 ([8]). *For all $n \geq 1$, there are as many partitions with perimeter n and odd parts as there are partitions with perimeter n and distinct parts.*

Naturally, for any theorem like Euler's we can attempt to replace size with perimeter and show that an analogous result holds. For instance, Glaisher's theorem, presented below as Theorem 3, generalizes Euler's to allow an arbitrary modulus. Theorem 1 is retrieved when $m = 2$.

Theorem 3 ([2], Corollary 1.3). *For all $m \geq 2$ and $n \geq 1$, there are as many partitions of n with parts not divisible by m as there are partitions of n with parts repeating fewer than m times.*

Let $g_m(n)$ and $h_m(n)$ be the sets of partitions with perimeter n having parts not divisible by m and parts repeating fewer than m times, respectively. Amdeberhan, Andrews, and Ballantine [1] showed that Straub's theorem becomes the following inequality when generalized in the direction of Glaisher's theorem.

Theorem 4 ([1]). *We have $h_m(n) - g_m(n) \geq 0$ for all $m \geq 2$ and $n \geq 1$.*

They proved this result by manipulating generating functions, and asked if a combinatorial proof exists. In this work, we provide such a proof. The technique we use appears to have additional utility in this kind of research. As an example, we provide another application to prove a refinement of the following theorem of Fu and Tang. Here, by *gaps between parts* we mean the values $\lambda_i - \lambda_{i+1}$ for $1 \leq i < \ell(\lambda)$.

Theorem 5 ([3]). *For all $m, n \geq 1$, there are as many partitions with perimeter n and parts congruent to 1 modulo $m+1$ as there are partitions with perimeter n and gaps between parts at least m.*

Our refinement, which we state below, adds the additional parameter k, and reduces to Fu and Tang's theorem when $k = 0$, and then to Straub's theorem when $m = 1$. An example may be found in Table 1.

Theorem 6. *For all $n, m \geq 1$ and $k \geq 0$, the following families of partitions with perimeter n are equinumerous:*

(1) Partitions λ with the property that if $\lambda_{\ell(\lambda)} \equiv 1 \pmod{m+1}$, then the the parts must be grouped into $k+1$ non-empty and disjoint sets of sequential parts $\{\lambda_1, \ldots, \lambda_{t_1}\}$, $\{\lambda_{t_1+1}, \ldots, \lambda_{t_2}\}$, $\ldots$, $\{\lambda_{t_k+1}, \ldots, \lambda_{t_{k+1}=\ell(\lambda)}\}$, and otherwise we must have that $\lambda_{\ell(\lambda)} > m+1$ and the parts of λ form k such sets. Each set has parts all belonging to the same congruence class modulo $m+1$, with

neighboring sets having different congruence classes and gaps $\lambda_{t_i} - \lambda_{t_{i+1}} > m$ between them.

(2) *Partitions λ with gaps between parts at least m with exactly k exceptions $\lambda_i - \lambda_{i+1} < m$, always preceded by a gap $\lambda_{i-1} - \lambda_i \geq m$, and followed by a gap $\lambda_{i+1} - \lambda_{i+2} > m$ unless $\ell(\lambda) = i + 1$.*

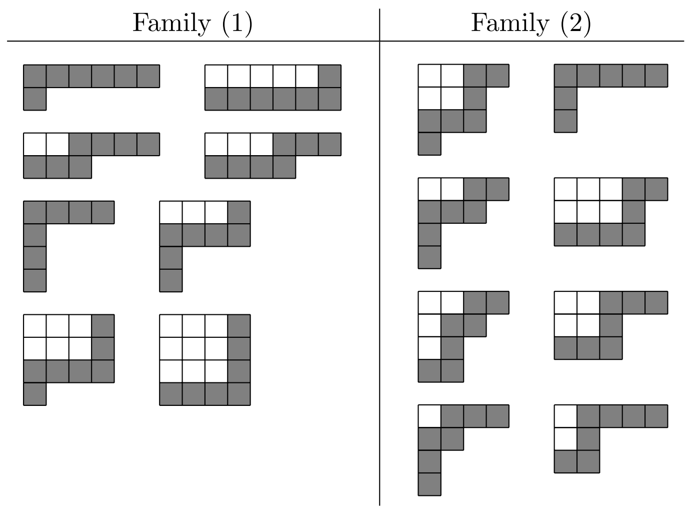

Table 1: Example of the partitions enumerated in Theorem 6 with the parameters $n = 7$, $m = 1$, and $k = 1$.

The rest of this paper is organized into two sections. Section 2 contains a description of the technique used in the proofs, as well as the proof of Theorem 6. The combinatorial proof of Theorem 4 is in section 3, as well as some commentaries.

2. Compositions and the Partition Perimeter

A *composition* of the integer $n \geq 1$ is a partition $c = (c_1, c_2, \ldots, c_\ell)$ of n without the requirement that $c_1 \geq c_2 \geq \cdots \geq c_\ell$. We borrow the same terminology and tools for compositions from partitions when appropriate.

We may visually represent c as a *Young diagram*, which is a grid of squares containing c_i squares in the ith row from the top, aligned on the left. The m-*modular diagram* of c for fixed $m \geq 2$ is a Young diagram where the squares furthest to the right contain a value between 1 and m, and all others contain m, having the property that the sum of the values in the ith row is c_i.

For any partition λ, the *conjugate* λ' is a new partition with λ'_i defined to be the

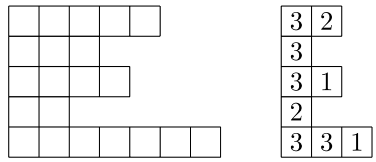

Figure 1: The Young diagram and 3-modular diagram of $c = (5, 3, 4, 2, 7)$.

ith column of λ's Young diagram, from the left. Conjugation is an involution that preserves the perimeter since λ_1 and $\ell(\lambda)$ are swapped.

Remark 1. Conjugation is only defined for partitions, although an analogous operation exists for compositions in general (see the work of Munagi such as [6] and the references therein for recent work).

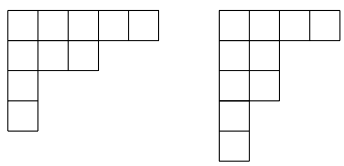

Figure 2: The conjugate partition pair $(5, 3, 1, 1)$ and $(4, 2, 2, 1, 1)$.

The perimeter of a partition λ can be equivalently defined as the number of squares with an edge or corner on the bottom or right side of λ's Young diagram. Let

$$c = (\lambda_1 - \lambda_2 + 1, \lambda_2 - \lambda_3 + 1, \ldots, \lambda_{\ell(\lambda)-1} - \lambda_{\ell(\lambda)} + 1, \lambda_{\ell(\lambda)}).$$

Clearly, c_i counts the number of squares from the ith row of λ's Young diagram that contribute to the perimeter, so c is a composition with the properties $|c| = \lambda_1 + \ell(\lambda) - 1$ and $\ell(c) = \ell(\lambda)$. Indeed, this defines a bijection since each part of λ can easily be recovered using the formula

$$\lambda_i = 1 + \sum_{j=i}^{\ell(c)} (c_j - 1).$$

Let $\mathcal{C}$ and $\mathcal{P}$ be the sets of all compositions and partitions, respectively, and $\pi \colon \mathcal{C} \to \mathcal{P}$ be this bijection. We can also interpret $\pi(c)$ as the partition formed by sliding the rows of c over so they overlap at exactly one square. This is shown in Figure 3.

Using π and these Young diagram interpretations, we can encode information about the partition perimeter into a composition, which allows us to then work entirely with compositions, only switching back with π^{-1} when done.

For an example of how this technique may be used, consider the following composition result of Munagi.

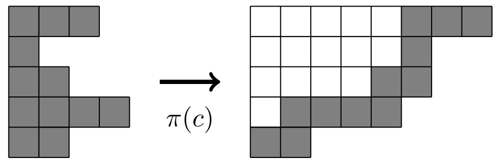

Figure 3: An example of the bijection π with $c = (3, 1, 2, 4, 2)$ and $\pi(c) = (8, 6, 6, 5, 2)$.

Theorem 7 ([7])**.** *For all $n, m \geq 1$ there are as many compositions of n with parts congruent to 1 modulo m as there are compositions of $n + m - 1$ with parts at least m.*

Replacing m with $m + 1$, the composition parts from the first family listed in Theorem 7 become under π partition parts congruent to 1 modulo $m+1$ since they all have the effect of preserving the congruence of the part below. Deleting m from the second family's last part, which is guaranteed to be at least $m + 1$, brings the size down to n. Sliding the parts over as in Figure 3, we obtain partitions with gaps between parts at least m, and so we arrive at Theorem 5.

The proof of Theorem 6 is similar to this, using the following refinement of Munagi's theorem from Huang.

Theorem 8 ([4])**.** *For all $n, m \geq 1$, and $k \geq 0$, there are as many compositions of n with exactly k parts not congruent to 1 modulo m, each larger than m, as there are compositions of $n + m - 1$ with exactly k parts less than m, each proceeded by a part at least m and followed by either a last part or a part larger than m.*

Proof of Theorem 6. Proceed as above, replacing m with $m + 1$ and now using Theorem 8.

For Family (1), any composition parts that are not congruent to 1 modulo $m+1$ have the effect under π of changing the corresponding part's congruence class to be different than the part below's, if there is a part below, so each of the $k+1$ or k sets are formed from a part that is not congruent to 1 modulo $m+1$, followed above by any amount of parts that are congruent to 1 modulo $m+1$.

For Family (2), delete m from the final part of each composition. Each of the k parts that are less than $m + 1$ along with the guaranteed part above that is at least $m+1$ and below that is larger than $m+1$ produce under π a gap at least m, smaller than m, and then larger than m.

The omitted details are straightforward to verify. $\square$

Remark 2. Huang provides a formula (Theorem 1.7 in [4]) that the partitions in Theorem 6 inherit.

One can use this technique in the other direction too, yielding composition results. Although this idea is not pursued in this work, we provide the following simple example.

Proposition 1. *For all $m \geq 2$ and $n, k \geq 1$, there are as many compositions of n with k parts, all at most m and last part less than m, as there are compositions of n with $n-k+1$ parts and no continuous sequence of $m-1$ or more 1's, excluding the last part.*

Proof. Partitions with perimeter n, k parts, and gaps between parts smaller than m conjugate to give partitions with perimeter n, $n-k+1$ parts since $k-1$ squares in the Young diagram have a perimeter contributing square below them, and parts repeating fewer than m times. Interpreting these families through π^{-1} gives the result. $\square$

3. Combinatorial Proof of Theorem 4

Before proceeding, we need the following lemma that we provide a simple bijective proof of. An example is shown in Figure 4.

Lemma 1. *Let $m \geq 2$, $n \geq 1$, and $R \subseteq \{1, 2, \ldots, m-1\}$ be non-empty. There are as many compositions of n with parts each congruent to some element of R modulo m as there are compositions of n with parts in $R \cup \{m\}$ and last part in R.*

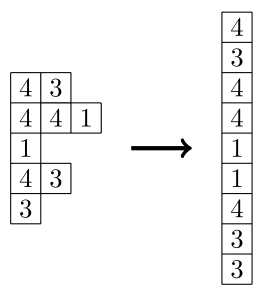

Figure 4: With $m = 4$ and $R = \{1, 3\}$, the composition $(7, 9, 1, 7, 3)$ under the bijection in Lemma 1 maps to $(4, 3, 4, 4, 1, 1, 4, 3, 3)$.

Proof. Let c be a composition with each part congruent to some element of R modulo m. Writing c as an m-modular diagram, rotate each part one quarter turn clockwise, leaving the m-modular diagram of a composition with parts in $R \cup \{m\}$ and last part in R. Since the numbers are unchanged, the resulting composition has the same size.

For the inverse, given a composition c with parts in $R \cup \{m\}$ and last part in R, iterate upwards in the m-modular diagram starting from c's last part. Collect the square not containing m and then all squares containing m directly above, then rotate counter-clockwise to form a part congruent to some element of R modulo m. Repeat until all squares are exhausted. □

Combinatorial proof of Theorem 4. Fix the integers m and n. The proof is built using a series of simple bijections and one injection between six families of compositions and partitions. Partitions with perimeter n and parts not divisible by m become compositions c of n with the property that

$$1 + \sum_{j=i}^{\ell(c)} (c_j - 1) \not\equiv 0 \pmod{m} \text{ for all } 1 \le i \le \ell(c) \tag{$*$}$$

through π^{-1}. The injection φ defined below sends these to compositions of n with parts not divisible by m. Next, using Lemma 1 with $R = \{1, \ldots, m-1\}$, these become compositions of n with all parts at most m and last part smaller than m. Applying π, we then get partitions with perimeter n and gaps between parts (as well as the final part) smaller than m, and then, finally, partitions with perimeter n and parts repeating fewer than m times, by conjugation.

The injection φ is defined as follows. Let c be a composition with the property $(*)$. From c we construct $\varphi(c) = d$ using the following iterative algorithm, starting at c's last part. The algorithm is designed to preserve parts in c that are already not divisible by m, and transform the others in such a way that a part in $\pi(d)$, up to a small technical detail, becomes divisible by m, violating $(*)$ with each occurrence. Here, we use the notation () to mean the *empty composition*, which has no parts and size 0.

1: Initialize with $i = \ell(c)$, $j = 1$, and $d = ()$
2: **while** $i \neq 0$ **do**
3: Let $j = j + c_i - 1$.
4: **if** $c_i \not\equiv 0 \pmod{m}$ **then**
5: Append c_i to the start of d (c_i becomes the new first part of d)
6: **else**
7: Let r be the remainder after dividing j by m
8: Append the part $m - r$ to the start of d
9: Append the part $c_i - (m - r)$ to the start of d
10: **end if**
11: Let $i = i - 1$
12: **end while**

From this construction, that $|c| = |d|$ is immediate. Moreover, from $(*)$, $j \not\equiv 0 \pmod{m}$ will hold at every step, so in the case of lines 7 through 9, r will never be 0, which implies that the parts $m-r$ and $c_i-(m-r)$ are not divisible by m as well. Therefore each part of d is not divisible by m.

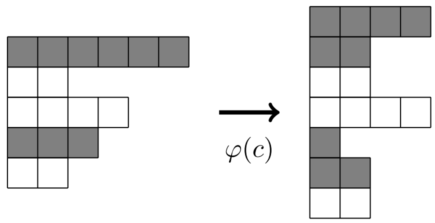

Figure 5: With $m=3$, φ sends $c=(6,2,4,3,2)$ to $d=(4,2,2,4,1,2,2)$.

Suppose that some composition d is in the image of φ. Like in the algorithm, we iterate d_i from d's last part to the top, with the goal of uniquely determining the parts of the preimage. Start with $c=()$ and $j=1$, and increment j by d_i-1 at each step.

If $j \not\equiv 0 \pmod{m}$, then we must be in the case of line 5, so append d_i to the top of c. Otherwise, we are in the case of lines 7 through 9 and so we must append d_i+d_{i-1} to the start of c. We must then subtract 1 from i an additional time, and also add 1 to j since the action of splitting the composition part into two in c has the effect of decreasing j by 1 in d. Continue until all parts of d are exhausted, which leaves a uniquely determined c. This shows injectivity which completes the proof. $\square$

3.1. Closing Remarks

Let $e_m(n)$ count the number of compositions with parts not divisible by m that are not in φ's image. As a consequence of the proof above, we can strengthen Theorem 4 to an equality.

Corollary 1. *We have $h_m(n)-g_m(n)=e_m(n)$ for all $m \geq 2$ and $n \geq 1$.*

A characterization of the compositions d counted by $e_m(n)$ is provided here for the interested reader. Let $1 \leq t_1 < \ell(d)$ be the largest index of d such that $\pi(d)_{t_1} \equiv 0 \pmod{m}$, then let t_2 be the largest index such that $t_1-t_2 \geq 2$ and $\pi(d)_{t_2}+1 \equiv 0 \pmod{m}$, and so on until reaching d's first part with final index $t_k \geq 1$. Since d may not be a fixed point of φ, d does not satisfy $(*)$ so we are guaranteed the sequence $t_1,\ldots,t_k$ exists with at least $k \geq 1$. For an example of this, in Figure 5 we have $k=2$, with $t_1=6$ and $t_2=2$. Once this sequence of indices is obtained, the condition for d to be counted by $e_m(n)$ is $t_k=1$, or for at least one value $1 \leq i \leq k$, $d_{t_i} > m$ or $d_{t_i}+d_{t_r-i} \not\equiv 0 \pmod{m}$.

If $n > m > 2$ in Theorem 4 then the inequality becomes strict since the compositions of size n

$$(m-1, 2, \underbrace{1, \ldots, 1}_{n-m-1 \text{ many } 1' \text{s}})$$

cannot be in the image of φ. This fact can also be obtained from the original proof. If $n \leq m$ both families of partitions coincide giving an equality. Also, if $m = 2$, φ becomes the identity map, which provides another proof of Straub's theorem.

Many papers that treat compositions have noted that the following families of compositions are all enumerated by the nth Fibonacci number F_n (recall that $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for any $n \geq 2$):

(1) compositions of n with odd parts;

(2) compositions of n with parts 1 and 2, and last part 1;

(3) compositions of $n + 1$ with parts greater than 1;

Lemma 1 with $m = 2$ and $R = \{1\}$ gives a bijection from (1) to (2), and by identifying (3) with compositions of n where only the last part may be 1, Proposition 1 with $m = 2$ gives a bijection from (2) to (3).

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