COMBINATORIAL IDENTITIES FOR VACILLATING TABLEAUX

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Abstract

Vacillating tableaux, which are sequences of integer partitions that satisfy specific conditions, arise in the representation theory of the partition algebra and the combinatorial theory of crossings and nestings of matchings and set partitions. By exploring a correspondence between vacillating tableaux and pairs comprising a set partition and a partial Young tableau, we derive combinatorial identities that involve the number of vacillating tableaux, the number of standard Young tableaux, and Schur functions. We also examine analogous problems concerning limiting vacillating tableaux and vacillating tableaux of odd lengths. Various integer sequences arise as counting sequences for the associated combinatorial structures.

1. Introduction

A vacillating tableau is a sequence of integer partitions that must adhere to specific conditions and can be visualized as particular walks on Young's lattice, a lattice

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that consists of all integer partitions ordered by the containment of diagrams. The concept of vacillating tableaux was independently introduced by two research teams around the same time. Halverson and Lewandowski [8] introduced one definition to provide combinatorial proofs of identities arising from the representation theory of the partition algebra $\mathcal{CA}_k(n)$. Meanwhile, Chen, Deng, Du, Stanley, and Yan [4] proposed another definition to characterize the maximal crossings and nestings in the arc-diagrams associated with perfect matchings and set partitions of $[k] := \{1, 2, \ldots, k\}$. It is worth noting that these two definitions are equivalent when $n \geq 2k$. Additionally, in our previous work [3] we introduced the concept of *limiting vacillating tableaux* while studying the intricate properties of a bijection constructed in [8]. These results further motivated our investigation on the enumeration of vacillating tableaux, resulting in a number of combinatorial identities and integer sequences, which we will present here.

We begin by providing the necessary definitions. A *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_t)$ of integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 0$ and $|\lambda| \coloneqq \lambda_1 + \cdots + \lambda_t = n$. We also say that the size of λ is n and denote it as $\lambda \vdash n$. In addition, we let the empty partition $\emptyset$ to be the only integer partition of 0.

A partition λ is visually represented by the Young diagram, which contains λ_j boxes in the j-th row. We adopt the English notation in which the diagrams are aligned in the upper-left corner. A *Young tableau* of shape λ is an array obtained by filling each box of the Young diagram of λ with an integer. A Young tableau is *semistandard* if the entries are weakly increasing in every row and strictly increasing in every column. A semistandard Young tableau (SSYT) is *partial* if the entries are all distinct, and we call it a partial tableau. The *content* of a Young tableau T, denoted as $\mathrm{content}(T)$, is the multiset of all the entries in T. If the content of an SSYT T with shape $\lambda \vdash n$ is exactly $[n]$, then we say T is a *standard Young tableau* (SYT). Throughout this paper, we use f^λ to denote the number of SYTs of shape λ. By convention, we set $f^\emptyset = 1$.

Below is the definition of a vacillating tableau introduced in [8]. To emphasize the dependency on the parameter n, we call such tableaux n-vacillating tableaux.

Definition 1 ([8])**.** For integers $k \geq 0$ and $n \geq 1$, an *n-vacillating tableau of shape λ and length $2k$* is a sequence of $2k+1$ integer partitions

$$((n) = \lambda^{(0)}, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, \lambda^{(1\frac{1}{2})}, \ldots, \lambda^{(k-\frac{1}{2})}, \lambda^{(k)} = \lambda)$$

so that for each $j = 0, 1, \ldots, k-1$,

(a) $\lambda^{(j)} \supseteq \lambda^{(j+\frac{1}{2})}$ and $|\lambda^{(j)}/\lambda^{(j+\frac{1}{2})}| = 1$,

(b) $\lambda^{(j+\frac{1}{2})} \subseteq \lambda^{(j+1)}$ and $|\lambda^{(j+1)}/\lambda^{(j+\frac{1}{2})}| = 1$.

In other words, an n-vacillating tableau of shape λ is a walk on Young's lattice from (n) to λ, where a box is removed in each odd step and added in each even step.

Let $\Lambda_n^k = \{\lambda \vdash n : \lambda_1 \geq n-k\}$ and $\mathcal{VT}_{n,k}(\lambda)$ be the set of all n-vacillating tableaux of shape λ and length $2k$. Note that for any vacillating tableau in $\mathcal{VT}_{n,k}(\lambda)$, $\lambda^{(j)} \in \Lambda_n^k$ and $\lambda^{(j+\frac{1}{2})} \in \Lambda_{n-1}^k$. Let $m_{n,k}^\lambda$ be the cardinality of $\mathcal{VT}_{n,k}(\lambda)$. When $n \geq 2k$, the value of $m_{n,k}^\lambda$ does not depend on n. In that case, there is an equivalent notion of vacillating tableau that does not use the parameter n, which was mentioned in [8] but originally introduced in [4] to study maximal monotone substructures in matchings and set partitions. To distinguish from the n-vacillating tableau, we call this second notion the *simplified vacillating tableau*. Given an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \vdash n$, let λ^* be obtained from λ by removing its first part λ_1, i.e., $\lambda^* = (\lambda_2, \ldots, \lambda_t) \vdash (n - \lambda_1)$. For an n-vacillating tableau $(\lambda^{(j)} : j = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots, k)$ in $\mathcal{VT}_{n,k}(\lambda)$, the corresponding *simplified vacillating tableau* is the sequence $(\mu^{(j)} = (\lambda^{(j)})^* : j = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots, k)$. One can also define the simplified vacillating tableau directly in terms of integer partitions.

Definition 2 ([4])**.** A *simplified vacillating tableau* of shape μ and length $2k$ is a sequence of $2k+1$ integer partitions

$$(\mu^{(0)} = \emptyset, \mu^{(\frac{1}{2})}, \mu^{(1)}, \mu^{(1\frac{1}{2})}, \ldots, \mu^{(k-\frac{1}{2})}, \mu^{(k)} = \mu)$$

so that for each integer $j = 0, 1, \ldots, k-1$,

(a) $\mu^{(j)} \supseteq \mu^{(j+\frac{1}{2})}$ and $|\mu^{(j)}/\mu^{(j+\frac{1}{2})}| = 0$ or 1,

(b) $\mu^{(j+\frac{1}{2})} \subseteq \mu^{(j+1)}$ and $|\mu^{(j+1)}/\mu^{(j+\frac{1}{2})}| = 0$ or 1.

Note that the definition forces $\mu^{(\frac{1}{2})} = \emptyset$ and $|\mu| \leq k$. Equivalently, a simplified vacillating tableau of shape λ is a walk on Young's lattice from $\emptyset$ to λ where in each odd step either zero or one box is removed, and in each even step either zero or one box is added. Let $\mathcal{SVT}_k(\mu)$ be the set of all simplified vacillating tableaux of shape μ and length $2k$, and let $g_k(\mu)$ be the cardinality of $\mathcal{SVT}_k(\mu)$. In general, $m_{n,k}^\lambda \leq g_k(\lambda^*)$, with equality holding when $n \geq 2k$.

The diagram in Figure 1 gives the simplified vacillating tableaux of length $2k$, for $0 \leq k \leq 3$. The number next to each integer partition μ is the value of $g_k(\mu)$.

A limiting vacillating tableau is a special kind of simplified vacillating tableau, which was introduced in [3] from the study of images of a bijection from [8] between sequences in $\{(i_1, \ldots, i_k) : \ 1 \leq i_j \leq n\}$ and pairs (T, P), where T is an SYT of some shape $\lambda \in \Lambda_n^k$, and P is an n-vacillating tableau of shape λ and length $2k$.

Definition 3 ([3])**.** A *limiting vacillating tableau* of shape μ and length $2k$ is a sequence of $2k+1$ integer partitions

$$(\mu^{(0)} = \emptyset, \mu^{(\frac{1}{2})}, \mu^{(1)}, \mu^{(1\frac{1}{2})}, \ldots, \mu^{(k-\frac{1}{2})}, \mu^{(k)} = \mu)$$

so that for each integer $j = 0, 1, \ldots, k-1$,

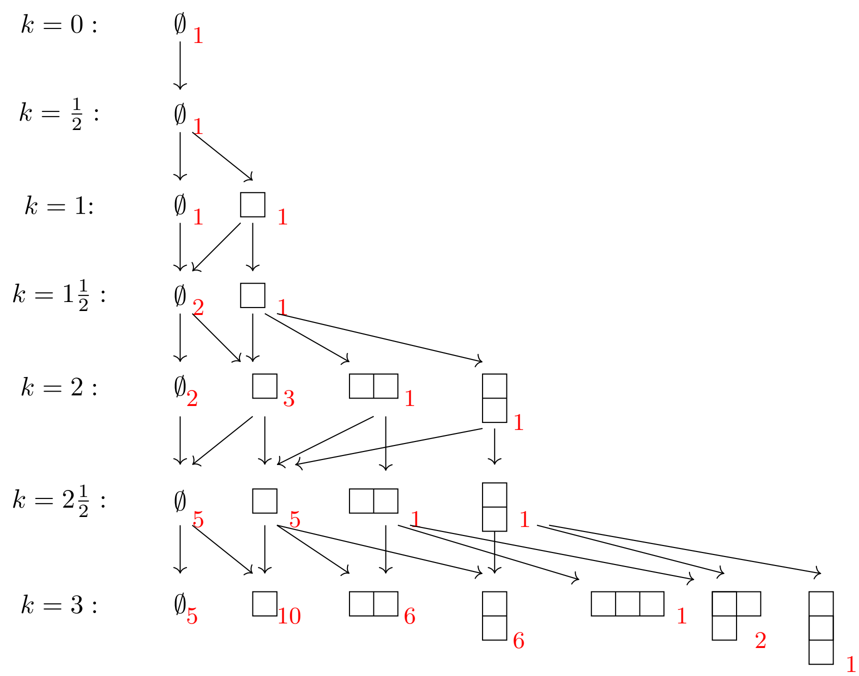


Figure 1: Simplified vacillating tableaux of length $2k$, up to $k = 3$.

(a) $\mu^{(j)} \supseteq \mu^{(j+\frac{1}{2})}$ and $|\mu^{(j)}/\mu^{(j+\frac{1}{2})}| = 0$ or 1,

(b) $\mu^{(j+\frac{1}{2})} \subseteq \mu^{(j+1)}$ and $|\mu^{(j+1)}/\mu^{(j+\frac{1}{2})}| = 1$.

Equivalently, a limiting vacillating tableau of shape λ is a walk on Young's lattice from $\emptyset$ to λ where in each odd step either zero or one box is removed, and in each even step exactly one box is added.

Figure 2 gives the limiting vacillating tableaux of length $2k$, for $0 \leq k \leq 3$. The number next to each integer partition μ is the number of limiting vacillating tableaux ending at μ, which is denoted by $a_k(\mu)$.

The above definitions suggest that we can also consider the numbers of vacillating tableaux and their simplified analogs of odd length, namely, $m^{\lambda}_{n,k+\frac{1}{2}}$ and $g_{k+\frac{1}{2}}(\mu)$, where $g_{k+\frac{1}{2}}(\mu) = m^{\lambda}_{n,k+\frac{1}{2}}$ if $n \geq 2k$ and $\mu = \lambda^*$. Similarly, we define $a_{k+\frac{1}{2}}(\mu)$ as the number of limiting vacillating tableaux of shape μ and length $2k + 1$.

In Section 2, we review some combinatorial algorithms and bijections related to vacillating tableaux. Then we consider various summations concerning $g_k(\mu)$, $g_{k+\frac{1}{2}}(\mu)$, $a_k(\mu)$, and $a_{k+\frac{1}{2}}(\mu)$, obtain new identities, and present combinatorial interpretations. Sections 3 and 4 focus on simplified vacillating tableaux. We will

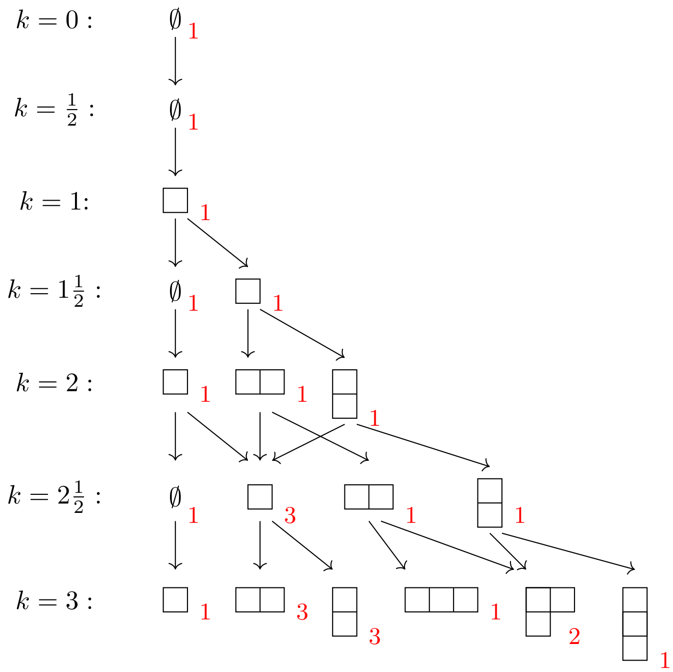


Figure 2: Limiting vacillating tableaux of length $2k$, up to $k = 3$.

specify when there is an analogous result for $m^{\lambda}_{n,k}$ for general n, k (without the constraint that $n \geq 2k$). Sections 5 and 6 focus on limiting vacillating tableaux. More precisely, we consider the following integer sequences.

- The number of simplified vacillating tableaux of lengths $2k$ and $2k+1$: $g_k = \sum_{\mu} g_k(\mu)$ and $g_{k+\frac{1}{2}} = \sum_{\mu} g_{k+\frac{1}{2}}(\mu)$.
- The number of limiting vacillating tableaux of lengths $2k$ and $2k+1$: $a_k = \sum_{\mu} a_k(\mu)$ and $a_{k+\frac{1}{2}} = \sum_{\mu} a_{k+\frac{1}{2}}(\mu)$.
- Sums of products of numbers of simplified vacillating tableaux of shape μ and SYTs of shape μ: $u_k = \sum_{\mu} g_k(\mu) f^{\mu}$ and $u_{k+\frac{1}{2}} = \sum_{\mu} g_{k+\frac{1}{2}}(\mu) f^{\mu}$.
- Sums of products of numbers of limiting vacillating tableaux of shape μ and SYTs of shape μ: $v_k = \sum_{\mu} a_k(\mu) f^{\mu}$ and $v_{k+\frac{1}{2}} = \sum_{\mu} a_{k+\frac{1}{2}}(\mu) f^{\mu}$.

In addition, we present results for the products of the numbers of vacillating tableaux with each other, with f^{μ}, and with Schur function s_{μ}. In Section 7, we discuss the relation between the aforementioned integer sequences and conclude this paper with some final remarks and future research projects.

2. Algorithms and Bijections on Tableaux

In this section, we review combinatorial algorithms and bijections related to Young tableaux and vacillating tableaux.

2.1. The RSK Insertion Algorithm

The well-known RSK row insertion algorithm constructs a pair of tableaux of the same shape from an integer sequence, or more generally, a two-line array of integers. There are two major components: the row insertion procedure, and the RSK algorithm that is an iteration of row insertions.

We first recall the row insertion procedure. Let T be an SSYT of shape λ and x be an integer. The operation $x \xrightarrow{RSK} T$ is defined as follows.

(a) Let R be the first row of T.

(b) While x is less than some entries in R, do:

i) let y be the smallest entry of R greater than x;

ii) replace $y \in R$ with x;

iii) let $x \coloneqq y$ and let R be the next row.

(c) Place x at the end of R (which is possibly empty).

The result is a semistandard tableau of shape μ such that $|\mu/\lambda| = 1$. For each occurrence of step (b), we say that x *bumps* y to the next row.

We will use Knuth's construction [10] that iterates the above row insertion procedure on a two-line array of integers

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}, \tag{1}$$

where (u_j, v_j) are arranged in non-decreasing lexicographic order from left to right, that is, $u_1 \le u_2 \le \cdots \le u_n$ and $v_j \le v_{j+1}$ if $u_j = u_{j+1}$.

The RSK algorithm is described as follows. Given the two-line array (1), construct a pair of Young tableaux (P, Q) of the same shape by starting with $P = Q = \emptyset$. For $j = 1, 2, \ldots, n$,

(a) Insert v_j into tableau P using the row insertion procedure. This operation adds a new box to the shape of P. Assume the new box is at the end of the i-th row of P.

(b) Add a new box with entry u_j at the end of the i-th row of Q.

We call P the *insertion tableau* and Q the *recording tableau*.

Let A and B be two totally ordered alphabets. If $u_j \in A$ and $v_j \in B$ for all j, we say that the two-line array (1) is a *generalized permutation from A to B*.

Theorem 1 ([10])**.** *There exists a one-to-one correspondence between generalized permutations from A to B and pairs of SSYTs (P, Q) of the same shape, where content$(P) \subseteq B$ and content$(Q) \subseteq A$.*

Knuth's correspondence, when restricted to two special families of two-line arrays, gives the correspondences discovered by Robinson [16] and Schensted [20]. These two cases will be used frequently in our proofs.

(i) When $(u_1\ u_2\ \ldots\ u_n) = (1\ 2\ \ldots\ n)$ and $(v_1\ \ldots\ v_n)$ ranges over all permutations of $[n]$, the correspondence gives a bijection between permutations of length n and pairs of SYTs of the same shape. Under this bijection a pair of identical SYTs corresponds to an involution π, that is, a permutation satisfying $\pi^2 = e$ where e is the identity element in $\mathfrak{S}_n$.

(ii) When $(u_1\ u_2\ \ldots\ u_n) = (1\ 2\ \ldots\ n)$ and $v_i \in \mathbb{Z}^+$, the correspondence gives a bijection between sequences of positive integers of length n and pairs of Young tableaux of the same shape $\lambda \vdash n$, where P is an SSYT with content in $\mathbb{Z}^+$, and Q is an SYT.

2.2. Vacillating Tableaux and Set Partitions

An important tool to study the combinatorial properties of simplified vacillating tableaux is the bijection ψ defined in Section 2 of [4], which extends a map of Sundaram [21, Lemma 2.2] on up-down tableaux of arbitrary shapes. The bijection ψ maps simplified vacillating tableaux of shape μ and length $2k$ to pairs $(\boldsymbol{B}, T)$, where $\boldsymbol{B}$ is a partition of $[k]$ and T is a partial tableau of shape μ such that content$(T) \subseteq \max(\boldsymbol{B})$; here $\max(\boldsymbol{B})$ is the set consisting of the maximal element in each block of $\boldsymbol{B}$. In this paper, we modify the definition of ψ so that the partial tableau in the image is always standard. To precisely define the map ψ we begin by stating needed definitions and setting our notation.

Let S be a finite set. A (set) partition of S is a collection $\boldsymbol{B} = \{B_1, B_2, \ldots, B_k\}$ of pairwise disjoint non-empty subsets of S such that $B_1 \cup B_2 \cup \cdots \cup B_k = S$. Each B_i is called a block of $\boldsymbol{B}$. Let $\Pi(S)$ be the set of all partitions of S. For $0 \le j \le k$, let

$$\begin{aligned} \Pi(k,j) &= \{\boldsymbol{B} :\ \boldsymbol{B} \in \Pi([k]) \text{ and } \boldsymbol{B} \text{ has exactly } j \text{ blocks}\}, \\ \Pi^*(k,j) &= \{\boldsymbol{B}^* = (\boldsymbol{B}, A) : \boldsymbol{B} \in \Pi([k]) \text{ and } A \text{ is a subset of } j \text{ blocks of } \boldsymbol{B}\}. \end{aligned}$$

By definition, if $\boldsymbol{B}^* = (\boldsymbol{B}, A) \in \Pi^*(k,j)$, then $\boldsymbol{B}$ has at least j blocks. We say that each block in A is marked and call $\boldsymbol{B}^* \in \Pi^*(k,j)$ *a partition of $[k]$ with j marked blocks*. See Example 1 for an illustration.

Let $\mu \vdash j$. We define the bijection ψ from $\mathcal{SVT}_k(\mu)$ to $\Pi^*(k,j) \times \mathcal{SYT}(\mu)$ as follows. Given a simplified vacillating tableau

$$P = (\emptyset = \lambda^{(0)}, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, \ldots, \lambda^{(k)}),$$

where $\lambda^{(k)} = \mu$, we will recursively define a sequence $(E_0, T_0), (E_{\frac{1}{2}}, T_{\frac{1}{2}}), \ldots, (E_k, T_k)$, where for any index i, E_i is a set of ordered pairs of integers in $[k]$ (which are viewed as "edges"), and T_i is a partial tableau of shape $\lambda^{(i)}$. Let E_0 be the empty set and T_0 be the empty tableau. For each integer $j = 1, 2, \ldots, k$, assume (E_{j-1}, T_{j-1}) is known.

(a) If $\lambda^{(j-\frac{1}{2})} = \lambda^{(j-1)}$, then $(E_{j-\frac{1}{2}}, T_{j-\frac{1}{2}}) = (E_{j-1}, T_{j-1})$.

(b) If $\lambda^{(j-\frac{1}{2})} \subsetneq \lambda^{(j-1)}$, let $T_{j-\frac{1}{2}}$ be the unique partial tableau with the property that there exists an integer m such that $T_{j-1} = \left(m \xrightarrow{RSK} T_{j-\frac{1}{2}}\right)$. Note that m must be less than j. Let $E_{j-\frac{1}{2}}$ be obtained from E_{j-1} by adding the ordered pair (m, j).

(c) If $\lambda^{(j)} = \lambda^{(j-\frac{1}{2})}$, then $(E_j, T_j) = (E_{j-\frac{1}{2}}, T_{j-\frac{1}{2}})$.

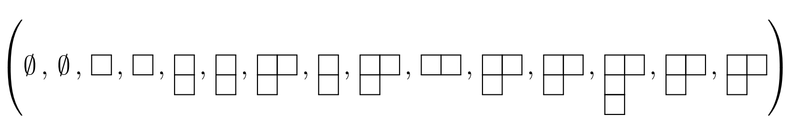
(d) If $\lambda^{(j)} \supsetneq \lambda^{(j-\frac{1}{2})}$, let $E_j = E_{j-\frac{1}{2}}$ and T_j be obtained from $T_{j-\frac{1}{2}}$ by adding the entry j in the box $\lambda^{(j)}/\lambda^{(j-\frac{1}{2})}$.

It is clear from the above construction that $E_0 \subseteq E_{\frac{1}{2}} \subseteq \cdots \subseteq E_k$. Let $G = (V, E_k)$ be a graph with vertex set $V = [k]$ and edge set E_k, and let $\boldsymbol{B}$ be the set partition of $[k]$ whose blocks are vertices of connected components of G.

Let μ be the shape of T_k. Note that if an integer i appears in T_k, then E_k cannot contain any ordered pair (i, j) with $i < j$. It follows that i is the maximal element in the block containing it. Hence, the content of T_k is a subset of $\max(\boldsymbol{B})$. We get a set partition $\boldsymbol{B}^*$ with marked blocks by putting a mark on each block X if $\max(X)$ is in T_k, and then replacing the integers in T_k with integers $1, 2, \ldots, j = |\mu|$, following numerical order. This results in an SYT T of shape μ.

Finally, we define $\psi(P) = (\boldsymbol{B}^*, T)$.

Example 1. As an example of the map ψ, let $k = 7$ and the simplified vacillating tableau be



Then the sequence of T_j and the corresponding new edge added to E_j at each step are given below.

j	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7
T_j	$\emptyset$	$\emptyset$	$\begin{array}{c}1\end{array}$	$\begin{array}{c}1\end{array}$	$\begin{array}{c}1\\2\end{array}$	$\begin{array}{c}1\\2\end{array}$	$\begin{array}{cc}1&3\\2&\end{array}$	$\begin{array}{c}1\\2\end{array}$	$\begin{array}{cc}1&4\\2&\end{array}$	$\begin{array}{cc}2&4\end{array}$	$\begin{array}{cc}2&4\\5&\end{array}$	$\begin{array}{cc}2&4\\5&\end{array}$	$\begin{array}{cc}2&4\\5&\\6&\end{array}$	$\begin{array}{cc}2&5\\6&\end{array}$	$\begin{array}{cc}2&5\\6&\end{array}$
new edge								(3,4)		(1,5)				(4,7)	

It follows that $E_k = \{(3,4),(1,5),(4,7)\}$ and

$$\boldsymbol{B}^* = \{1, 5^* \mid 2^* \mid 3, 4, 7 \mid 6^*\}, \qquad T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}.$$

The map ψ is bijective. This map and its restrictions are our main tool to study simplified and limiting vacillating tableaux. To better understand this map, we describe its inverse. First, given a set partition of $[k]$, we represent it by an arc-diagram on the vertex set $[k]$ whose edge set consists of arcs connecting consecutive elements of each block in numerical order. Such a diagram is called a *standard diagram* of the set partition. Figure 3 gives the standard diagram of the set partition $\{1, 5 \mid 2 \mid 3, 4, 7 \mid 6\}$.

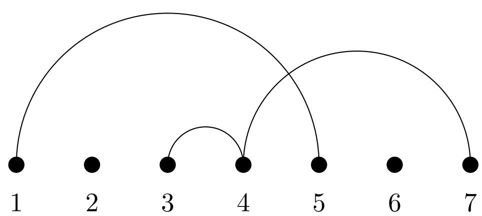


Figure 3: The standard diagram of $\{1, 5 \mid 2 \mid 3, 4, 7 \mid 6\}$.

Given $(\boldsymbol{B}^*, T) \in \Pi^*(k, j) \times \mathcal{SYT}(\mu)$, where $\mu \vdash j$, the unique vacillating tableau P such that $\psi(P) = (\boldsymbol{B}^*, T)$ can be constructed as follows. First we recover the partial tableau T_k: Let $a_1 < a_2 < \cdots < a_j$ be the maximal elements of the marked blocks in $\boldsymbol{B}^*$. Then T_k is obtained from T by replacing entry i with a_i for all i. Let G be the standard diagram of the set partition of $\boldsymbol{B}^*$. We work our way backwards from T_k, reconstructing the preceding tableaux, whose shapes form the vacillating tableau P. For $j = k, k-1, \ldots, 1$, if we know the tableau T_j, we can get the tableaux $T_{j-\frac{1}{2}}$ and T_{j-1} by the following rules.

(a′) If j is an entry of T_j, then $T_{j-\frac{1}{2}}$ is obtained from T_j by removing the box containing j. Otherwise, $T_{j-\frac{1}{2}} = T_j$.

(b′) If G has an edge of the form (i, j) with $i < j$, then

$$T_{j-1} = \left(i \xrightarrow{RSK} T_{j-\frac{1}{2}}\right).$$

Otherwise, $T_{j-1} = T_{j-\frac{1}{2}}$.

Example 2. Let $k = 3$ and $\lambda = (1, 1)$. There are six simplified vacillating tableaux of shape λ and length 6. The corresponding pairs of (E_k, T_k) and $(\boldsymbol{B}^*, T)$ are given in Table 1.

vacillating tableau	(E_k, T_k)	$(\boldsymbol{B}^*, T)$
$(\emptyset, \emptyset, \ydiagram{1}, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1})$	$(\{(1,2)\}, \ytableaushort{2,3})$	$(\{1, 2^* \mid 3^*\}, \ytableaushort{1,2})$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1}, \ydiagram{1,1})$	$(\{(1,3)\}, \ytableaushort{2,3})$	$(\{1, 3^* \mid 2^*\}, \ytableaushort{1,2})$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{1}, \ydiagram{1,1})$	$(\{(2,3)\}, \ytableaushort{1,3})$	$(\{1^* \mid 2, 3^*\}, \ytableaushort{1,2})$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{1,1})$	$(\emptyset, \ytableaushort{1,2})$	$(\{1^* \mid 2^* \mid 3\}, \ytableaushort{1,2})$
$(\emptyset, \emptyset, \emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1})$	$(\emptyset, \ytableaushort{2,3})$	$(\{1 \mid 2^* \mid 3^*\}, \ytableaushort{1,2})$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1})$	$(\emptyset, \ytableaushort{1,3})$	$(\{1^* \mid 2 \mid 3^*\}, \ytableaushort{1,2})$

Table 1: The simplified vacillating tableaux of shape $\lambda = (1, 1)$ and length 6.

3. Vacillating Tableaux of Even Length

This section pertains to $g_k(\mu)$, the number of simplified vacillating tableaux of shape μ and length $2k$. We will remark on instances where a result is applicable to $m^{\lambda}_{n,k}$ for arbitrary integers $n \geq 1$ and $k \geq 0$.

3.1. Formula for $g_k(\mu)$

Theorem 2 is an immediate consequence of the bijection ψ.

Theorem 2. *For all integers $k \geq 0$,*

$$g_k(\mu) = B(k, |\mu|) f^{\mu},$$

where $B(k, j)$ is the number of set partitions of $[k]$ with j marked blocks.

It is easy to see that for $0 \leq j \leq k$,

$$B(k, j) = \sum_{r=j}^{k} \binom{r}{j} S(k, r), \tag{2}$$

where $S(k, r)$ is the Stirling number of the second kind that counts the number of set partitions of $[k]$ with r non-empty blocks. By convention, we let $S(0, 0) = 1$ and

$S(0,r) = 0$ for $r \geq 1$. The array $B(k,j)$ is given by OEIS sequence A049020. The exponential generating function of $B(k,j)$ for a fixed j is

$$\sum_{k\geq 0} B(k,j)\frac{x^k}{k!} = \frac{1}{j!}(e^x-1)^j \exp(e^x-1). \tag{3}$$

Theorem 2 and Equation (3) are given in [4, Theorem 2.4] and Equation (2) is given in [1, Equation (5.11)]. When $\mu = \emptyset$, Theorem 2 implies $g_k(\emptyset) = \mathrm{Bell}(k)$, the k-th Bell number that counts the number of set partitions of $[k]$. Further refinement between simplified vacillating tableaux of shape $\emptyset$ and set partitions was studied in [4].

Remark 1. The special case $\mu = \emptyset$ of Theorem 2, written in terms of $m^{\lambda}_{n,k}$, gives $m^{(n)}_{n,k} = \mathrm{Bell}(k)$ for $n \geq 2k$. Martin and Rollet [13] proved a more general identity:

$$\sum_{j=1}^{n} S(k,j) = m^{(n)}_{n,k}, \qquad \text{for } n,k \geq 1. \tag{4}$$

Identity (4) with even k is also proved by Benkart and Halverson [1]. Using growth diagrams, Krattenthaler [12] gave a simple proof of Identity (4) as well as the identity $S(k,n) + S(k,n-1) = m^{(1^n)}_{n,k}$ for $n \geq 1$. Note that the latter identity is trivial when $n \geq 2k$.

Using the same construction as ψ but starting with a Young tableau of shape (n) filled with n zeros, Benkart, Halverson and Harman [2] gave a bijection from the set of n-vacillating tableaux in $\mathcal{VT}_{n,k}(\lambda)$ to pairs $(\boldsymbol{B}, \hat{T})$, where $\boldsymbol{B} \in \Pi(k,j)$ is a partition of $[k]$ into j blocks, and $\hat{T}$ is an SSYT of shape λ with content $\{0^{n-j}\} \cup \max(\boldsymbol{B})$. We will use this result in Subsections 3.6 and 4.6 to get the expansion of $\sum_{\lambda} m^{\lambda}_{n,k} s_{\lambda}(\boldsymbol{x})$ in terms of complete symmetric functions. See Theorems 6 and 11.

3.2. The Sum $\sum_{\mu} g_{k_1}(\mu) g_{k_2}(\mu)$

Note that there is a symmetry between the even steps and the odd steps in the definition of simplified vacillating tableaux. Thus any walk on Young's lattice, as described in Definition 2, from the empty shape to itself in $2(k_1+k_2)$ steps can be viewed as a walk from $\emptyset$ to some shape μ in $2k_1$ steps, then followed by the reverse of a walk from $\emptyset$ to μ in $2k_2$ steps. It follows that for all integers $k_1, k_2 \geq 0$,

$$\sum_{\mu} g_{k_1}(\mu) g_{k_2}(\mu) = g_{k_1+k_2}(\emptyset) = \mathrm{Bell}(k_1+k_2). \tag{5}$$

For the special case where $k_1 = k_2$, Identity (5) is proved in [8] in the form $\sum_{\lambda\in\Lambda^k_n} (m^{\lambda}_{n,k})^2 = \mathrm{Bell}(2k)$ for $n \geq 2k$.

3.3. The Sum $g_k = \sum_\mu g_k(\mu)$

Let $g_k = \sum_\mu g_k(\mu)$ be the number of simplified vacillating tableaux of length $2k$. The initial terms of the sequence $(g_k)_{k=0}^{\infty}$ are $1, 2, 7, 31, 164, 999, \ldots$. This sequence is cataloged as A002872 in OEIS.

Let $[-k]$ be the set of integers $\{-k, \ldots, -2, -1\}$. A set partition of $[-k] \cup [k]$ is *symmetric* if $-B$ is a block whenever B is a block of the set partition. Theorem 3 was first proved in [8, Equation (5.5)], using the symmetry of the bijection ψ when one restricts ψ to a simplified vacillating tableau of shape $\emptyset$ and its reverse. Here we give a purely combinatorial proof using the pairs $(\boldsymbol{B}^*, T)$ in $\Pi^*(k, j) \times \mathcal{SYT}(\mu)$, where $|\mu| = j$.

Theorem 3. *Let $k \geq 1$ be an integer. Then g_k counts the number of symmetric partitions of $[-k] \cup [k]$.*

Proof. Each simplified vacillating tableau of length $2k$ is uniquely represented by a pair $(\boldsymbol{B}^*, T) \in \Pi^*(k, j) \times \mathcal{SYT}(\mu)$ with $|\mu| = j$. Draw the standard diagram of the partition $\boldsymbol{B}$ on vertices $1, 2, \ldots, k$, followed by the reverse of this diagram on vertices with labels $-k, \ldots, -2, -1$. If the marked blocks of $\boldsymbol{B}^*$ are $X_1, \ldots, X_j$ (say, ordered by the values of their maximal elements), then the corresponding marked blocks on $[-k]$ are $-X_1, \ldots, -X_j$. For the SYT T, applying the inverse of the RSK algorithm to the pair (T, T), we obtain an involution $\sigma \in \mathfrak{S}_j$. Now connect the maximal element of block X_i with the minimal element of block $-X_{\sigma(i)}$, for each $i = 1, \ldots, j$. Since σ is an involution, the maximal element of block $X_{\sigma(i)}$ is connected to the minimal element of $-X_i$. Therefore, we get a symmetric diagram that defines a symmetric set partition of $[-k] \cup [k]$. Note that the marked blocks of $\boldsymbol{B}^*$ correspond to the blocks of the symmetric set partition containing positive and negative elements as well. Each step of the above construction can be easily reversed. Hence, our construction is a bijection. $\square$

Example 3. As an example, let $B^* = \{1^* \mid 2, 4 \mid 5^* \mid 3, 6^*\}$ and $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. The marked blocks are ordered by their maximal elements, therefore, $X_1 = \{1\}$, $X_2 = \{5\}$, and $X_3 = \{3, 6\}$. The involution determined by T is $\sigma = 213$ (in one-line notation). Hence, we merge the blocks X_1 with $-X_2$, X_2 with $-X_1$, and X_3 with $-X_3$, as shown in Figure 4. The resulting symmetric set partition is $\{1, -5 \mid 5, -1 \mid 2, 4 \mid -2, -4 \mid 3, 6, -3, -6\}$

3.4. The Sum $\sum_\mu g_k(\mu) f^\mu$

Let $u_k = \sum_\mu g_k(\mu) f^\mu$. The initial values of $(u_k)_{k=0}^{\infty}$ are $1, 2, 7, 33, 198, \ldots$, which coincide with the initial terms of OEIS sequence A059099 that was studied by

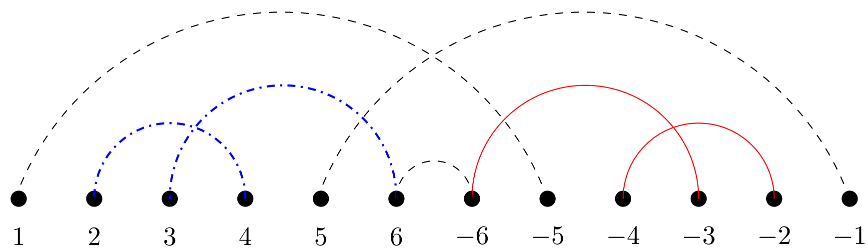


Figure 4: The arc diagram of the symmetric partition in Example 3.

Nkonkobe and Murali [14] as one case of "restricted barred preferential arrangements." We will present a simple combinatorial proof that $(u_k)_{k=0}^{\infty}$ is indeed the OEIS sequence A059099.

Recall that an *ordered partition*, or a *preferential arrangement* of a set S, is a partition of S into disjoint non-empty blocks, together with a linear order on the blocks. For $n \in \mathbb{N}$, ordered partitions of $[n]$ are counted by *Fubini numbers*, which have exponential generating function $1/(2-e^x)$ and appear as A000670 in OEIS.

We say that a set partition of S is *partly ordered* if it has a (possibly empty) subset of marked blocks that are linearly ordered. For example, the following set partition of $\{1, 2, \ldots, 7\}$ is partly ordered:

$$\{(2, 7^* \mid 1, 3^* \ ) \mid 4, 6 \mid 5\}.$$

The first two blocks with marks are linearly ordered and listed inside the parentheses; the last two blocks are not marked and are unordered. If we switch the first two blocks, we obtain a different partly ordered set partition, while switching the last two blocks gives the same partly ordered set partition. That is,

$$\{(1, 3^* \mid 2, 7^* \ ) \mid 4, 6 \mid 5\} \neq \{(2, 7^* \mid 1, 3^* \ ) \mid 4, 6 \mid 5\} = \{(2, 7^* \mid 1, 3^* \ ) \mid 5 \mid 4, 6\}.$$

Theorem 4. *Let $k \geq 1$ be an integer. The number u_k counts the number of elements in the set of partly ordered set partitions of $[k]$.*

Proof. Again we use the bijection ψ to represent a simplified vacillating tableau of length $2k$ by a pair $(\boldsymbol{B}^*, T)$, where $\boldsymbol{B}^*$ is a set partition of $[k]$ with j marked blocks, and T is an SYT of some shape μ with $|\mu| = j$. Now, u_k counts the pairs in $\mathcal{SVT}_k(\mu) \times \mathcal{SYT}(\mu)$, each of which corresponds to a triple $(\boldsymbol{B}^*, T, S)$ where both T and S are SYTs of shape μ. Via the inverse of the RSK algorithm, (T, S) uniquely determines a permutation $\sigma \in \mathfrak{S}_j$, which gives the linear order on the marked blocks of $\boldsymbol{B}^*$. $\square$

From Theorem 2 we have $u_k = \sum_j j! B(k, j)$, where $B(k, j)$ is given by Equation (2). By definition of partly ordered set partition, the exponential generating function of $(u_k)_{k=0}^{\infty}$ can be expressed as the product of the exponential generating functions of Bell numbers and Fubini numbers. Explicitly,

$$\sum_{k\geq 0} u_k \frac{x^k}{k!} = \frac{\exp(e^x - 1)}{2 - e^x}. \tag{6}$$

3.5. Product of $g_k(\mu)$ with Schur Functions $\sum_\mu g_k(\mu)s_\mu(x)$

Next we connect vacillating tableaux to symmetric functions. Let μ be a partition. For an SSYT T of shape μ and content$(T) \subseteq \mathbb{Z}^+$, let $\alpha_i(T)$ be the number of times the integer i appears in T, and write

$$\boldsymbol{x}^T = x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \cdots .$$

The Schur function s_μ in the variables $\boldsymbol{x} = (x_1, x_2, \dots)$ is the formal power series

$$s_\mu(\boldsymbol{x}) = \sum_T \boldsymbol{x}^T,$$

where T ranges over all SSYT of shape μ and content$(T) \subseteq \mathbb{Z}^+$. By convention, set $s_\emptyset = 1$.

Theorem 5. *Let $k \geq 0$ be an integer. We have the identity*

$$\sum_\mu g_k(\mu)s_\mu(\boldsymbol{x}) = \sum_{j=0}^{k} B(k,j)h_1(\boldsymbol{x})^j, \tag{7}$$

where $B(k,j)$ is the number of set partitions of $[k]$ with j marked blocks and $h_1(\boldsymbol{x}) = \sum_i x_i$.

Proof. The left side of Identity (7) is a summation of $\boldsymbol{x}^S$ over the set of triples $(\boldsymbol{B}^*, T, S)$ where $\boldsymbol{B}^*$ is a set partition of $[k]$ with j marked blocks, T is an SYT of some shape μ of size j, and S is a SSYT of shape μ and content$(S) \subseteq \mathbb{Z}^+$. The pair (T, S), under the inverse of the RSK algorithm, corresponds uniquely to a two-line array of the form (1), where the top line is $(1, 2, \dots, j)$, and the second line is a sequence $\boldsymbol{i} = (i_1, i_2, \dots, i_j)$ of positive integers. Note that

$$\boldsymbol{x}^S = x_{i_1} x_{i_2} \cdots x_{i_j}.$$

Hence,

$$\sum_{(\boldsymbol{B}^*,T,S)} \boldsymbol{x}^S = \sum_j B(k,j) \sum_{\boldsymbol{i}\in(\mathbb{Z}^+)^j} x_{i_1} x_{i_2} \cdots x_{i_j} = \sum_j B(k,j) \left(\sum_i x_i\right)^j. \qquad \square$$

Let m be a positive integer and $x_i = q^{i-1}$ for $i = 1, 2, \ldots, m$ and $x_j = 0$ for $j > m$. Then Identity (7) becomes

$$\sum_{\mu} g_k(\mu) s_\mu(1, q, \ldots, q^{m-1}) = \sum_{j=0}^{k} B(k, j)[m]_q^j, \tag{8}$$

where $[m]_q = 1+q+\cdots+q^{m-1}$ is the q-integer. The polynomial $s_\mu(1, q, \ldots, q^{m-1})$ is called *the principal specialization of* s_μ, which can be computed by Stanley's hook-content formula [19, Theorem 7.21.2]. For an integer partition $\mu = (\mu_1, \ldots, \mu_t)$, let $\mu' = (\mu'_1, ..., \mu'_{\mu_1})$ be the conjugate of μ, where μ'_j is the number of boxes in the j-th column of the Young's diagram of μ. For a box $u = (i, j)$ in the i-th row and j-th column of the Young diagram of μ, define the *hook length* $h(u)$ of μ at u by

$$h(u) = \mu_i + \mu'_j - i - j + 1.$$

In addition, define

$$b(\mu) = \sum_{i\geq 1}(i-1)\mu_i = \sum_{i\geq 1}\binom{\mu'_i}{2}.$$

Then the hook-content formula is

$$s_\mu(1, q, \ldots, q^{m-1}) = q^{b(\mu)} \prod_{u=(i,j)\in\mu} \frac{[m+j-i]_q}{[h(u)]_q}. \tag{9}$$

3.6. Product of $m^\lambda_{n,k}$ with Schur Functions $\sum_\lambda m^\lambda_{n,k} s_\lambda(x)$

To compare with the result of the preceding subsection, we also consider the product of Schur functions with the number of n-vacillating tableaux for general $n \geq 1$ and $k \geq 0$. Note that this summation is over all integer partitions $\lambda \in \Lambda^k_n$, while for $g_k(\mu)$, μ ranges over all integer partitions of size at most k.

For a nonnegative integer n, the complete symmetric functions h_λ are defined by the formulas

$$\begin{aligned} h_0 &= 1, \\ h_n &= \sum_{i_1\leq\cdots\leq i_n} x_{i_1}\cdots x_{i_n}, \qquad (n\geq 1) \\ h_\lambda &= h_{\lambda_1}h_{\lambda_2}\cdots \qquad \text{if } \lambda = (\lambda_1, \lambda_2, \ldots). \end{aligned}$$

The sum of products of $m^\lambda_{n,k}$ and Schur functions has a nice expansion in complete symmetric functions.

Theorem 6. *For integers $k \geq 0$ and $n \geq 1$,*

$$\sum_{\lambda\in\Lambda^k_n} m^\lambda_{n,k} s_\lambda(\boldsymbol{x}) = \sum_{j=0}^{\min(n,k)} S(k, j) h_{(n-j,1^j)}(\boldsymbol{x}), \tag{10}$$

where if $j = n$, the integer partition $(n-j, 1^j)$ is just (1^n).

Proof. We use a bijection of Benkart, Halverson, and Harman [2] that maps the set of n-vacillating tableaux of shape λ and length $2k$ to the set of pairs $(\boldsymbol{B}, \hat{T})$ where $\boldsymbol{B} \in \Pi(k, j)$ is a partition of $[k]$ into j blocks for some $j \leq n$, and $\hat{T}$ is an SSYT of shape λ filled with $n-j$ zeros and j distinct positive integers, which are the maximal elements of the blocks of $\boldsymbol{B}$.

We extend the above bijection to an injection from $\mathcal{VT}_{n,k}(\lambda)$ into a subset of $\Pi([k]) \times \mathcal{SYT}(\lambda)$ and then analyze the images. First, we replace the SSYT $\hat{T}$ with an SYT T of shape λ as follows: Replace the $n-j$ zeros, all of which must be in the first row of $\hat{T}$, by $1, 2, \ldots, n-j$ from left to right. Then replace the remaining j entries of $\hat{T}$ by $n-j+1, \ldots, n$ according to their numerical order. Conversely, given $\boldsymbol{B} = \{B_1, \ldots, B_j\}$ and T, we can recover $\hat{T}$ whose content is $\{0^{n-j}\} \cup \{\max(B_i) : i = 1, 2, \ldots, j\}$ simply by replacing each of $1, \ldots, n-j$ in T with 0 and replacing $n-j+i$ in T with the i-th smallest element in $\max(\boldsymbol{B})$.

Now we can represent an n-vacillating tableau by a pair $(\boldsymbol{B}, T)$ where $\boldsymbol{B}$ is a partition of $[k]$ with j blocks, and T is an SYT of some shape $\lambda \vdash n$ with the property that $\lambda_1 \geq n-j$ and the entries $1, \ldots, n-j$ are in the first row of T.

The formula $\sum_{\lambda \in \Lambda_n^k} m_{n,k}^{\lambda} s_{\lambda}(\boldsymbol{x})$ is the formal power series that sums $\boldsymbol{x}^S$ over the set of triples $(\boldsymbol{B}, T, S)$ where $(\boldsymbol{B}, T)$ is as described above, and S is an SSYT of shape λ with positive integer entries. Fix the set partition $\boldsymbol{B} \in \Pi(k, j)$ and consider all the tableaux T and S that can appear with $\boldsymbol{B}$ in some triples. Under the inverse of the RSK insertion, taking S as the insertion tableau and T as the recording tableau, the pair (S, T) uniquely corresponds to a sequence $\boldsymbol{i} = (i_1, \ldots, i_n)$ of positive integers, where $\boldsymbol{x}^S = x_{i_1} \cdots x_{i_n}$. Since T is the recording tableau, the condition that the entries $1, \ldots, n-j$ appear in the first row of T holds if and only if $i_1 \leq \cdots \leq i_{n-j}$. Let

$$\mathcal{D}_{n,n-j} = \{(i_1, \ldots, i_n) \in (\mathbb{Z}^+)^n : \ i_1 \leq \cdots \leq i_{n-j}\}.$$

Then the formal power series $\sum_{\lambda \in \Lambda_n^k} m_{n,k}^{\lambda} s_{\lambda}(\boldsymbol{x})$ can be expressed as

$$\begin{aligned}\sum_j S(k,j) \sum_{\boldsymbol{i} \in \mathcal{D}_{n,n-j}} x_{i_1} \cdots x_{i_n} &= \sum_j S(k,j) \left(\sum_{i_1 \leq \cdots \leq i_{n-j}} x_{i_1} \cdots x_{i_{n-j}} \right) \left(\sum_i x_i \right)^j \\ &= \sum_j S(k,j) h_{(n-j,1^j)}(\boldsymbol{x}),\end{aligned}$$

where the range of j is $0 \leq j \leq \min(n,k)$. □

Example 4. We illustrate Theorem 6 with some examples. First, let $n = 2$ and $k \geq 1$. From [12] we have that $m_{n,k}^{(2)} = m_{n,k}^{(1,1)} = S(k,1) + S(k,2) = 2^{k-1}$. Hence, Theorem 6 becomes

$$\sum_{\lambda \in \Lambda_2^k} m_{2,k}^{\lambda} s_{\lambda}(\boldsymbol{x}) = 2^{k-1}(s_{(2)} + s_{(1,1)}) = 2^{k-1} h_{(1,1)} = S(k,1) h_{(1,1)} + S(k,2) h_{(1,1)}.$$

As another example, let $n = 6$ and $k = 3$. Then we have

$$\begin{aligned}\sum_{\lambda\in\Lambda_6^3} m_{6,3}^{\lambda} s_\lambda(\boldsymbol{x}) &= 5s_{(6)} + 10s_{(5,1)} + 6s_{(4,2)} + 6s_{(4,1,1)} + s_{(3,3)} + 2s_{(3,2,1)} + s_{(3,1,1,1)} \\ &= h_{(5,1)} + 3h_{(4,1,1)} + h_{(3,1,1,1)} \\ &= S(3,1)h_{(5,1)} + S(3,2)h_{(4,1,1)} + S(3,3)h_{(3,1,1,1)}.\end{aligned}$$

For the principal specialization, note that $h_n = s_{(n)}$. Using Stanley's hook-content formula, we have $h_n(1, q, \ldots, q^{m-1}) = \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q$. Therefore, Theorem 6 implies

$$\sum_{\lambda\in\Lambda_n^k} m_{n,k}^{\lambda} s_\lambda(1, q, \ldots, q^{m-1}) = \sum_{j\le\min(k,n)} S(k, j) \begin{bmatrix} m+n-j-1 \\ n-j \end{bmatrix}_q [m]_q^j. \tag{11}$$

Corollary 1. *For integers $k \ge 0$ and $n \ge 1$,*

$$n^k = \sum_{\lambda\in\Lambda_n^k} f^\lambda m_{n,k}^{\lambda}. \tag{12}$$

Proof. Take the coefficient of $x_1x_2\cdots x_n$ on both sides of Identity (10). From the left side we get $\sum_{\lambda\in\Lambda_n^k} f^\lambda m_{n,k}^\lambda$. On the right side, we can obtain a monomial $x_1x_2\cdots x_n$ by choosing $n-j$ variables from the factor $h_{n-j} = \sum x_{i_1}\cdots x_{i_{n-j}}$, and then obtain the product of the remaining j variables from the factor $h_1^j = (\sum_i x_i)^j$. The latter step can be done in $j!$ ways. Hence, the total number of $x_1x_2\cdots x_n$ is $\sum_j S(k, j)\binom{n}{j} j!$. By the well-known identity $x^k = \sum_r S(k, r)(x)_r$, where $(x)_r = x(x-1)\cdots(x-r+1)$, we obtain that the coefficient of $x_1x_2\cdots x_n$ on the right side is n^k. This finishes the proof. □

Corollary 1 was first proved by Halverson and Lewandowski [8] via a bijection that uses a combination of the RSK row insertion and jeu de taquin. Recently Krattenthaler [12] proved a generalization of Identity (12) using growth diagrams:

$$n^k = \sum_{\lambda\vdash n} f^\lambda m_\mu^\lambda(k), \tag{13}$$

where μ is any fixed integer partition of n and $m_\mu^\lambda(k)$ is the number of n-vacillating tableaux from shape μ to shape λ in $2k$ steps. In this paper, we only consider n-vacillating tableaux starting at the shape (n) and simplified vacillating tableaux starting at the empty shape.

4. Vacillating Tableaux of Odd Length

4.1. Formula for $g_{k+\frac{1}{2}}(\mu)$

In the description of the bijection ψ in Subsection 2.2, comparing the pairs (E_i, T_i) for the last two steps where $i = k - \frac{1}{2}$ and $i = k$, one observes that $E_{k-\frac{1}{2}} = E_k$ and $T_{k-\frac{1}{2}}$ is either the same as T_k or missing one corner box containing k. That is, without the last pair (E_k, T_k), we still get the set partition $\boldsymbol{B}$ of $[k]$ with some j blocks marked, and an SYT $T = T_{k-\frac{1}{2}}$ of shape μ with $|\mu| = j$, with the additional condition that the block containing element k is not marked.

Let

$$\tilde{\Pi}^*(k,j) = \left\{ \boldsymbol{B}^* = (\boldsymbol{B}, A) : \begin{array}{c} \boldsymbol{B} \in \Pi([k]) \text{ and } A \text{ is a subset of } j \\ \text{blocks of } \boldsymbol{B} \text{ not containing } k \end{array} \right\}.$$

Denote by $\tilde{B}(k,j)$ the cardinality of $\tilde{\Pi}^*(k,j)$. Then the map ψ induces a bijection

$$\mathcal{SVT}_{k-\frac{1}{2}}(\mu) \longleftrightarrow \tilde{\Pi}^*(k, |\mu|) \times \mathcal{SYT}(\mu),$$

where $\mathcal{SVT}_{k-\frac{1}{2}}(\mu)$ is the set of simplified vacillating tableaux of shape μ and length $2k-1$. It follows that $g_{k-\frac{1}{2}}(\mu) = \tilde{B}(k, |\mu|) f^\mu$, or equivalently,

$$g_{k+\frac{1}{2}}(\mu) = \tilde{B}(k+1, |\mu|) f^\mu, \tag{14}$$

where for $0 \leq j \leq k$,

$$\tilde{B}(k+1, j) = \sum_r \binom{r}{j} S(k+1, r+1). \tag{15}$$

The array $\tilde{B}(k,j)$ is given in A137597 in OEIS. Equation (14) appears in [1, Equation (5.12)]. When $\mu = \emptyset$, Equation (14) gives that $g_{k+\frac{1}{2}}(\emptyset) = \mathrm{Bell}(k+1)$, the number of set partitions of $[k+1]$.

4.2. The Sum $\sum_\mu g_{k_1+\frac{1}{2}}(\mu) g_{k_2+\frac{1}{2}}(\mu)$

Theorem 7. *For all integers $k_1, k_2 \geq 0$,*

$$\sum_\mu g_{k_1+\frac{1}{2}}(\mu) g_{k_2+\frac{1}{2}}(\mu) = \mathrm{Bell}(k_1 + k_2 + 1). \tag{16}$$

First proof of Theorem 7. Observe that if a simplified vacillating tableau of shape μ and length $2k_1+1$ is followed by the reverse of another simplified vacillating tableau of same shape μ and length $2k_2+1$, we obtain exactly a simplified vacillating tableau of the empty shape $\emptyset$ and length $2k_1+2k_2+2$. Since $g_{k_1+k_2+1}(\emptyset) = \mathrm{Bell}(k_1+k_2+1)$, Identity (16) follows. $\square$

In addition to the above simple argument, we will give another proof that uses the pair $(\boldsymbol{B}^*, T)$ in $\tilde{\Pi}^*(k,j) \times \mathcal{SYT}(\mu)$. This new argument can be applied to compute $\sum_\mu g_{k_1}(\mu) g_{k_2+\frac{1}{2}}(\mu)$ and $\sum_\mu g_{k_1}(\mu) a_{k_2}(\mu)$, etc.

Second proof of Theorem 7. The left side of Identity (16) counts the quadruple $(\boldsymbol{B}_1^*, T_1; \boldsymbol{B}_2^*, T_2)$ where for $i = 1, 2$, T_i is an SYT of the shape μ with $|\mu| = j$ for some integer j, and $\boldsymbol{B}_i^*$ is a partition of $[k_i+1]$ with j marked blocks such that the block containing $k_i + 1$ is not marked. Order the marked blocks of $\boldsymbol{B}_i^*$ by their maximal elements, and let $\sigma \in \mathfrak{S}_j$ be the permutation corresponding to the pair (T_1, T_2) under the RSK algorithm.

Draw the standard diagram of the partition $\boldsymbol{B}_1^*$, followed by the reverse of the standard diagram of $\boldsymbol{B}_2^*$. Then, for each $i \le j$, add an edge connecting the maximal element of the i-th marked block of $\boldsymbol{B}_1^*$ to the maximal element of the $\sigma(i)$-th marked block of $\boldsymbol{B}_2^*$, and then identify the vertex k_1+1 of $\boldsymbol{B}_1^*$ with the vertex k_2+1 of $\boldsymbol{B}_2^*$. Now we get a diagram on k_1+k_2+1 vertices, which corresponds to a set partition of $[k_1+k_2+1]$. The procedure can be easily reversed. Hence, it is a bijection. $\square$

Example 5. This example illustrates the construction in the second proof. Let $k_1 = 7$ and $k_2 = 6$. Take $\boldsymbol{B}_1^* = \{2^* \mid 1,3 \mid 6^* \mid 5,7^* \mid 4,8\}$. To distinguish the elements, we put a bar on each element in the partition $\boldsymbol{B}_2^*$. Let $\boldsymbol{B}_2^* = \{\bar{1},\bar{4}^* \mid \bar{5}^* \mid \bar{3},\bar{6}^* \mid \bar{2},\bar{7}\}$, and

$$T_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad , \quad T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \, .$$

The marked blocks are ordered by their maximal elements. For $\boldsymbol{B}_1^*$, the marked blocks are $X_1 = \{2\}$, $X_2 = \{6\}$ and $X_3 = \{5,7\}$; for $\boldsymbol{B}_2^*$, the marked blocks are $Y_1 = \{\bar{1},\bar{4}\}$, $Y_2 = \{\bar{5}\}$ and $Y_3 = \{\bar{3},\bar{6}\}$. The permutation determined by (T_1, T_2) is $\sigma = 231$ (in one-line notation). Hence, we merge the blocks X_1 with Y_2, X_2 with Y_3, and X_3 with Y_1, and identify the element 8 of $\boldsymbol{B}_1^*$ with the element $\bar{7}$ of $\boldsymbol{B}_2^*$, as shown in Figure 5. Finally, replacing the integer $\bar{i}$ with k_1+k_2+2-i, we get the partition $\{1,3 \mid 2,10 \mid 4,8,13 \mid 5,7,11,14 \mid 6,9,12\}$.

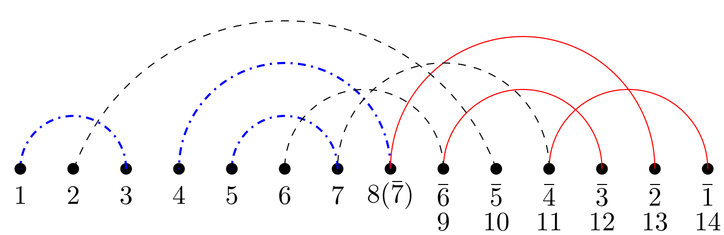


Figure 5: The arc diagram of the set partition in Example 5.

The special case of Identity (16) with $k_1 = k_2$ was obtained in [8, Equation (5.1)]. With a similar argument as in the second proof of Theorem 7, but without identifying the last elements of the partitions $\boldsymbol{B}_1^*$ and $\boldsymbol{B}_2^*$, we obtain Theorem 8.

Theorem 8. *For all integers $k_1, k_2 \geq 0$, the number $\sum_\mu g_{k_1+\frac{1}{2}}(\mu) g_{k_2}(\mu)$ counts the number of set partitions of $[k_1+k_2+1]$ with the condition that k_1+1 is the maximal element in the block containing it.*

4.3. The Sum $\sum_\mu g_{k+\frac{1}{2}}(\mu)$

Let $g_{k+\frac{1}{2}}$ be the sum $\sum_\mu g_{k+\frac{1}{2}}(\mu)$. We will prove in Theorem 9 that $g_{k+\frac{1}{2}}$ is the same as the number of symmetric partitions of the integers in the interval $[-k, k] = \{-k, \ldots, -1, 0, 1, \ldots, k\}$. The sequence $(g_{k+\frac{1}{2}})_{k=0}^{\infty}$ is A080337 in OEIS, with the initial values $1, 3, 12, 59, 339 \ldots$.

Indeed, combining with the sequence A002872 and considering g_0, $g_{\frac{1}{2}}$, g_1, $g_{1\frac{1}{2}}$, g_2, $\ldots$, we get the sequence A080107 in OEIS, which is described as the number of fixed points of set partitions under the involution $i \leftrightarrow n+1-i$. Such fixed points are clearly equivalent to symmetric set partitions.

Theorem 9. *For all integers $k \geq 0$, $g_{k+\frac{1}{2}}$ counts the number of symmetric partitions of the set $[-k, k]$.*

Proof. As in the second proof of Theorem 7, for a pair $(\boldsymbol{B}^*, T)$ counted by $g_{k+\frac{1}{2}}(\mu)$, draw the standard diagram of the partition $\boldsymbol{B}^*$ on vertices $1, 2, \ldots, k+1$, followed by the reverse of this diagram on vertices with labels $(k+1)', \ldots, 2', 1'$ from left to right. If the marked blocks of $\boldsymbol{B}^*$ are $X_1, \ldots, X_j$, then the corresponding marked blocks on $[(k+1)']$ are $X_1', \ldots, X_j'$. The SYT T corresponds uniquely to an involution $\sigma \in \mathfrak{S}_j$. Note that $k+1$ and $(k+1)'$ are not in any marked blocks. Now identifying $k+1$ with $(k+1)'$, merging the block X_i with $X'_{\sigma(i)}$ by adding an edge between their maximal elements, and then relabeling the vertices as $-k, \ldots, -1, 0, 1, \ldots, k$ from left to right, we get a symmetric partition of the set $[-k, k]$. The blocks containing both positive and negative integers but not 0 come from the marked blocks. $\square$

Example 6. This example illustrates the construction in the proof of Theorem 9. Let $k = 7$. Take $\boldsymbol{B}^* = \{2^* \mid 1, 3 \mid 6^* \mid 5, 7^* \mid 4, 8\ \}$ and $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$. The marked blocks are $X_1 = \{2\}$, $X_2 = \{6\}$ and $X_3 = \{5, 7\}$. The involution determined by T is $\sigma = 213$. Hence, we merge the blocks X_1 with X_2', X_2 with X_1', and X_3 with X_3', and identify the element 8 of $\boldsymbol{B}^*$ with the element $8'$, as shown in Figure 6. Finally, relabeling the vertices from left to right as $-7, \ldots, -1, 0, 1, \ldots, 7$, we get the symmetric partition $\{-7, -5 \mid -6, 2 \mid -4, 0, 4 \mid -3, -1, 1, 3 \mid -2, 6 \mid 5, 7\}$ of the set $[-7, 7]$.

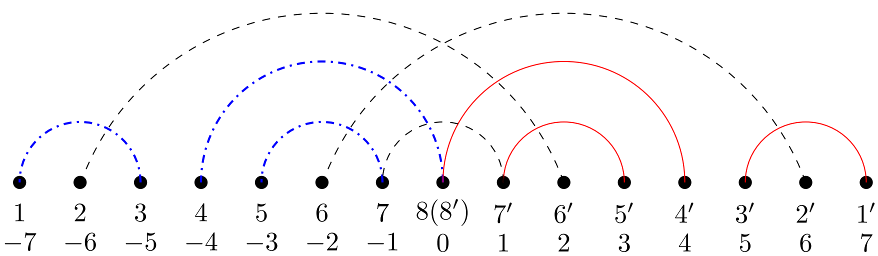


Figure 6: The arc diagram of the symmetric partition in Example 6

Remark 2. For two sequences $(a_k)_{k=0}^{\infty}$ and $(b_k)_{k=0}^{\infty}$, we say that $(b_k)_{k=0}^{\infty}$ is the *binomial transform* of $(a_k)_{k=0}^{\infty}$ if $b_k = \sum_{j=0}^{k} \binom{k}{j} a_j$ for all k. In this case, we have $\sum_k b_k x^k/k! = e^x \sum_k a_k x^k/k!$.

As an example, we note that the sequence $(g_{k+\frac{1}{2}})_{k=0}^{\infty}$ is the binomial transform of $(g_k)_{k=0}^{\infty}$, that is, $g_{k+\frac{1}{2}} = \sum_{j=0}^{k} \binom{k}{j} g_j$. Indeed, in order to form a symmetric partition of $[-k, k]$, we need exactly one self-symmetric block that contains 0, and form a symmetric set partition with the remaining integers. If there are i positive integers (and, hence, i negative integers by symmetry) in the block containing 0, then there are $\binom{k}{i}$ ways to form such a block and there are g_{k-i} ways to form a symmetric partition of the set of the remaining numbers. Hence, $g_{k+\frac{1}{2}} = \sum_{i=0}^{k} \binom{k}{i} g_{k-i} = \sum_{j=0}^{k} \binom{k}{j} g_j$. A similar relation holds for several other pairs from the enumeration of vacillating tableaux. See Section 7 for a summary.

4.4. The Sum $\sum_\mu g_{k+\frac{1}{2}}(\mu) f^\mu$

Let $u_{k+\frac{1}{2}} = \sum_\mu g_{k+\frac{1}{2}}(\mu) f^\mu$. Similar to the discussion for u_k in Theorem 4, one can prove that $u_{k+\frac{1}{2}}$ counts the number of partly ordered set partitions of $[k+1]$, with the additional condition that the block containing $k+1$ is not marked and, hence, not in the linear order.

The initial values of $u_{k+\frac{1}{2}}$ are $1, 3, 12, 61, 381, 2854, \ldots$. This sequence is not included in OEIS. The following properties are easy to obtain.

(a) $u_{k+\frac{1}{2}} = \sum_{j \geq 0} j! \tilde{B}(k+1, j)$, where $\tilde{B}(k+1, j)$ is given in Identity (15).

(b) The sequence $(u_{k+\frac{1}{2}})_{k=0}^{\infty}$ is the binomial transform of the sequence $(u_k)_{k=0}^{\infty}$ (A059099). The binomial coefficient in the binomial transform corresponds to the number of ways to form the block containing the largest element $k+1$.

The exponential generating function of $(u_{k+\frac{1}{2}})_{k=0}^{\infty}$ is

$$\sum_{k\geq 0} u_{k+\frac{1}{2}} \frac{x^k}{k!} = e^x \cdot \frac{\exp(e^x - 1)}{2 - e^x}.$$

4.5. Product of $g_{k+\frac{1}{2}}(\mu)$ with Schur Functions $\sum_{\mu} g_{k+\frac{1}{2}}(\mu) s_{\mu}(x)$

Theorem 10 is an analog of Theorem 5. We omit the proof of Theorem 10 because it is almost identical to that of Theorem 5.

Theorem 10. *Let $k \geq 0$ be an integer. We have the identity*

$$\sum_{\mu} g_{k+\frac{1}{2}}(\mu) s_{\mu}(\boldsymbol{x}) = \sum_{j\geq 0} \tilde{B}(k+1, j) h_1(\boldsymbol{x})^j, \tag{17}$$

where $\tilde{B}(k, j)$ is the number of set partitions of $[k+1]$ with j marked blocks such that the block containing $k+1$ is not marked.

The principal specialization of Identity (17) is

$$\sum_{\mu} g_{k+\frac{1}{2}} s_{\mu}(1, q, \ldots, q^m) \quad = \quad \sum_{j\geq 0} \tilde{B}(k+1, j)[m]_q^j.$$

4.6. Product of $m^{\lambda}_{n,k-\frac{1}{2}}$ with Schur Functions $\sum_{\lambda} m_{k-\frac{1}{2}}(\mu) s_{\lambda}(x)$

Theorem 11 is an analog of Theorem 6.

Theorem 11. *For all positive integers n and k,*

$$\sum_{\lambda \in \Lambda^k_{n-1}} m^{\lambda}_{n,k-\frac{1}{2}} s_{\lambda}(\boldsymbol{x}) \quad = \quad \sum_{j=1}^{\min(n,k)} S(k, j) h_{(n-j, 1^{j-1})}(\boldsymbol{x}). \tag{18}$$

Proof. The proof is similar to that of Theorem 6 except that we need to explain what one gets when applying the bijection constructed in [2] to n-vacillating tableaux of shape $\lambda \in \Lambda^k_{n-1}$ and length $2k-1$. Analyzing the bijection of [2], we see that in this case we would obtain a pair $(\boldsymbol{B}, \hat{T})$ where $\boldsymbol{B}$ is a partition of $[k]$ into j blocks for some $j \leq n$, and $\hat{T}$ is a semistandard tableau of shape $\lambda \vdash (n-1)$ filled with $n-j$ zeros and $j-1$ distinct positive integers, which are the maximal elements of the blocks of $\boldsymbol{B}$ except k.

Using the same argument as for Theorem 6, we can represent an n-vacillating tableaux of shape $\lambda \in \Lambda^k_{n-1}$ and length $2k-1$ by a pair $(\boldsymbol{B}, T)$ where $\boldsymbol{B} \in \Pi(k, j)$, $T \in \mathcal{SYT}(\lambda)$, and $\lambda_1 \geq n-j$. (Note that $\lambda \vdash (n-1)$ instead of $\lambda \vdash n$.) The remaining of the proof is the same as that of Theorem 6. $\square$

The principal specialization of Identity (18) is

$$\sum_{\lambda\in\Lambda_{n-1}^k} m^{\lambda}_{n,k-\frac{1}{2}} s_\lambda(1,q,\ldots,q^m) \quad = \quad \sum_{j=1}^{\min(n,k)} S(k,j)\begin{bmatrix} m+n-j-1 \\ n-j\end{bmatrix}_q [m]_q^{j-1}.$$

Extracting the coefficient of $x_1x_2\cdots x_{n-1}$ on both sides of Identity (18) and shifting the index, we obtain an analog of Corollary 1.

Corollary 2. *For all integers $n \geq 1$ and $k \geq 0$,*

$$n^k = \sum_{\lambda\in\Lambda_{n-1}^k} f^\lambda m^{\lambda}_{n,k+\frac{1}{2}}. \tag{19}$$

Corollary 2 can also be proved by an adaptation of the deletion-insertion algorithm of Halverson and Lewandowski [8].

5. Limiting Vacillating Tableaux of Even Length

Sections 5 and 6 focus on limiting vacillating tableaux. Restricting the map ψ of Subsection 2.2 to limiting vacillating tableaux, we proved in [3, Proposition 7] that the map ψ induces a bijection

$$\mathcal{SVT}_k(\mu) \longleftrightarrow \Pi(k,|\mu|)\times\mathcal{SYT}(\mu),$$

where $\Pi(k,j)$ is the set of partitions of $[k]$ with exactly j blocks and $\mathcal{SVT}(\mu)$ is the set of SYTs of shape μ. Indeed, the set partition $\boldsymbol{B}$ and the tableaux T_k constructed when applying the algorithm of ψ to a limiting vacillating tableau satisfy the property that content(T_k) = max$(\boldsymbol{B})$. Hence, $\boldsymbol{B}^*$ would have all its blocks marked. Therefore, we can represent a limiting vacillating tableau of shape μ and length $2k$ simply by a pair $(\boldsymbol{B},T)$, where $\boldsymbol{B}$ is a set partition of $[k]$ with $j=|\mu|$ blocks and T is an SYT of shape μ.

Example 7. Let $k = 4$ and $\mu = (2,1)$. There are twelve limiting vacillating tableaux of shape $(2,1)$ and length 8. The corresponding pairs $(\boldsymbol{B},T)$ are given in Table 2.

5.1. Formula for $a_k(\mu)$

Let $a_k(\mu)$ be the number of limiting vacillating tableau of shape μ and length $2k$. It follows from the above bijection that for $k \geq 0$,

$$a_k(\mu) \quad = \quad S(k,|\mu|)f^\mu. \tag{20}$$

Note that $a_0(\emptyset) = 1$ and $a_k(\emptyset) = 0$ for $k > 0$.

limiting vacillating tableau	$(\boldsymbol{B}, T)$
$(\emptyset, \emptyset, \ydiagram{1}, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{2}, \ydiagram{2,1})$	$(\{1,2 \mid 3 \mid 4\}, \young(12,3))$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{2}, \ydiagram{2,1})$	$(\{1,3 \mid 2 \mid 4\}, \young(12,3))$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2}, \ydiagram{2,1})$	$(\{1,4 \mid 2 \mid 3\}, \young(12,3))$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{1}, \ydiagram{2}, \ydiagram{2}, \ydiagram{2,1})$	$(\{2,3 \mid 1 \mid 4\}, \young(12,3))$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{2}, \ydiagram{2,1})$	$(\{2,4 \mid 1 \mid 3\}, \young(12,3))$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{2}, \ydiagram{3}, \ydiagram{2}, \ydiagram{2,1})$	$(\{3,4 \mid 1 \mid 2\}, \young(12,3))$
$(\emptyset, \emptyset, \ydiagram{1}, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{2,1})$	$(\{1,2 \mid 3 \mid 4\}, \young(13,2))$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{2,1})$	$(\{1,3 \mid 2 \mid 4\}, \young(13,2))$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{1,1,1}, \ydiagram{1,1}, \ydiagram{2,1})$	$(\{1,4 \mid 2 \mid 3\}, \young(13,2))$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{2,1})$	$(\{2,3 \mid 1 \mid 4\}, \young(13,2))$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{1,1}, \ydiagram{2,1})$	$(\{2,4 \mid 1 \mid 3\}, \young(13,2))$
$(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{1,1}, \ydiagram{2,1})$	$(\{3,4 \mid 1 \mid 2\}, \young(13,2))$

Table 2: The limiting vacillating tableaux of shape $(2,1)$ and length 8.

5.2. The Sum $\sum_{\mu} a_{k_1}(\mu) a_{k_2}(\mu)$

Let k_1, k_2 be two positive integers and $k = k_1 + k_2$. We say that a set partition of $[k]$ is (k_1, k_2)-*connecting* if for any block X of the partition, $\min X \le k_1 < \max X$, or equivalently, $X \cap [k_1] \neq \emptyset$ and $X \cap \{k_1+1, \ldots, k_1+k_2\} \neq \emptyset$.

Example 8. Let $k_1 = k_2 = 2$. Then the set partitions $\{1,3 \mid 2,4\}$ and $\{1,4 \mid 2,3\}$ are $(2,2)$-connecting while $\{1,2 \mid 3,4\}$ is not. There are altogether three $(2,2)$-connecting set partitions of $[k]$ for $k = 4$; the third one is $\{1,2,3,4\}$.

Theorem 12. *For all integers $k_1, k_2 \geq 1$, the sum $\sum_{\mu} a_{k_1}(\mu) a_{k_2}(\mu)$ is the number of (k_1, k_2)-connecting set partitions of $[k_1 + k_2]$.*

Proof. For $i = 1, 2$, let P_i be a limiting vacillating tableau of shape μ and length

$2k_i$. Then P_i can be represented as a pair $(\boldsymbol{B}_i, T_i)$, where $\boldsymbol{B}_i$ is a set partition of $[k_i]$ with $j = |\mu|$ blocks, and T_i is an SYT of shape μ.

Draw the standard diagram of $\boldsymbol{B}_1$ followed by the reverse of the standard diagram of $\boldsymbol{B}_2$. Assume the blocks of $\boldsymbol{B}_i$ are $X_1^i, X_2^i, \ldots, X_j^i$, ordered by their maximal elements. Using the RSK algorithm, the pair (T_1, T_2) uniquely determines permutation σ. Now merge the blocks of $\boldsymbol{B}_1$ with those of $\boldsymbol{B}_2$ by connecting the maximal element of the blocks X_t^1 with that of $X_{\sigma(t)}^2$, for all $t \leq j$. Finally, by relabeling the vertices as $1, 2, \ldots, k_1 + k_2$ from left to right, we get a (k_1, k_2)-connecting set partition.

Conversely, given a (k_1, k_2)-connecting set partition, one can easily recover $\boldsymbol{B}_1, \boldsymbol{B}_2$ and the permutation σ. Applying the RSK algorithm to σ, we get the SYTs T_1, T_2. Therefore, the above procedure is a bijection. $\square$

The sum $\sum_\mu a_{k_1}(\mu) a_{k_2}(\mu)$ can be computed as

$$\sum_{\mu} a_{k_1}(\mu) a_{k_2}(\mu) = \sum_{j \geq 0} j! S(k_1, j) S(k_2, j). \tag{21}$$

When $k_1 = k_2$, the sum is sequence A023997 in OEIS.

5.3. The Sum $a_k = \sum_\mu a_k(\mu)$

Let a_k be the number of limiting vacillating tableaux of length $2k$. Theorem 13 follows immediately from Equation (20).

Theorem 13 ([3]). *For $k \geq 0$,*

$$a_k = \sum_{j=0}^{k} \left(S(k,j) \sum_{\mu \vdash j} f^\mu \right) = \sum_{j=0}^{k} S(k,j) I_j, \tag{22}$$

where I_j is the number of involutions in $\mathfrak{S}_j$ and $I_0 = 1$.

The sequence $(a_k)_{k=0}^\infty$ is A004211 in OEIS with the initial values $1, 1, 3, 11, 49, \ldots$. It has the following combinatorial interpretations relating to set partitions. For all integers $k \geq 1$,

(a) a_k is the number of set partitions of $[k]$ such that each element is colored either red or blue, and for each block the minimal element is colored red (see [6]);

(b) a_k is the number of pairs $(\boldsymbol{B}, \sigma)$ where $\boldsymbol{B}$ is a set partition of $[k]$ and σ is an involution defined on the blocks of $\boldsymbol{B}$ (see [3]).

The proof of Theorem 12 provides a new combinatorial interpretation. In that proof when $(\boldsymbol{B}_1, T_1) = (\boldsymbol{B}_2, T_2)$, the pair (T_1, T_2) corresponds to an involution.

Hence, the construction yields a symmetric set partition of $[2k]$ (symmetric under the map $i \leftrightarrow 2k+1-i$).

Corollary 3. *For $k \geq 1$, the sum $a_k = \sum_{\mu} a_k(\mu)$ is the number of symmetric (k,k)-connecting set partitions of $[2k]$.*

5.4. The Sum $\sum_{\mu} a_k(\mu) f^{\mu}$

Theorem 14. *Let $v_k = \sum_{\mu} a_k(\mu) f^{\mu}$. Then v_k is the k-th Fubini number that counts the number of ordered set partitions of $[k]$.*

Proof. This is because v_k counts the number of triples $(\boldsymbol{B}, T, S)$ where $\boldsymbol{B}$ is a partition of $[k]$ with $j = |\mu|$ blocks, and T and S are SYTs of shape μ. Via the RSK algorithm, the pair (T, S) corresponds to a permutation in $\mathfrak{S}_j$, which gives the ordering on the blocks of $\boldsymbol{B}$. $\square$

5.5. Product of $a_k(\mu)$ with Schur Functions $\sum_{\mu} a_k(\mu) s_{\mu}(\boldsymbol{x})$

Next, we connect the limiting vacillating tableaux to symmetric functions.

Theorem 15. *For all integers $k \geq 0$,*

$$\sum_{\mu} a_k(\mu) s_{\mu}(\boldsymbol{x}) = \sum_{j \geq 0} S(k,j) h_1(\boldsymbol{x})^j. \tag{23}$$

Proof. The left side of Identity (23) is a summation of $\boldsymbol{x}^S$ over the set of triples $(\boldsymbol{B}, T, S)$ where $\boldsymbol{B}$ is a set partition of $[k]$ into j blocks, T is an SYT of some shape μ of size j, and S is an SSYT of shape μ filled with positive integers. The pair (T, S), under the inverse of the RSK algorithm, corresponds uniquely to the sequence of positive integers $\boldsymbol{i} = (i_1, i_2, \ldots, i_j)$, where

$$\boldsymbol{x}^S = x_{i_1} x_{i_2} \cdots x_{i_j}.$$

Hence,

$$\begin{aligned}\sum_{(\boldsymbol{B},T,S)} \boldsymbol{x}^S &= \sum_{j} S(k,j) \sum_{\boldsymbol{i} \in (\mathbb{Z}^+)^j} x_{i_1} x_{i_2} \cdots x_{i_j} \\ &= \sum_{j} S(k,j) \left(\sum_{i} x_i \right)^j = \sum_{j} S(k,j) h_1(\boldsymbol{x})^j.\end{aligned}$$ $\square$

The principle specialization of Identity (23) is

$$\sum_{\mu} a_k(\mu) s_{\mu}(1, q, \ldots, q^{n-1}) = \sum_{j \geq 0} S(k,j) [n]_q^j. \tag{24}$$

In particular, for $q = 1$ we get

$$\sum_{\mu} a_k(\mu) s_\mu(1^n) = \sum_{j\geq 0} S(k,j) n^j, \tag{25}$$

where

$$s_\mu(1^n) = \prod_{u\in\mu} \frac{n+c(u)}{h(u)}.$$

Note that the right side of Identity (25) is the evaluation of the k-th Bell polynomial $B_k(x) := \sum_j S(k,j)x^j$ at $x = n$. It is interesting to compare Identities (24), (25), and the q-hook-content formula (9) with the following identity of Halverson and Thiem [9, Corollary 2.4]: for all positive integers n, k,

$$\sum_{\lambda\in\Lambda_n^k} f^\lambda(q) m_{n,k}^\lambda = \sum_j S(k,j)[n]_q[n-1]_q \cdots [n-j+1]_q, \tag{26}$$

where $f^\lambda(q)$ is a q-analog of f^λ and is given by the q-hook length formula

$$f^\lambda(q) = q^{b(\lambda)}[n]_q! \prod_{u\in\lambda} \frac{1}{[h(u)]_q}.$$

When $q = 1$, Identity (26) reduces to Identity (12) by using $x^k = \sum_r S(k,r)(x)_r$.

6. Limiting Vacillating Tableaux of Odd Length

Consider the case when one applies the map ψ to a limiting vacillating tableau of length $2k$. If we stop at the step $2k-1$ with the index $k-\frac{1}{2}$ but skip the last step, then we will have the edge set $E_{k-\frac{1}{2}} = E_k$ and the partial tableau $T_{k-\frac{1}{2}}$ whose content does not contain k. Adjusting indices, we get that limiting vacillating tableaux of shape μ and length $2k+1$ are in one-to-one correspondence with the pairs $(\boldsymbol{B}, T)$, where $\boldsymbol{B}$ is a set partition of $[k+1]$ with $j = |\mu| + 1$ blocks, and T is an SYT of shape μ.

6.1. Formula for $a_{k+\frac{1}{2}}(\mu)$

Let $a_{k+\frac{1}{2}}(\mu)$ be the number of limiting vacillating tableaux of shape μ and length $2k+1$. From the above correspondence, we have that for $k \geq 0$,

$$a_{k+\frac{1}{2}}(\mu) = S(k+1, |\mu|+1) f^\mu. \tag{27}$$

Note that $a_{k+\frac{1}{2}}(\emptyset) = 1$.

6.2. The Sum $\sum_\mu a_{k_1+\frac{1}{2}}(\mu)a_{k_2+\frac{1}{2}}(\mu)$

Let $\ell \le k$ be positive integers. We say that a set partition of $[k]$ is *ℓ-connecting* if for any block X of the partition, $\min(X) \le \ell \le \max(X)$. Equivalently, a set partition of $[k]$ is ℓ-connecting if and only if for any block X, $X \cap [\ell] \ne \emptyset$ and $X \cap \{\ell, \dots, k\} \ne \emptyset$.

Theorem 16. *For all integers $k_1, k_2 \ge 0$, the sum $\sum_\mu a_{k_1+\frac{1}{2}}(\mu)a_{k_2+\frac{1}{2}}(\mu)$ is the number of (k_1+1)-connecting set partitions of $[k_1+k_2+1]$.*

Proof. The proof is similar to that of Theorem 12, except that we start with two set partitions of $[k_1+1]$ and $[k_2+1]$ with $j+1$ blocks, respectively, where $j = |\mu|$. Note that the block containing (k_i+1) is the $(j+1)$-th block of $\boldsymbol{B}_i$ for $i = 1, 2$. After merging the blocks according to the permutation determined by the two SYTs and identifying the element k_1+1 of $\boldsymbol{B}_1$ with $k_2'+1$ of $\boldsymbol{B}_2$, we get a set partition of $[k_1+k_2+1]$. $\square$

The sum $\sum_\mu a_{k_1+\frac{1}{2}}(\mu)a_{k_2+\frac{1}{2}}(\mu)$ can be computed as

$$\sum_\mu a_{k_1+\frac{1}{2}}(\mu)a_{k_2+\frac{1}{2}}(\mu) = \sum_{j\ge 0} j!S(k_1+1, j+1)S(k_2+1, j+1). \tag{28}$$

When $k_1 = k_2$, the sum is the sequence A014235 in OEIS. A similar argument gives Corollary 4

Corollary 4. *(i) For all integers $k_1, k_2 \ge 0$, $\sum_\mu a_{k_1+\frac{1}{2}}(\mu)a_{k_2}(\mu)$ is the number of set partitions of $[k_1+k_2+1]$ such that $\min(X) \le k_1+1 \le \max(X)$ for all blocks X, and k_1+1 is the maximal element in its block. This sum can be computed as*

$$\sum_\mu a_{k_1+\frac{1}{2}}(\mu)a_{k_2}(\mu) = \sum_{j\ge 0} j!S(k_1+1, j+1)S(k_2, j).$$

(ii) For all integers $k_1 \ge 1$ and $k_2 \ge 0$, $\sum_\mu g_{k_1}(\mu)a_{k_2}(\mu) = \sum_\mu g_{k_1-\frac{1}{2}}(\mu)a_{k_2+\frac{1}{2}}(\mu)$ is the number of set partitions of $[k_1+k_2]$ such that $\min(X) \le k_1$ for all blocks X. This sum can be computed as

$$\begin{aligned}\sum_\mu g_{k_1}(\mu)a_{k_2}(\mu) &= \sum_\mu g_{k_1-\frac{1}{2}}(\mu)a_{k_2+\frac{1}{2}}(\mu)\\ &= \sum_{j\ge 0} j!B(k_1, j)S(k_2, j)\\ &= \sum_{j\ge 0} j!\tilde{B}(k_1, j)S(k_2+1, j+1).\end{aligned}$$

(iii) *For all integers $k_1, k_2 \geq 0$, $\sum_\mu g_{k_1+\frac{1}{2}}(\mu) a_{k_2}(\mu)$ is the number of set partitions of $[k_1+k_2+1]$ such that $\min(X) \leq k_1+1$ for all blocks X, and k_1+1 is the maximal element in its block. This sum can be computed as*

$$\sum_\mu g_{k_1+\frac{1}{2}}(\mu) a_{k_2}(\mu) = \sum_{j\geq 0} j!\tilde{B}(k_1+1, j) S(k_2, j).$$

(iv) *For all integers $k_1, k_2 \geq 0$, $\sum_\mu g_{k_1}(\mu) a_{k_2+\frac{1}{2}}(\mu)$ is the number of set partitions of $[k_1+k_2+1]$ such that $\min(X) \leq k_1+1$ for all blocks X, and k_1+1 is the minimal element in its block. This sum can be computed as*

$$\sum_\mu g_{k_1}(\mu) a_{k_2+\frac{1}{2}}(\mu) = \sum_{j\geq 0} j!B(k, j) S(k+1, j+1).$$

6.3. The Sum $\sum_\mu a_{k+\frac{1}{2}}(\mu)$

Let $a_{k+\frac{1}{2}} = \sum_\mu a_{k+\frac{1}{2}}(\mu)$ be the number of limiting vacillating tableaux of shape μ and length $2k+1$. Theorem 17 follows immediately from Equation (27).

Theorem 17. *For $k \geq 0$,*

$$a_{k+\frac{1}{2}} = \sum_{j=0}^{k} S(k+1, j+1) \sum_{\mu \vdash j} f^\mu = \sum_{j=0}^{k} S(k+1, j+1) I_j \tag{29}$$

where I_j is the number of involutions in $\mathfrak{S}_j$ and $I_0 = 1$.

The numbers of limiting vacillating tableaux of odd and even length, respectively, are closely related.

Theorem 18. *The sequence $(a_{k+\frac{1}{2}})_{k=0}^\infty$ is the binomial transform of $(a_k)_{k=0}^\infty$. That is,*

$$a_{k+\frac{1}{2}} = \sum_{r=0}^{k} \binom{k}{r} a_r. \tag{30}$$

Proof. $a_{k+\frac{1}{2}}$ counts the number of pairs of $(\boldsymbol{B}, T)$, where $\boldsymbol{B}$ is a set partition of $[k+1]$ with $j+1$ blocks, and T is an SYT of shape μ with $\mu \vdash j$. For a partition of $[k+1]$ with $j+1$ blocks, assume the block containing $k+1$ has $k+1-r$ elements. There are $\binom{k}{r}$ ways to pick those elements. The remaining r elements then form a partition with j blocks, which, together with the SYT T of shape μ, corresponds to a limiting vacillating tableau of length $2r$. □

The sequence $(a_{k+\frac{1}{2}})_{k=0}^\infty$ is A007405 in OEIS, whose initial terms are 1, 2, 6, 24, 116, 648, ... Interestingly, $(a_k)_{k=0}^\infty$ can also be obtained from the binomial transform of $(a_{k+\frac{1}{2}})_{k=0}^\infty$.

Theorem 19. *For $k \geq 0$,*

$$a_{k+1} = \sum_{j=0}^{k} \binom{k}{j} a_{j+\frac{1}{2}}. \tag{31}$$

Proof. By Equation (29) a combinatorial interpretation of $a_{j+\frac{1}{2}}$ is the number of pairs $(\boldsymbol{B}', \sigma')$, where $\boldsymbol{B}'$ is a set partition of $[j+1]$ with r blocks and $\sigma' \in \mathfrak{S}_r$ is an involution on the blocks of $\boldsymbol{B}'$ such that the block containing $j+1$ is a 1-cycle.

Let $(\boldsymbol{B}, T)$ be a pair counted by a_{k+1}, that is, $\boldsymbol{B}$ is a partition of $[k+1]$ whose blocks are $X_1, X_2, \dots, X_t$ (ordered by their maximal elements), and T is an SYT of some shape λ with $|\lambda| = t$. Let σ be the involution corresponding to (T, T) under the RSK algorithm. Note that X_t is the block of $\boldsymbol{B}$ containing the element $k+1$. Assume $|X_t| = 1+k-j$. Remove the block X_t from $\boldsymbol{B}$ and introduce a new element $\star$. Using the following steps, we construct a set partition $\boldsymbol{B}'$ on $([k+1] - X_t) \cup \{\star\}$ together with an involution σ' on the blocks of $\boldsymbol{B}'$, such that the block containing $\star$ is a 1-cycle of σ'.

(i) If $\sigma(t) = t$, that is, X_t is not paired with another block of $\boldsymbol{B}$ under σ, then $\{\star\}$ is the t-th block of $\boldsymbol{B}'$, and this block is a 1-cycle of σ'.

(ii) If $\sigma(t) = r$ for some $r < t$, that is, (X_t, X_r) is a 2-cycle of σ, then add $\star$ to the block X_r to form the r-th block of $\boldsymbol{B}'$, and let $\sigma'(r) = r$.

(iii) For any block X_a of $\boldsymbol{B}$ with $\sigma(a) = b$ where $a, b \neq t$, X_a and X_b are both blocks of $\boldsymbol{B}'$ with $\sigma'(a) = b$. Note that it is possible that $a = b$ in this step.

The resulting pairs $(\boldsymbol{B}', \sigma')$ are counted by $a_{j+\frac{1}{2}}$. There are $\binom{k}{j}$ ways to choose the elements in X_t. Summing over j completes the proof. □

Example 9. Let $k = 8$. As an example, assume that $\boldsymbol{B} = \{2, 5 \mid 1, 3, 6 \mid 4, 8 \mid 7, 9\}$, where the blocks are ordered by their maximal elements. If $\sigma = (1)(23)(4)$ in cycle notation, then the operation yields the set partition $\boldsymbol{B}' = \{2, 5 \mid 1, 3, 6 \mid 4, 8 \mid \star\}$ with involution $\sigma' = (1)(23)(4)$. If $\sigma = (12)(34)$ in cycle notation, then the operation yields the set partition $\boldsymbol{B}' = \{2, 5 \mid 1, 3, 6 \mid 4, 8, \star\}$ with involution $\sigma' = (12)(3)$.

Combining Theorems 18 and 19, one can derive that $(a_k)_{k=0}^{\infty}$ satisfies the recurrence relation

$$a_{k+1} = \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} a_j.$$

This can be interpreted combinatorially using the model of bi-colored set partitions defined in Subsection 5.3, item (a) after Theorem 13, where j is the number of elements not in the same block as $k+1$.

We remark that the sequences $(a_k)_{k=0}^{\infty}$ and $(a_{k+\frac{1}{2}})_{k=0}^{\infty}$ are the unique pair of sequences that start at $s_0 = 1$ and are recursively defined by the binomial transforms $t_k = \sum_{j=0}^{k} \binom{k}{j} s_j$ and $s_{k+1} = \sum_{j=0}^{k} \binom{k}{j} t_j$, respectively.

Another combinatorial interpretation of $a_{k+\frac{1}{2}}$ is the number of type B set partitions, which was first introduced by Reiner [15] in the study of intersection lattice for the classical reflection groups of type B.

Definition 4 ([15])**.** A set partition of type B is a partition π of the set $[-k] \cup [k]$ into blocks satisfying the following conditions:

(a) for any block X of π, its opposite $-X$ is also a block of π;

(b) there is at most one zero-block, which is defined to be a block X such that $X = -X$.

Theorem 20. *For $k \geq 1$, the integer $a_{k+\frac{1}{2}}$ is the number of type B partitions of the set $[-k] \cup [k]$.*

Proof. Using Equation (29), we interpret $a_{k+\frac{1}{2}}$ as the number of pairs $(\boldsymbol{B}', \sigma')$ where $\boldsymbol{B}'$ is a set partition of $[k+1]$ with $j+1$ blocks and $\sigma' \in \mathfrak{S}_j$ is an involution of size j.

Take two identical copies of the set partition $\boldsymbol{B}'$, the first on the elements $1, 2, \ldots, k+1$, and the second on the elements $-1, -2, \ldots, -(k+1)$. Let $X_1, \ldots, X_{j+1}$ be the blocks in the first copy of $\boldsymbol{B}'$ with $k+1 \in X_{j+1}$, and $-X_1, \ldots, -X_{j+1}$ the corresponding blocks in the second copy. We form a type B partition of $\{\pm 1, \ldots, \pm k\}$ using the following steps.

(i) If $\sigma'(a) = b$ while $a \neq b$, then merge block X_a with $-X_b$, and block X_b with $-X_a$.

(ii) If $\sigma'(a) = a$, then leave both X_a and $-X_a$ unchanged.

(iii) Merge X_{j+1} with $-X_{j+1}$ and then remove the elements $\pm(k+1)$. If the resulting set is not empty, then it is the zero block. Otherwise, discard the empty set.

The collection of all the blocks obtained by these steps is the desired type B partition. We leave it to the reader to check that the above construction indeed gives a bijection. $\square$

Example 10. As an example, assume $\boldsymbol{B}' = \{2 \mid 1, 3 \mid 6 \mid 5, 7 \mid 4, 8\}$, where the blocks are ordered by their maximal elements, and $\sigma' = (1)(23)(4)$ in cycle notation. Then the operations yield the type B partition $\{2 \mid -2 \mid 1, 3, -6 \mid -1, -3, 6 \mid 5, 7 \mid -5, -7, \mid 4, -4\}$.

6.4. The Sum $\sum_\mu a_{k+\frac{1}{2}}(\mu) f^\mu$

For a set S, a *cyclically ordered set partition* of S is a set partition $\boldsymbol{B}$ of S together with a cyclic ordering on the blocks of $\boldsymbol{B}$. For $S = [k+1]$, the number of cyclically ordered set partitions can be computed by the summation $\sum_{j\geq 0} j! S(k+1, j+1)$.

Theorem 21. *Let $v_{k+\frac{1}{2}} = \sum_\mu a_{k+\frac{1}{2}}(\mu) f^\mu$ for $k \geq 0$. Then $v_{k+\frac{1}{2}}$ is the number of the following structures:*

(a) cyclically ordered set partitions of $[k+1]$, or

(b) set partitions of $[k+1]$ with a linear order on all the blocks except the block containing $k+1$.

The proof is omitted since it is similar to that of Theorem 14. The sequence $(v_{k+\frac{1}{2}})_{k=0}^\infty$ is A000629 in OEIS, which is the binomial transform of Fubini numbers.

6.5. Product of $a_{k+\frac{1}{2}}(\mu)$ with Schur Functions $\sum_\mu a_{k+\frac{1}{2}}(\mu) s_\mu(\boldsymbol{x})$

Theorem 22 gives the analogous result to Theorem 15. Again, we skip the proof due to its similarity to the proof of Theorem 15.

Theorem 22. *For $k \geq 0$,*

$$\sum_\mu a_{k+\frac{1}{2}}(\mu) s_\mu(\boldsymbol{x}) = \sum_{j\geq 0} S(k+1, j+1) h_1(\boldsymbol{x})^j. \tag{32}$$

Consequently,

$$\sum_\mu a_{k+\frac{1}{2}}(\mu) s_\mu(1, q, \ldots, q^{n-1}) = \sum_{j\geq 0} S(k+1, j+1)[n]_q^j.$$

7. Final Remarks and Future Projects

Comparing the results of Sections 3 and 4 and those of Sections 5 and 6, we notice that an integer sequence relating to vacillating tableaux of odd length is often the binomial transform of its analog of the vacillating tableaux of even length. This includes the following pairs:

- $g_k = \sum_\mu g_k(\mu)$ and $g_{k+\frac{1}{2}} = \sum_\mu g_{k+\frac{1}{2}}(\mu)$
- $u_k = \sum_\mu g_k(\mu) f^\mu$ and $u_{k+\frac{1}{2}} = \sum_\mu g_{k+\frac{1}{2}}(\mu) f^\mu$
- $a_k = \sum_\mu a_k(\mu)$ and $a_{k+\frac{1}{2}} = \sum_\mu a_{k+\frac{1}{2}}(\mu)$
- $v_k = \sum_\mu a_k(\mu) f^\mu$ and $v_{k+\frac{1}{2}} = \sum_\mu a_{k+\frac{1}{2}}(\mu) f^\mu$.

In addition, we observe that there are several instances that a sequence relating to simplified vacillating tableaux is the binomial convolution of the Bell numbers and the analogous sequence relating to limiting vacillating tableaux. Theorem 23 shows such an instance.

Theorem 23. *We have* $g_k = \sum_{j=0}^{k} \binom{k}{j} \mathrm{Bell}(j) a_{k-j}$, *where* $\mathrm{Bell}(j)$ *is the* j*-th Bell number.*

Proof. A vacillating tableau of length $2k$ is represented by a pair $(\boldsymbol{B}^*, T)$, where $\boldsymbol{B}^*$ is a partition of $[k]$ with j marked blocks, and T is an SYT of some shape μ with $|\mu| = j$. Let $S \subseteq [k]$ be the union of those unmarked blocks. Then $\boldsymbol{B}^*$ can be viewed as a disjoint union of two structures: a set partition $\boldsymbol{B}_1$ of S, and a set partition $\boldsymbol{B}_2$ of $[k] - S$ with exactly j blocks. The number of choices for $\boldsymbol{B}_1$ is $\mathrm{Bell}(|S|)$, while the pairs $(\boldsymbol{B}_2, T)$ correspond to limiting vacillating tableaux of length $2(k - |S|)$ that are counted by $a_{k-|S|}$. $\square$

It follows that the exponential generating function for $(g_k)_{k=0}^{\infty}$ is the product of those for the Bell numbers and for $(a_k)_{k=0}^{\infty}$, i.e.,

$$\sum_{k \geq 0} g_k \frac{x^k}{k!} = \exp(e^x - 1) \cdot \sum_{k \geq 0} a_k \frac{x^k}{k!}.$$

A similar relation holds between the following pairs of sequences:

- g_k and a_k,
- $g_{k+\frac{1}{2}}$ and $a_{k+\frac{1}{2}}$,
- u_k and v_k, and
- $u_{k+\frac{1}{2}}$ and $v_{k+\frac{1}{2}}$.

Table 3 summarizes the exponential generating functions of the above sequences.

We conclude this paper with some final remarks and future research projects.

Using growth diagrams, Krattenthaler [11] explored the results on simplified vacillating tableaux and their connections to crossings and nestings of set partitions with a broader context. This context involves the enumeration of fillings of Young diagrams, where certain restrictions are imposed on the increasing and decreasing chains of the fillings. Recently, Krattenthaler [12] extended Identity (12) to n-vacillating tableaux starting at an arbitrary shape by studying the growth diagrams associated with the Young diagram $((n+k)^n, n+k-1, \ldots, n+1, n)$.

The growth diagrams prove to be a valuable tool as they provide in-depth insights into tableau operations and facilitate visualizations of vacillating tableaux between arbitrary shapes μ and λ. It is intriguing to explore how the combinatorial identities considered in this paper would transform for such generalized vacillating tableaux.

	Sequence	OEIS ID	EGF
Limiting vacillating tableaux	$a_k = \sum_\mu a_k(\mu)$	A004211	$\exp\left(\frac{e^{2x}-1}{2}\right)$
	$a_{k+\frac{1}{2}} = \sum_\mu a_{k+\frac{1}{2}}(\mu)$	A007405	$\exp\left(x+\frac{e^{2x}-1}{2}\right)$
	$v_k = \sum_\mu a_k(\mu) f^\mu$	A000670	$\frac{1}{2-e^x}$
	$v_{k+\frac{1}{2}} = \sum_\mu a_{k+\frac{1}{2}}(\mu) f^\mu$	A000629	$\frac{e^x}{2-e^x}$
Simplified vacillating tableaux	$g_k = \sum_\mu g_k(\mu)$	A002872	$\exp\left(e^x+\frac{e^{2x}-3}{2}\right)$
	$g_{k+\frac{1}{2}} = \sum_\mu g_{k+\frac{1}{2}}(\mu)$	A080337	$\exp\left(x+e^x+\frac{e^{2x}-3}{2}\right)$
	$u_k = \sum_\mu g_k(\mu) f^\mu$	A059099	$\frac{\exp(e^x-1)}{2-e^x}$
	$u_{k+\frac{1}{2}} = \sum_\mu g_{k+\frac{1}{2}}(\mu) f^\mu$	not in OEIS	$\frac{\exp(x+e^x-1)}{2-e^x}$

Table 3: Summary of exponential generating functions.

In the context of fillings of Young diagrams and other general polyominoes, the growth diagrams frequently establish sequences of integer partitions under specific restrictions, bearing resemblance to vacillating tableaux. Notably, the shapes of these integer partitions play a significant role in characterizing the northeast and southeast chains within the fillings. This observation has been explored in various papers, including [11, 17, 7]. The paper [12] leveraged this property to define a sub-family of fillings that exhibit special combinatorial structures. One interesting direction is to develop q-analogs of those combinatorial identities to include the statistics of northeast and southeast chains.

Identity (12) arises from the representation theory of partition algebra. Several other identities presented in this paper also exhibit strong indications of a connection to representation theory. Unveiling and understanding such connections would be highly valuable. It is worth noting that, although vacillating tableaux are closely related to the partition algebra, the role of limiting vacillating tableaux in representation theory remains unclear and requires further exploration.

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