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HENRIK ERIKSSON'S BULGARIAN SOLITAIRE VARIANT

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Abstract

Bulgarian Solitaire, originally a recreational mathematics topic about pile sizes of various objects under a rearrangement procedure, is now studied as an operation on the integer partitions of a fixed number. Martin Gardner popularized the mathematical puzzle in one of his late *Scientific American* columns, but the unusual name comes from an earlier article by Henrik Eriksson in the Swedish language journal *Elementa*. That article ends with a variant game which does not yet seem to have been fully analyzed. Here we provide a thorough study of Henrik Eriksson's Bulgarian Solitaire variant and its immediate generalizations, also placing these in the context of several broad generalizations of Bulgarian Solitaire developed in the past ten years. Of particular interest, characterizing and enumerating the Garden of Eden states (those with no predecessors under the operation) uses a new "stretched" version of Dyson's rank statistic for integer partitions.

– In memory of Henrik Eriksson (1942–2017)

1. Bulgarian Solitaire

Given an integer $n \geq 0$, we say $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a partition of n if each λ_i is a positive integer, $\sum \lambda_i = n$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$ (there is also one empty partition of $n = 0$). The number of parts s is called the length of λ. We will also use the frequency notation $\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n}$ which indicates that λ has m_1 parts 1, m_2 parts 2, etc. Write $P(n)$ for the set of partitions of n and $p(n)$ for the number of partitions of n. (In the last section, we will expand the definition to allow partitions with certain positive rational parts.)

Definition 1. Given $n \geq 1$ and $\lambda = (\lambda_1, \ldots, \lambda_s) \in P(n)$, the Bulgarian Solitaire operation $B : P(n) \to P(n)$ is determined by

$$B(\lambda) = (s, \lambda_1 - 1, \ldots, \lambda_s - 1)$$

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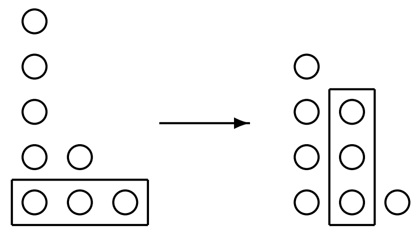

Figure 1: The image of $(5, 2, 1)$ under B is $(4, 3, 1)$.

where any zeros are removed and the parts may need to be reordered to be in nonincreasing order.

This operation can be visualized using the Ferrers diagram of a partition where each part corresponds to a column of dots. A move in Bulgarian Solitaire removes one dot from each column; it is helpful to think of removing the bottom row of the diagram. See Figure 1 for an example.

Applying B to all partitions of a fixed n creates a finite dynamical system; see Figure 2 showing the $n = 4$ case. Note that the resulting directed graph with vertices $P(4)$ has one (weakly) connected component with a 3-cycle and one partition with no preimage.

What became known as Bulgarian Solitaire dates to Moscow in 1980 and spread in mathematical circles around Europe. Henrik Eriksson heard of it from a colleague who had attended a conference in Sofia, thus his article "Bulgarisk Patiens" in the Swedish language journal *Elementa* [6]. He soon traveled to California and communicated the puzzle with the name Bulgarian Solitaire; from Donald Knuth and Ron Graham it passed to Martin Gardner who included it in a 1983 article [9]. See [11] for a more detailed history including Eriksson's declaration, "The silly name is my invention, silly because it is neither Bulgarian nor a solitaire."

The compelling question of Bulgarian Solitaire concerns longterm behavior, es-

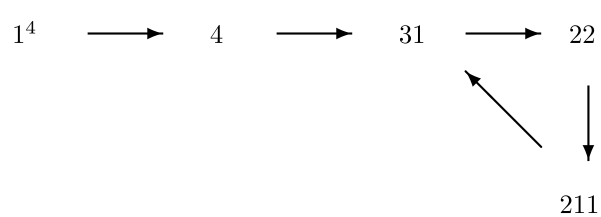

Figure 2: The Bulgarian Solitaire operation B on the partitions of 4. Parentheses and commas are omitted, and exponents are used to denote repetition.

pecially for certain numbers of objects (e.g., Gardner's 45 playing cards). As a dynamical system on a finite set, B applied to $P(n)$ for any n must eventually form cycles: Given $\lambda \in P(n)$, the partitions $\lambda, B(\lambda), B^2(\lambda) = B(B(\lambda)), B^3(\lambda), \ldots$ cannot all be distinct. Call a partition λ cyclic if $\lambda = B^j(\lambda)$ for some j. The following description of cyclic partitions was found independently by Bojanov [2], Eriksson [6], and Toom [17], all in problems sections of journals primarily intended for students, and by Brandt [3] in the *Proceedings of the American Mathematical Society*. Let T_m be the mth triangular number $m(m+1)/2$.

Theorem 1. *Given $n \geq 1$, write n uniquely as $T_m + m'$ for some integer m' with $0 \leq m' \leq m$. The cyclic partitions of n under the operation B have the form*

$$(m + r_m, m - 1 + r_{m-1}, \ldots, 1 + r_1, r_0)$$

where each integer r_i satisfies $0 \leq r_i \leq 1$ and $\sum_i r_i = m'$. Therefore, there are $\binom{m+1}{m'}$ cyclic partitions. In particular, n has a unique cyclic partition (a 1-cycle, also known as a fixed point) when each $r_i = 0$ and $n = T_m$.

It is clear that the described partitions are closed under the operation B. See Drensky [4] for an expository account of the necessity part of the proof, using a "cradle" model attributed to Björner. In Figure 2, the cyclic partitions of 4 for B are $(3,1)$, $(2,2)$, and $(2,1,1)$, which can be thought of as $(2,1)$ with an additional dot in the first, second, or third column, respectively. In general, the r_i all occupy the $(m+1)$st diagonal above the m full diagonals corresponding to the triangular partition $(m, \ldots, 1)$. Combinatorially, the r_i correspond to a necklace of $m+1$ white and black beads with m' being black, or a circular/cyclic weak composition of m' having $m+1$ parts each 0 or 1.

Brandt also applied Pólya enumeration to determine the number of connected components which is the same as the number of disjoint cycles [3, Theorem 5].

Theorem 2. *Given $n \geq 1$, write n uniquely as $T_m + m'$ for some integer m' with $0 \leq m' \leq m$. The number of connected components in $P(n)$ under the operation B is*

$$\frac{1}{m+1} \sum_{d \mid (m+1, m')} \varphi(d) \binom{(m+1)/d}{m'/d}$$

where $(m+1, m')$ denotes the greatest common divisor of $m+1$ and m', and $\varphi(d)$ is the Euler phi function.

The smallest n for which Bulgarian Solitaire disconnects $P(n)$ into multiple components is $n = 8$. However, there is a single connected components only when n is within one of a triangular number.

Other distinguished partitions in these dynamical systems are those without a preimage, such as $(1,1,1,1)$ in Figure 2. These are known as Garden of Eden (GE) partitions.

The most succinct characterization of GE partitions uses a partition statistic defined by Dyson [5]: The rank of $\lambda = (\lambda_1, \ldots, \lambda_s)$ is $\lambda_1 - s$. For instance, $\text{rank}((5,2,1)) = 5 - 3 = 2$.

Write $N(m,n)$ for the number of partitions λ of n with $\text{rank}(\lambda) = m$. We will need the following two generating functions, given with standard q-series notation such as the q-Pochhammer symbol $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$ for $n > 0$.

Proposition 3. *Given an integer m, two generating functions for $N(m,n)$ are*

$$\sum_{n\geq 0} N(m,n)q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{\frac{1}{2}n(3n-1)+mn}(1-q^n), \tag{1}$$

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} N(m,n)w^m q^n = \sum_{n\geq 0}\frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}. \tag{2}$$

Proof. Equation (1) was stated by Dyson [5] and proven by Atkin and Swinnerton-Dyer [1].

Equation (2) makes use of the Durfee square of a partition, the largest possible square starting from the lower left corner of the Ferrers diagram (as we draw them here). The subpartition above an $n \times n$ Durfee square has at most n parts and the subpartition to the right of the Durfee square has largest part at most n which gives Jacobi's result

$$\sum_{n\geq 0} p(n)q^n = \sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n^2}.$$

Inside the Durfee square, there are an equal number of dots in the left column and in the bottom row, so they are not necessary for the rank computation. In order for the exponent of w to record the rank, the dots above the Durfee square in the left column each have weight w while the dots to the right of the Durfee square in the bottom row each have weight w^{-1}. □

See Figure 3 for the partition $(5,2,1)$, its Durfee square size 2×2, and the weights w and w^{-1} contributing to its rank, 2.

The following characterization of Garden of Eden partitions was given by the author and Jones [12] soon followed by the description using rank and enumeration results by the author and Sellers [14]. Write $ge(n)$ for the number of GE partitions of n under B.

Theorem 4. *The Garden of Eden partitions of n under B are exactly the $\lambda \in P(n)$ for which* $\text{rank}(\lambda) \leq -2$.

A generating function for $ge(n)$ is

$$\sum_{n\geq 0} ge(n)q^n = \sum_{m\leq -2}\frac{1}{(q;q)_\infty}\sum_{r\geq 1}(-1)^{r-1}\left(q^{(3r^2-r)/2} + q^{(3r^2+r)/2}\right)q^{|m|r} \tag{3}$$

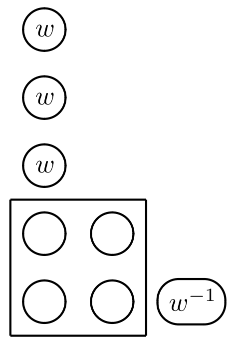

Figure 3: The Durfee square of $(5, 2, 1)$ and the weights on some dots.

and $ge(n)$ satisfies

$$ge(n) = p(n-3) - p(n-9) + p(n-18) - \cdots = \sum_{j \geq 1} (-1)^{j+1} p(n - 3T_j). \qquad (4)$$

The expression in Equation (3) comes from Equation (1) and the rank symmetry $N(m, n) = N(-m, n)$. This simplifies nicely to an expression that gives the expression in Equation (4).

See Equation (8) in Section 3 for another generating function for $ge(n)$.

Note that a GE partition for B must have length at least 3 since any two-part partition λ has $\operatorname{rank}((\lambda_1, \lambda_2)) = \lambda_1 - 2 \geq -1$ and any one-part partition has nonnegative rank.

The next section introduces Henrik Eriksson's variant of Bulgarian Solitaire and its immediate generalizations, the family of operations H_k. We also place these in the context of several broad generalizations of Bulgarian Solitaire. The last section gives analogues of the Bulgarian Solitaire results in this section for the H_k.

2. Henrik Eriksson's Variant and Its Generalizations

At the end of his 1981 article, Henrik Eriksson offered a modification of Bulgarian Solitaire. "As a variation on the rules, one may choose to take two cards from each pile and let them form a new pile. (From a one-card pile, you only take the card that is available.)" [6]

To define this and related operations symbolically, we use the partition frequency notation.

Definition 2. Given integers $k \geq 1$ and $n \geq 1$ with $\lambda = (\lambda_1, \ldots, \lambda_s) \in P(n)$ and $\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n}$, the generalized Henrik Eriksson operation $H_k : P(n) \to P(n)$

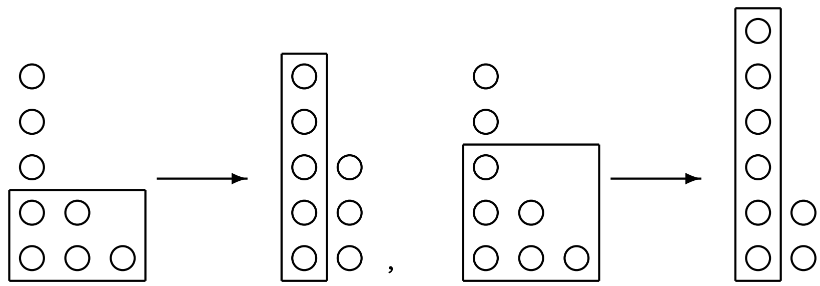

Figure 4: Representations of $H_2((5,2,1)) = (5,3)$ and $H_3((5,2,1)) = (6,2)$.

is determined by

$$H_k(\lambda) = (ks - (k-1)m_1 - (k-2)m_2 - \cdots - m_{k-1}, \lambda_1 - k, \ldots, \lambda_s - k)$$

where any zeros and negative values are removed and the parts may need to be reordered to be in nonincreasing order.

Examples on $(5,2,1) \sim 1^1 2^1 5^1$ include $H_2((5,2,1)) = (2 \cdot 3 - 1, 5 - 2) = (5,3)$ and $H_3((5,2,1)) = (3 \cdot 3 - 2 - 1, 5 - 3) = (6,2)$. The operations H_k are more easily understood graphically; see Figure 4 for the same examples.

Note that $H_1 = B$ so that H_k is a generalization of Bulgarian Solitaire. At the other extreme, H_n on $P(n)$ sends every partition to (n) giving a single component with $p(n) - 1$ Garden of Eden partitions each leading to the 1-cycle (n). Note that for $k \geq 2$, this means that there are GE partitions under H_k of length 2 (such as $(1,1)$) in contrast to Bulgarian Solitaire.

In the remainder of this section, we consider the H_k operations in terms of broad generalizations of Bulgarian Solitaire considered by Jeffrey Olson [15] and the team of Kimmo Eriksson (Henrik's son), Markus Jonsson, and Jonas Sjöstrand [7, 8]. We consider these in order of increasing generality. As appropriate, we mention results from these authors that address the analysis of the H_k operations.

In 2018, Kimmo Eriksson, Jonsson, and Sjöstrand introduced q-proportional Solitaire [7]: Given real $q \in (0,1]$, dots from rows $\{1, 1 + \lfloor 1/q \rfloor, 1 + \lfloor 2/q \rfloor, \ldots\}$ (counting from the bottom) are removed from each column ($\lfloor \cdot \rfloor$ indicates the integer floor). Given n, Bulgarian Solitaire corresponds to $q = 1/n$. Also, $q = 1$ removes dots from every row, matching H_n. For no other k, however, is H_k a q-proportional Solitaire. The choice $q = 2/n$, for instance, removes dots from rows 1 and $1 + \lfloor n/2 \rfloor$ which is not in general the bottom two rows.

In 2020, the same authors defined L-Solitaire [8]: For a positive integer n and a set $L \subseteq \{1, \ldots, n\}$, dots in the rows given by L are removed from each column. Bulgarian Solitaire corresponds to $L = \{1\}$. The generalized Henrik Eriksson operation H_k corresponds to $L = \{1, \ldots, k\}$. Also, each q-Solitaire is an L-Solitaire.

An L-Solitaire result relevant here is that, for each choice of n and L, there is at most one 1-cycle on $P(n)$ under that L-Solitaire [8, Theorem 3(b)]. For the H_k operations, this will also follow from Theorem 5 below.

In 2016, Olson defined a very general σ-Solitaire which only requires that the same number of dots be removed from columns of the same height [15]: Given a positive integer n, $\lambda = (\lambda_1, \ldots, \lambda_s) \in P(n)$, and rule

$$\sigma : \{1, \ldots, n\} \to \{0, \ldots, n-1\}$$

with $\sigma(i) < i$ for each $1 \leq i \leq n$, the operation $\bar{\sigma} : P(n) \to P(n)$ is determined by

$$\bar{\sigma}(\lambda) = \left(\sigma(\lambda_1), \ldots, \sigma(\lambda_s), n - \sum_{i=1}^{s} \sigma(\lambda_i)\right)$$

where any zeros are removed and the parts may need to be reordered. Every L-Solitaire (and thus every q-proportional Solitaire and each H_k) is a σ-Solitaire. In particular, the rule σ for H_k is

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots \\ 0 & 0 & \cdots & 0 & 1 & 2 & \cdots \end{pmatrix}.$$

One σ-Solitaire result is that, for every rule σ, any partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ with $\lambda_1 < s - 1$ is a Garden of Eden partition [15, Corollary 6]. By Theorem 4, this condition is necessary and sufficient for Bulgarian Solitaire. Theorem 8 below characterizes the GE partitions for each H_k.

There is another family of operations on $P(n)$ related to the H_k operations. Note that all the operations discussed so far create one new part/column with each application. The author and Kolitsch considered operations that remove k rows and create k columns in each step, matching Bulgarian Solitaire at $k = 1$ and partition conjugation at $k = n$ (and $k = n - 1$) [10, 13].

3. Analysis of the H_k Operations

There are significant differences between the dynamical system behavior of $B = H_1$ and the generalized Henrik Eriksson operations H_k for different k. Contrast, for example, Figure 2 with $P(4)$ under Bulgarian Solitaire and Figure 5 with the same partitions under H_2: Figure 5 shows two components, one with a 2-cycle and two Garden of Eden partitions, the other an isolated 1-cycle. (Note that $(3, 1)$ is its own predecessor, so it is not a GE partition for H_2.)

In this final section, we provide the analogues of Theorems 1, 2, and 4 for the H_k, fully characterizing the cyclic partitions and GE partitions, and counting the number of connected components. We also introduce and study a generalization of Dyson's rank statistic on partitions.

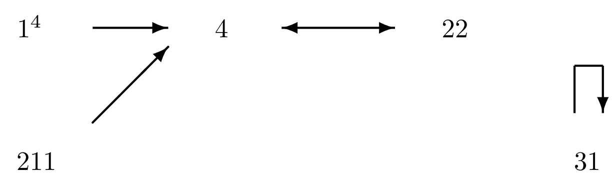

Figure 5: Henrik Eriksson's operation H_2 on the partitions of 4.

For the cyclic partitions under H_k, the partition $(m, m-1, \ldots, 1) \in P(T_m)$ of Theorem 1 is replaced by $(km, k(m-1), \ldots, k) \in P(kT_m)$, a scaled triangular partition. The enumeration uses Euler's polynomial coefficient or generalized binomial coefficient $\binom{n}{k}_\ell$, the coefficient of x^k in $(1 + x + \cdots + x^{\ell-1})^n$ (so that the standard binomial coefficient is $\binom{n}{k}_2$).

Theorem 5. *Given positive integers k and n, write n uniquely as $kT_m + m'$ for some integer m' with $0 \le m' \le k(m+1) - 1$. The cyclic partitions of n under the operation H_k have the form*

$$(km + r_m, k(m-1) + r_{m-1}, \ldots, k + r_1, r_0)$$

where each integer r_i satisfies $0 \le r_i \le k$ and $\sum_i r_i = m'$. There are $\binom{m+1}{m'}_{k+1}$ cyclic partitions. In particular, n has a 1-cycle when the r_i are equal which occurs when $n = kT_m + \ell(m+1)$ for $0 \le \ell \le k-1$.

The core ideas of the proof were given by Henrik Eriksson:

> The analysis [of H_2] is surprisingly almost as simple [as that of B]. One first notices that in each step the newly formed pile always has an even number of cards, unless one or more one-card piles disappear at the same time. The number of odd piles can therefore only decrease and reaches its minimum in the final cycle. Now replace each card with half a brick! The previous reasoning [about Bulgarian Solitaire] can be applied and shows as before that all levels are filled except possibly the top one and that all odd half-bricks must be on the top level. [6]

See Figure 6 for a vertical compression by 1/2 applied to the partition $(5, 2, 1)$. The H_2 operation then becomes the B operation applied to partitions allowed to have parts in $\frac{1}{2}\mathbb{Z}^+$, the positive half-integers. Also, notice in Figures 5 and 6 that the number of odd parts weakly decreases when H_2 is applied.

Proof. First, the bound on m' comes from each of the $m+1$ columns having up to k additional dots but avoiding the case where $r_i = k$ for all i since that would give $n = kT_{m+1}$.

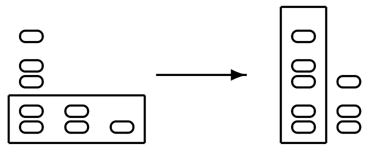

Figure 6: The compressed version of $H_2((5,2,1)) = (5,3)$ shown as Bulgarian Solitaire on half-integers with $B((2\frac{1}{2}, 1, \frac{1}{2})) = (2\frac{1}{2}, 1\frac{1}{2})$.

For sufficiency of the cyclic partitions,

$$\begin{aligned}H_k((km + r_m, k(m-1) + r_{m-1}, \ldots, k + r_1, r_0))\\ = (km + r_0, k(m-1) + r_m, k(m-2) + r_{m-1}, \ldots, r_1)\end{aligned}$$

since the first m parts are reduced by k and the operation also removes r_0 to make a new first part $km + r_0$.

For necessity, use vertical compression by $1/k$ to apply the Bulgarian Solitaire analysis to partitions with parts in $\frac{1}{k}\mathbb{Z}^+$. Each application of H_k removes all parts strictly less than k to make one new part, so that the number of nonmultiples of k is weakly decreasing. Equivalently, each application of B on partitions with parts in $\frac{1}{k}\mathbb{Z}^+$ removes all parts less than 1 to make one new part, so that the number of noninteger parts is weakly decreasing. Eventually, by the reasoning supporting Theorem 1 for standard Bulgarian Solitaire, each application of H_k involves at most one nonmultiple of k so that, in the compressed system, the first m diagonals are filled with "complete bricks" and any partial bricks are in the $(m+1)$st diagonal which corresponds to the r_i.

For the enumeration, the r_i correspond to a weak composition of m' with $m+1$ parts each satisfying $0 \le r_i \le k$. Adding one to each part gives a standard integer composition of $m' + m + 1$ with $m+1$ positive integer parts each at most $k+1$; these are counted by $\binom{m+1}{m'}_{k+1}$ [16, p. 124].

The statements about 1-cycles and the n for which they occur follow directly. $\square$

For example, $P(4)$ under the operation H_2 has $4 = 2T_1 + 2$ and $\binom{2}{2}_3$ cyclic partitions. Indeed, the coefficient of x^2 in $(1 + x + x^2)^2 = 1 + 2x + 3x^2 + 2x^3 + x^4$ is 3, matching the cyclic partitions shown in Figure 5.

For another example of the result that the number of nonmultiples of k is weakly decreasing under H_k, Figure 7 shows a subset of $P(8)$ under H_3 where nonmultiples of 3 are marked; note that both partitions in the cycle have one nonmultiple of 3.

The same Pólya enumeration approach used by Brandt for Theorem 2 applies for the generalized Eriksson operations.

Theorem 6. *Given positive integers k and n, write n uniquely as $kT_m + m'$ for some integer m' with $0 \le m' \le k(m+1) - 1$. The number of connected components*

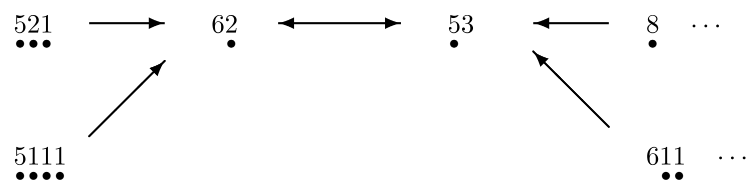

Figure 7: A subset of $P(8)$ under H_3 with nonmultiples of 3 marked.

in $P(n)$ under the operation H_k is

$$\frac{1}{m+1}\sum_{d|(m+1,m')}\varphi(d)\binom{(m+1)/d}{m'/d}_{k+1}$$

where $(m+1,m')$ denotes the greatest common divisor of $m+1$ and m', and $\varphi(d)$ is the Euler phi function.

The only difference from the proof of Theorem 2 is in the polynomial coefficient for counting compositions with a greater bound on their parts. For example, $P(4)$ under the operation H_2 with $4 = 2T_1 + 2$ has

$$\frac{1}{2}\sum_{d|(2,2)}\varphi(d)\binom{2/d}{2/d}_3 = \frac{1}{2}\left(1\cdot\binom{2}{2}_3 + 1\cdot\binom{1}{1}_3\right) = \frac{1}{2}(3+1) = 2$$

components, matching the state diagram of Figure 5.

The analysis of the Garden of Eden partitions for the H_k operations requires a new generalization of Dyson's rank statistic that we call the k-stretched rank.

Definition 3. Given a positive integer k and a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$, define the k-stretched rank as $\mathrm{rank}_k(\lambda) = \lambda_1 - ks$.

For example, $\mathrm{rank}_2((5,2,1)) = 5 - 2\cdot 3 = -1$. The $k = 1$ case is the standard rank.

Let $N_k(m,n)$ be the number of partitions λ of n with $\mathrm{rank}_k(\lambda) = m$. Analogous to Proposition 3, we determine two generating functions for $N_k(m,n)$.

Proposition 7. *Given an integer m, two generating functions for $N_k(m,n)$ are*

$$\sum_{n\geq 0} N_k(m,n)q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{\frac{1}{2}n((2k+1)n-1)+mn}\prod_{s=n}^{kn}(1-q^s), \quad (5)$$

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} N_k(m,n)w^m q^n = \sum_{n\geq 0}\frac{q^{n^2}w^{(1-k)n}}{(wq;q)_n(w^{-k}q;q)_n}. \quad (6)$$

Proof. To show Equation (5), we modify the Atkin–Swinnerton-Dyer proof of Equation (1) [1, Lemma 1].

A partition $\lambda = (\lambda_1, \ldots, \lambda_s) \in P(n)$ with $\mathrm{rank}_k(\lambda) = m$ has $\lambda_1 = ks + m$. The number of such partitions equals the number of partitions of $n - ks - m$ with exactly $s - 1$ parts, none of which exceeds $ks + m$. It is therefore the coefficient of $x^{n-ks-m}z^{s-1}$ in

$$\prod_{u=1}^{ks+m}(1 - zx^u)^{-1}$$

which, following their application of two identities of Hardy and Wright, is the coefficient of x^n in

$$x^{(k+1)s+m-1}\sum_{u=0}^{s-1}(-1)^u x^{\frac{1}{2}u(u+1)+u(ks+m-1)}\prod_{r=1}^{u}(1 - x^r)^{-1}\prod_{t=1}^{s-u-1}(1 - x^t)^{-1}.$$

This gives

$$\begin{aligned}
&\sum_{n\geq 0} N_k(m,n)x^n \prod_{u\geq 1}(1 - x^u)\\
&= \sum_{u\geq 1}(-1)^{u-1}x^{\frac{1}{2}u(u-1)+mu}(1 - x^u)\left[\sum_{s\geq u} x^{ksu+s-u}\prod_{t=1}^{s-u}(1 - x^t)^{-1}\right]\prod_{r\geq u+1}(1 - x^r)\\
&= \sum_{u\geq 1}(-1)^{u-1}x^{\frac{1}{2}u(u-1)+mu}\left[x^{ku^2}\prod_{r\geq ku+1}(1 - x^r)^{-1}\right](1 - x^u)\prod_{r\geq u+1}(1 - x^r)\\
&= \sum_{u\geq 1}(-1)^{u-1}x^{\frac{1}{2}u((2k+1)u-1)+mu}\prod_{r=u}^{ku}(1 - x^r)
\end{aligned}$$

from which Equation (5) follows.

For Equation (6), the proof of Equation (2) is modified by the dots in the bottom row each having weight w^{-k} rather than w^{-1}. Also, the dots in the left column and bottom row of the $n \times n$ Durfee square no longer cancel out, rather they contribute the factor $w^n w^{-kn} = w^{(1-k)n}$. $\square$

See Figure 8 for the weights showing $\mathrm{rank}_2((5,2,1)) = -1$. Notice that the bottom left dot, being in both the left column and bottom row, has weight $w \cdot w^{-2} = w^{-1}$. The dots in the 2×2 Durfee square have combined weight w^{-2}, matching the factor in the numerator of Equation (6). All the weights combine to w^{-1}, matching the 2-stretched rank of $(5, 2, 1)$.

Our last results use the q-binomial coefficient or Gaussian polynomial $\begin{bmatrix} m+n \\ n \end{bmatrix}$ which gives the number of partitions that fit inside an $m \times n$ rectangle. More specifically, the coefficient of q^ℓ in the polynomial gives the number of partitions of ℓ with at most n parts each at most m.

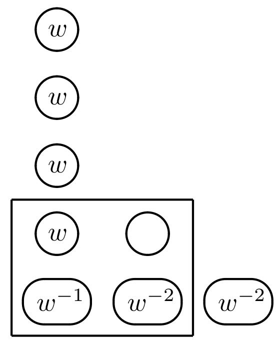

Figure 8: The weights on some dots of $(5, 2, 1)$ for the 2-stretched rank.

Write $ge_k(n)$ for the number of Garden of Eden partitions of n under the operation H_k.

Theorem 8. *Given an integer $k \geq 2$, the Garden of Eden partitions of n under H_k are exactly the $\lambda \in P(n)$ for which* $\text{rank}_k(\lambda) \leq -k - 1$. *A generating function for $ge_k(n)$ is*

$$\sum_{n \geq 0} ge_k(n) q^n = \sum_{s \geq 2} q^s \begin{bmatrix} (k+1)(s-1) - 1 \\ s \end{bmatrix}. \tag{7}$$

Proof. A partition $\lambda = (\lambda_1, \ldots, \lambda_s) \in P(n)$ has preimages under H_k for each part λ_i large enough to contribute at least k dots to each of the other $s - 1$ parts. For example, if $\lambda_i \geq k(s-1)$, then

$$H_k((\lambda_1 + k, \ldots, \lambda_{i-1} + k, \lambda_{i+1} + k, \ldots, \lambda_s + k, 1^{\lambda_i - k(s-1)})) = \lambda.$$

Therefore a partition λ with no preimage has no $\lambda_i \geq k(s-1)$. In particular, $\lambda_1 \leq k(s-1) - 1$ which is equivalent to $\text{rank}_k(\lambda) \leq -k - 1$.

To establish Equation (7), suppose a Garden of Eden partition λ of n under H_k has s parts. As noted in the previous section, we know $s \geq 2$. We have seen that the k-stretched rank condition is equivalent to $\lambda_1 \leq k(s-1) - 1$ which implies that the subpartition of λ above the bottom row has at most s parts with each part at most $k(s-1) - 2$. That is, the remainder of λ fits in an $(ks - k - 2) \times s$ rectangle. Therefore the GE partition corresponds to the product of q^s (the bottom row) and the given q-binomial coefficient (the remainder). □

Note that the GE condition for $k = 1$ is $\text{rank}_1(\lambda) \leq -2$, matching the Bulgarian Solitaire GE condition in Theorem 4. The expression analogous to Equation (7) for $k = 1$ varies only by starting with $s = 3$ since, as noted in the first section, a GE partition under B has length at least 3. Thus we have another generating function for the number of GE partitions under B, namely

$$\sum_{n\geq 0} ge(n)q^n = \sum_{s\geq 3} q^s \begin{bmatrix} 2s-3 \\ s \end{bmatrix}. \tag{8}$$

The number of partitions of n that have at least one preimage under H_k is clearly given by $p(n) - ge_k(n)$. With the techniques of the previous proof, we can give a direct generating function for this count.

Proposition 9. *Given a positive integer k, the partitions of n under H_k with at least one preimage are exactly those $\lambda \in P(n)$ for which $\mathrm{rank}_k(\lambda) \geq -k$. A generating function for the number of these partitions is*

$$\sum_{n\geq 0} (p(n) - ge_k(n))q^n = \sum_{t\geq 0} q^t \begin{bmatrix} t + \lfloor \frac{t}{k} \rfloor \\ t \end{bmatrix}. \tag{9}$$

Proof. The first claim is the complement of the k-stretched rank characterization of Theorem 8.

For the generating function, by the definition of the k-stretched rank, such partitions with $\lambda_1 = t$ have at most $1 + t/k$ parts. In other words, a partition with a preimage under H_k and first part t is followed by a subpartition that fits inside a $t \times \lfloor t/k \rfloor$ rectangle. Therefore the partition corresponds to the product of q^t (the bottom row) and the given q-binomial coefficient (the remainder). □

Combining Equations (7), (8), and (9) gives unusual generating functions for $p(n)$ such as

$$\sum_{n\geq 0} p(n)q^n = \sum_{s\geq 3} q^s \begin{bmatrix} 2s-3 \\ s \end{bmatrix} + \sum_{t\geq 0} q^t \begin{bmatrix} 2t \\ t \end{bmatrix} \tag{10}$$

$$= \sum_{s\geq 2} q^s \begin{bmatrix} 3s-4 \\ s \end{bmatrix} + \sum_{t\geq 0} q^t \begin{bmatrix} t + \lfloor \frac{t}{2} \rfloor \\ t \end{bmatrix} \tag{11}$$

$$= \sum_{s\geq 2} q^s \begin{bmatrix} 4s-5 \\ s \end{bmatrix} + \sum_{t\geq 0} q^t \begin{bmatrix} t + \lfloor \frac{t}{3} \rfloor \\ t \end{bmatrix}. \tag{12}$$

Finally, we have yet to find a formula similar to Equation (4) for $ge_k(n)$. Looking at $\sum_{m\leq -k-1} N_k(m,n)$ with the expressions derived above has not been fruitful. One complication is that, unlike Dyson's rank, the k-stretched rank is not symmetric, that is, $N_k(m,n) \neq N_k(-m,n)$ in general.

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