# POWERED NUMBERS IN SHORT INTERVALS

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**Abstract**

Generalizing the powerful numbers, Barry Mazur introduced the concept of powered numbers. In this note, we study powered numbers over short intervals.

## 1. Introduction and Main Results

A number $n$ is *squarefull* or *powerful* if its prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ satisfies $a_i \geq 2$ for all $1 \leq i \leq r$. Similarly, $n$ is *k-full* if $a_i \geq k$ for all $1 \leq i \leq r$. In contrast, $n$ is *squarefree* if $a_i = 1$ for all $i$. For example, $72 = 2^3 \cdot 3^2$ is squarefull, $243 = 3^5$ is 5-full, and $30 = 2 \cdot 3 \cdot 5$ is squarefree. Let $Q_k(x)$ denote the number of positive $k$-full numbers up to $x$. It is known that

$$Q_k(x) = \prod_p \Big(1 + \sum_{m=k+1}^{2k-1} \frac{1}{p^{m/h}}\Big) x^{1/k} + O(x^{1/(k+1)}),$$

where the product is over all primes (see [4] for example). Recently, the author [2] considered powerful numbers in short intervals $(x, x+y]$ and proved that

$$Q_2(x+y) - Q_2(x) \ll \frac{y}{\log(y+1)} \quad \text{uniformly for} \quad 1 \leq y \leq x, \tag{1}$$

which slightly improves upon a result of De Koninck, Luca and Shparlinski [3].

Let us recall the famous *abc*-conjecture: For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that, for any integers $a, b, c$ with $a + b = c$ and $\gcd(a, b) = 1$, the bound

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \kappa(abc)^{1+\epsilon}$$

holds with $\kappa(n) := \prod_{p|n} p$, the *squarefree kernel* of $n$. Conditional on the *abc* conjecture, the author presented at the INTEGERS conference 2023 that,

$$Q_2(x+y) - Q_2(x) \ll \frac{y}{\exp(\log 2 \cdot (\log y)^{0.09})}, \tag{2}$$

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based on a very recent breakthrough result of Kelley and Meka [5] on the density of integer sequence without 3-term arithmetic progressions.

Generalizing and smoothing the $k$-full numbers, Mazur [6] proposed the study of *powered numbers*. Let $l \geq 1$ be a real number (not necessarily an integer). A positive integer $n$ is an *$l$-powered number* if

$$\kappa(n) \leq n^{1/l}.$$

Clearly, every number is 1-powered, and every $k$-full number is also a $k$-powered number. For $0 < \theta < 1$, define

$$S_\theta(x) := \#\{n \leq x : \kappa(n) \leq n^\theta\}.$$

By an elementary argument, one has $x^\theta \ll S_\theta(x) \ll_\epsilon x^{\theta+\epsilon}$ for any $\epsilon > 0$. Recently, Brüdern and Robert [1] did a finer study on $S_\theta(x)$ and proved that

$$S_\theta(x) = (1 + o(1))\, x^\theta \cdot F((1-\theta)\log x) \cdot \frac{1}{\theta}\sqrt{\frac{2}{1-\theta}} \cdot \frac{1}{\sqrt{\log x \cdot \log\log x}} \tag{3}$$

where

$$F(v) = \frac{6}{\pi^2} \sum_{m \geq 1} \frac{1}{\prod_{p|m}(p+1)} \min\Big(1, \frac{e^v}{m}\Big) \quad \text{and} \quad \log F(v) = (1 + o(1))\sqrt{\frac{8v}{\log v}}.$$

Thus, it is natural to ask if one can say something about $l$-powered numbers in short intervals.

**Question 1.** Given $l > 1$, find a uniform upper bound for

$$S_{1/l}(x + y) - S_{1/l}(x)$$

for all $1 \leq y \leq x$ that is independent of $x$.

For any integer $n$ with prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, let

$$q(n) := \prod_{i \text{ with } a_i = 1} p_i \quad \text{be its “squarefree” part,}$$

and $P(n) := \max p_i$ be its largest prime factor. For example, $q(120) = 3 \cdot 5 = 15$, and $P(120) = 5$. When $P(q(n))$ is small, one has the following general observation.

**Proposition 1.** *For $1 \leq y \leq x$, we have*

$$\#\{x < n \leq x + y : P(q(n)) \leq \log y \log\log y\} \ll \frac{y \log\log(y+2)}{\log(y+2)}.$$

Hence, to obtain non-trivial upper bounds for Question 1, we need a better understanding of those $l$-powered numbers in $(x, x+y]$ with large prime factors in their "squarefree" parts. Although we lack such knowledge at the moment, one can prove the following conditional result.

**Theorem 1.** *Let $l > 3/2$. Under the abc-conjecture, there exists some constant $c_l > 0$ such that*

$$S_{1/l}(x+y) - S_{1/l}(x) \ll_l \frac{y}{\exp((c_l \log y)^{0.09})}$$

*for all $1 \le y \le x$.*

Theorem 1 goes beyond squarefull or 2-powered numbers and shows that few such $l$-powered numbers exist in short intervals when $l > 3/2$. It also corrects an inaccuracy in (2) as the implicit constant may not be exactly $\log 2$ when $l = 2$. Since $l$-powered numbers behave like $l$-full numbers, we have the following conjecture.

**Conjecture 1.** Given $l > 1$ and any $\epsilon > 0$,

$$S_{1/l}(x+y) - S_{1/l}(x) \ll_{\epsilon,l} y^{1/l+\epsilon}$$

for all $1 \le y \le x$.

Even stronger, we suspect the following to be true.

**Conjecture 2.** Given $l > 1$,

$$S_{1/l}(x+y) \le S_{1/l}(x) + S_{1/l}(y)$$

for all sufficiently large $x$ and $y$ (in terms of $l$).

In another direction, one can study short-interval behavior when $l$ is very close to 1. We have the following result.

**Theorem 2.** *Let $\omega(n)$ be any increasing function with $\omega(n) \to \infty$ and $\frac{n}{\omega(n)} \to \infty$ as $n \to \infty$. Let*

$$S_\omega(x) := \#\Big\{ n \le x : \kappa(n) \le n^{1 - \frac{1}{4\omega(\log n / \log\log n)}} \Big\}.$$

*Then, for sufficiently large integer $y$, there exist infinitely many integers $x$ such that $S_\omega(x+y) - S_\omega(x) = y$.*

This shows that it is impossible to obtain any $o(y)$-bound for integers $n \in (x, x+y]$ satisfying $\kappa(n) \le n^{1-o(1)}$ with any slow decaying function $o(1)$.

Throughout the paper, $p$ and $p_i$ stand for prime numbers. The symbols $f(x) = O(g(x))$, $f(x) \ll g(x)$, and $g(x) \gg f(x)$ are equivalent to $|f(x)| \le Cg(x)$ for some constant $C > 0$. Also, $f(x) = O_{\lambda_1,\dots,\lambda_r}(g(x))$ and $f(x) \ll_{\lambda_1,\dots,\lambda_r} g(x)$ mean that the implicit constant may depend on $\lambda_1, \dots, \lambda_r$. Furthermore, $f(x) = o(g(x))$ means $\lim_{x\to\infty} f(x)/g(x) = 0$.

## 2. Proof of Proposition 1

*Proof.* We may assume that $y$ is sufficiently large as the theorem is clearly true for bounded $y$ by picking a suitably large implicit constant. Note that any positive integer $n$ can be factored uniquely as $n = a_n b_n$ where $a_n$ is squarefree and $b_n$ is squarefull with $\gcd(a_n, b_n) = 1$. By the Prime Number Theorem, the number of primes up to $\log y \log\log y$ is at most $1.001 \log y$ for $y$ large enough. Since $P(q(n)) \leq \log y \log\log y$, the number of squarefree numbers $a_n$ is at most

$$2^{1.001 \log y} = y^{1.001 \log 2} \leq y^{0.7}. \tag{4}$$

Now, we are going to split $n$'s according to the size of $a_n$. For $a_n > y^{0.9}$, we have $x/a_n < b_n \leq x/a_n + y/a_n < x/a_n + y^{0.1}$. Hence, there are at most $y^{0.1} + 1$ choices for $b_n$, and there are at most

$$\sum_{a_n}{}^{*} (y^{0.1} + 1) \ll y^{0.8} \tag{5}$$

such $n$'s in $(x, x + y]$ by (4). Here and below, $\Sigma^*$ denotes a sum over squarefree numbers with prime factors at most $\log y \log\log y$.

For $a_n \leq y^{0.9}$, the number of such $n$'s is at most

$$\begin{aligned}\sum_{a \leq y^{0.9}}{}^{*} Q_2\Big(\frac{x+y}{a}\Big) - Q_2\Big(\frac{x}{a}\Big) &\ll \sum_{a \leq y^{0.9}}{}^{*} \frac{y/a}{\log(y/a)} \ll \frac{y}{\log y} \sum_{a \leq y^{0.9}}{}^{*} \frac{1}{a} \\ &\ll \frac{y}{\log y} \prod_{p \leq \log y \log\log y} \Big(1 + \frac{1}{p}\Big) \\ &\leq \frac{y}{\log y} \exp\Big(\sum_{p \leq \log y \log\log y} \frac{1}{p}\Big) \ll \frac{y \log\log y}{\log y}\end{aligned} \tag{6}$$

by (1), the inequality $e^x \geq 1 + x$, and Merten's estimate $\sum_{p \leq x} \frac{1}{p} = \log\log x + M + O(\frac{1}{\log x})$ for some absolute constant $M$. Then Proposition 1 follows from (5) and (6). $\square$

## 3. Proof of Theorem 1

We need a recent groundbreaking result of Kelley and Meka [5]: Let $N \geq 2$ and $A \subset \{1, 2, ..., N\}$ be a set with no non-trivial three-term arithmetic progressions, i.e., solutions to $x + y = 2z$ with $x \neq y$. Then

$$|A| \ll \frac{N}{2^{(\log N)^{0.09}}}. \tag{7}$$

*Proof of Theorem 1.* Fix $l > 3/2$ and set $\epsilon = \frac{l}{3} - \frac{1}{2} > 0$. First, we suppose that $y \leq x^{\epsilon/l}$. We claim that there is no non-trivial three term arithmetic progression among the $l$-powered numbers in the interval $(x, x+y]$ under the *abc*-conjecture. Suppose the contrary. Then we have three $l$-powered numbers in arithmetic progression $x < a_1b_1 < a_2b_2 < a_3b_3 \leq x+y$ with $a_1, a_2, a_3$ squarefree, $b_1, b_2, b_3$ squarefull and $\gcd(a_i, b_i) = 1$ for $1 \leq i \leq 3$. Note that, by definition of $l$-powered number,

$$a_i\kappa(b_i) = \kappa(a_ib_i) \leq (a_ib_i)^{1/l} \quad \text{or} \quad a_i \leq \frac{b_i^{1/(l-1)}}{\kappa(b_i)^{l/(l-1)}} \quad \text{or} \quad a_ib_i \leq \Big(\frac{b_i}{\kappa(b_i)}\Big)^{l/(l-1)}.$$

This implies

$$\frac{b_i}{\kappa(b_i)} \geq x^{1-1/l}. \tag{8}$$

Say,

$$a_1b_1 = a_2b_2 - d \quad \text{and} \quad a_3b_3 = a_2b_2 + d$$

for some positive integer $d$ with $2d \leq y$. Multiplying the above two equations, we get

$$a_1b_1a_3b_3 + d^2 = a_2^2b_2^2.$$

Say $D^2 = \gcd(a_2^2b_2^2, d^2)$ as the numbers are perfect squares. Then, the three integers $\frac{a_1a_3b_1b_3}{D^2}$, $\frac{d^2}{D^2}$, and $\frac{a_2^2b_2^2}{D^2}$ are pairwise relatively prime, and we have the equation

$$\frac{a_1a_3b_1b_3}{D^2} + \frac{d^2}{D^2} = \frac{a_2^2b_2^2}{D^2}.$$

Now, by the *abc*-conjecture and (8), we obtain

$$\begin{aligned}\frac{x^2}{D^2} \leq \frac{a_2^2b_2^2}{D^2} &\ll_l \kappa\Big(\frac{a_1a_3b_1b_3}{D^2}\frac{d^2}{D^2}\frac{a_2^2b_2^2}{D^2}\Big)^{1+\epsilon} \\ &\ll_l \big(a_1a_2a_3\kappa(b_1)\kappa(b_2)\kappa(b_3)\big)^{1+\epsilon}\kappa\Big(\frac{d}{D}\Big)^{1+\epsilon} \\ &\ll_l \Big(x^3\frac{\kappa(b_1)}{b_1}\frac{\kappa(b_2)}{b_2}\frac{\kappa(b_3)}{b_3}\Big)^{1+\epsilon}\Big(\frac{d}{D}\Big)^{1+\epsilon} \ll x^{3(1+\epsilon)/l}\frac{y^{1+\epsilon}}{D^{1+\epsilon}}.\end{aligned}$$

Since $1 \leq D \leq d \leq y \leq x^{\epsilon/l}$, the above implies

$$x^{2-3(1+\epsilon)/l} \ll_l D^{1-\epsilon}y^{1+\epsilon} \ll y^2 \leq x^{2\epsilon/l},$$

which is a contradiction when $x \geq y > C_l$ for some sufficiently large constant $C_l$ since

$$2 - \frac{3(1+\epsilon)}{l} = 2 - \frac{3(1+\frac{l}{3}-\frac{1}{2})}{l} = 1 - \frac{3}{2l} > \frac{2}{3}\Big(1-\frac{3}{2l}\Big) = \frac{2}{l}\Big(\frac{l}{3}-\frac{1}{2}\Big) = \frac{2\epsilon}{l}.$$

Clearly, the theorem is true for $1 \leq y \leq C_l$ by picking an appropriate implicit constant. So, we may assume $y > C_l$. Since arithmetic progressions are invariant under translation, we may shift all $l$-powered numbers in $(x, x+y]$ to numbers in $(0, y]$ without 3-term arithmetic progressions. By (7), we have

$$Q_2(x+y) - Q_2(x) \ll \frac{y}{\exp(\log 2 \cdot (\log y)^{0.09})}$$

which gives the theorem.

Now, suppose $y > x^{\epsilon/l}$. Then, the interval $(x, x+y]$ is a subset of the union of subintervals of length $x^{\epsilon/l}$:

$$(x, x+x^{\epsilon/l}] \cup (x+x^{\epsilon/l}, x+2x^{\epsilon/l}] \cup \cdots \cup \left(x + \left\lfloor \frac{y}{x^{\epsilon/l}} \right\rfloor x^{\epsilon/l}, x + \left(\left\lfloor \frac{y}{x^{\epsilon/l}} \right\rfloor + 1\right) x^{\epsilon/l}\right].$$

Over each subinterval $(x + jx^{\epsilon/l}, x + (j+1)x^{\epsilon/l}]$, we have the bound

$$Q_2(x+(j+1)x^{\epsilon/l}) - Q_2(x+jx^{\epsilon/l}) \ll \frac{x^{\epsilon/l}}{\exp(\log 2 (\log x^{\epsilon/l})^{0.09})}.$$

Summing over $\lfloor \frac{y}{x^{\epsilon/l}} \rfloor + 1$ of these intervals, we have

$$\begin{aligned} Q_2(x+y) - Q_2(x) \ll & \frac{y}{x^{\epsilon/l}} \cdot \frac{x^{\epsilon/l}}{\exp(\log 2 (\log x^{\epsilon/l})^{0.09})} \\ \ll & \frac{y}{\exp(c_l (\log x)^{0.09})} \le \frac{y}{\exp(c_l (\log y)^{0.09})} \end{aligned}$$

with $c_l = \log 2 \cdot (\frac{1}{3} - \frac{1}{2l})^{0.09}$. This gives the theorem as well. □

## 4. Proof of Theorem 2

*Proof.* Let $2 = p_1 < p_2 < \ldots < p_k$ be the first $k$ prime numbers with $\omega(k)$ large. Let $l$ be an integer such that $\frac{2k}{3\omega(k)} \le l \le \frac{k}{\omega(k)+1}$. This is possible as $k/\omega(k) \to \infty$. Consider the moduli

$$m_j := \prod_{i=1}^{l} p_{jl+i}^2 \quad \text{for} \quad 0 \le j \le u := \left\lfloor \frac{k}{l} - 1 \right\rfloor,$$

and the system of congruence equations

$$n + j \equiv 0 \pmod{m_j} \quad \text{for} \quad 0 \le j \le u. \tag{9}$$

The above congruence system has a solution $n \pmod{m_1 m_2 \cdots m_u}$ by the Chinese Remainder Theorem. As $k \to \infty$, the Prime Number Theorem gives

$$p_k = (1+o(1)) k \log k, \quad \text{and} \quad p_{k-l} = (1+o(1))(k-l)\log(k-l) = (1+o(1)) k \log k$$

since $l \leq k/\omega(k)$ and $\omega(k) \to \infty$. Also,

$$e^{2\sum_{i\leq k-l}\log p_i} \leq m_0 m_1 \cdots m_{\lfloor k/l-1\rfloor} \leq e^{2\sum_{i\leq k}\log p_i}.$$

Hence, there is an integer $n_0 \in (m_0 m_1 \cdots m_u, 2m_0 m_1 \cdots m_u]$ satisfying (9). Then

$$n_0 + i \quad \text{with} \quad \left\lfloor \frac{2u}{3} \right\rfloor \leq i \leq u$$

give $u - \lfloor \frac{2u}{3} \rfloor + 1$ consecutive integers of size $e^{(2+o(1))k\log k}$. Note that $k = (\frac{1}{2} + o(1))\frac{\log(n_0+i)}{\log\log(n_0+i)}$ and

$$m_{\lfloor \frac{2u}{3} \rfloor} \geq (p_{\lfloor \frac{2u}{3} \rfloor l})^{2l} \geq (0.6k\log 0.6k)^{2l} \geq e^{1.1\frac{k}{\omega(k)}\log k} \geq (n_0+i)^{\frac{1}{2\omega(k)}}$$

by the Prime Number Theorem, $\frac{2k}{3\omega(k)} \leq l$, and $\omega$ being increasing. Thus, each $n_0 + i$ has squarefree kernel

$$\kappa(n_0+i) \leq \frac{n_0+i}{\sqrt{m_{\lfloor \frac{2u}{3} \rfloor}}} \leq (n_0+i)^{1-\frac{1}{4\omega(\log(n_0+i)/\log\log(n_0+i))}}.$$

This gives the theorem by setting $y = u - \lfloor \frac{2u}{3} \rfloor + 1$ and $x = n_0 + \lfloor \frac{2u}{3} \rfloor$. Since $k/\omega(k)$, $y$, and $x$ grow as $k \to \infty$, we have longer and longer stretches of consecutive integers satisfying $\kappa(n) \leq n^{1-1/(4\omega(\log n/\log\log n))}$. This gives infinitely many $x$'s satisfying the condition of Theorem 2 for any given fixed large integer $y$. □

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