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EXPANSIONS IN NONINTEGER BASES

Vilmos Komornik[1]
Département de mathématique, Université de Strasbourg, Strasbourg, France
vilmos.komornik@math.unistra.fr

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Abstract

The purpose of these notes is to give an overview of part of number theory which grew out of the uniqueness of expansions. We present a number of elementary but powerful proofs and we illustrate the theorems by many examples. Some results and proofs are published here for the first time. We end the paper with a list of open problems.

1. Introduction

The familiar integer base expansions were extended to noninteger bases in a seminal paper of Rényi [58] more than fifty years ago. Since then many surprising phenomena were discovered and a great number of papers were devoted to unexpected connections with probability and ergodic theory, combinatorics, symbolic dynamics, measure theory, topology and number theory.

It was generally believed that for any given $1 < q < 2$ there are infinitely many expansions of the form

$$1 = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots$$

with digits $c_i \in \{0, 1\}$. Twenty years ago, Erdős, Horváth and Joó [24] made the startling discovery that for a continuum of bases $1 < q < 2$ there is only one such expansion. This result gave a new impetus to this research field.

For a complementary approach based on ergodic theory we refer to a recent survey of Sidorov [63].

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[1]Dedicated to the memory of P. Erdős and I. Joó

2. Number of Expansions

Given an integer $q \geq 2$, it is well known that every real number $x \in [0,1]$ has an expansion of the form

$$x = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots \tag{1}$$

with *digits* $c_i \in \{0, 1, \ldots, m := q-1\}$. If $x = 0, 1$ or if none of the numbers $x, qx, q^2x, \ldots$ is integer, then this expansion is unique. Otherwise x have exactly two different expansions, ending with 0^∞ and m^∞, respectively.

We extend this notion to arbitrary real bases $q > 1$:

Definition 1. Let $q > 1$ be a real number and let us denote by m the greatest integer $< q$. By an *expansion* of a real number x in *base* q we mean a sequence of integers $c_i \in \{0, 1, \ldots, m\}$ satisfying (1).

In order to have an expansion, x has to belong to the interval

$$J_q := \left[0, \sum_{i=1}^{\infty} \frac{m}{q^i}\right] = \left[0, \frac{m}{q-1}\right].$$

(Note that $[0,1] \subset J_q$ with equality if and only if q is integer.) Conversely, every $x \in J_q$ has at least one expansion:

Theorem 2. *(Rényi [58]) Given $q > 1$ and $x \in J_q$, we define a sequence of integers $(b_i) = (b_i(q, x))$ by the* greedy algorithm*: if $b_1, \ldots, b_{n-1}$ have already been defined (no assumption if $n = 1$), then let b_n be the largest integer $< q$ satisfying the inequality*

$$\frac{b_1}{q} + \cdots + \frac{b_n}{q^n} \leq x. \tag{2}$$

Then (b_i) is an expansion of x.

Definition 3. The expansion (b_i) of Theorem 2 is called the *greedy expansion* or the *β-expansion* of x in base q.

Remark 4. It follows from the definition that the greedy expansion $(b_i(q, x))$ is the lexicographically largest expansion of x in base q.

Proof of Theorem 2. The definition is meaningful because $x \geq 0$. If there are infinitely many digits $b_n < m$, then

$$\frac{b_1}{q} + \cdots + \frac{b_n}{q^n} + \frac{1}{q^n} > x \tag{3}$$

for all such indices by construction, and we conclude by letting $n \to \infty$ in (2) and (3).

Furthermore, letting $n \to \infty$ in (2) and using the assumption $x \in J_q$ we get

$$\sum_{i=1}^{\infty} \frac{b_i}{q^i} \le x \le \sum_{i=1}^{\infty} \frac{m}{q^i}. \tag{4}$$

If all digits are equal to m, then we conclude again that (b_i) is an expansion of x.

We complete the proof by showing there there cannot be a last digit $b_n < m$. Indeed, in such a case we would have

$$x < \left(\sum_{i=1}^{n} \frac{b_i}{q^i}\right) + \frac{1}{q^n} \le \left(\sum_{i=1}^{n} \frac{b_i}{q^i}\right) + \left(\sum_{i=n+1}^{\infty} \frac{m}{q^i}\right) = \sum_{i=1}^{\infty} \frac{b_i}{q^i},$$

contradicting (4). □

Remark 5. Given a sequence (p_i) of positive numbers satisfying $p_n \to 0$ and

$$p_n \le p_{n+1} + p_{n+2} + \cdots \quad \text{for all} \quad n = 1, 2, \ldots,$$

the adaptation of the above proof yields the following classical theorem of Kakeya [38], [39] (see also [57], Part 1, Exercise 131): every real number x satisfying the inequalities

$$0 \le x \le p_1 + p_2 + \cdots$$

has an expansion

$$x = \sum_{n=1}^{\infty} c_n p_n$$

with coefficients $c_n \in \{0, 1\}$.

Contrary to the integer case, in noninteger bases most numbers have infinitely many different expansions. In the following theorem, as in many other results in the sequel, the *Golden ratio* $(1+\sqrt{5})/2 \approx 1.618$ plays an important role. For brevity the Golden ratio will be denoted in this paper by the letter G. We recall the well-known relations

$$1 = \frac{1}{G} + \frac{1}{G^2} = \frac{1}{G} + \frac{1}{G^3} + \frac{1}{G^5} + \cdots . \tag{5}$$

Theorem 6. *Fix a base $q > 1$.*

(a) *(Erdős, Horváth, Joó [24]; [26]) If $q < G$, then each interior point of J_q has a continuum of distinct expansions.*[2]

(b) *(Sidorov [59], Dajani, de Vries [10]) If q is not integer, then (Lebesgue-)almost every $x \in J_q$ has a continuum of distinct expansions.*

[2]The expansions 0^∞ and m^∞ of the endpoints of J_q are unique in all bases $q > 1$. Theorem 6 (a) improved an earlier result of Eggan and Vanden Eynden [23].

Proof. (a) Since

$$0 < x < \frac{1}{q} + \frac{1}{q^2} + \cdots, \quad \text{and} \quad 1 < \frac{1}{q^2} + \frac{1}{q^3} + \cdots$$

because $q < G$, we may fix a large integer k such that

$$\frac{1}{q^k} + \frac{1}{q^{2k}} + \cdots \le x \le \sum_{k \nmid j} \frac{1}{q^j} \tag{6}$$

(j runs over the positive integers which are not multiples of k) and

$$1 \le \frac{1}{q^2} + \cdots + \frac{1}{q^k}. \tag{7}$$

Since there are continuum many choices of the digits $c_k, c_{2k}, c_{3k}, \ldots \in \{0,1\}$, the proof will be completed if we show that for each such choice we can find suitable digits $c_j \in \{0,1\}$ for all $k \nmid j$ such that

$$x - \left(\frac{c_k}{q^k} + \frac{c_{2k}}{q^{2k}} + \frac{c_{3k}}{q^{3k}} + \cdots\right) = \sum_{k \nmid j} \frac{c_j}{q^j}.$$

This follows by applying Kakeya's above mentioned theorem with $(p_i) = (q^{-j})_{k \nmid j}$. This is possible because $p_n \to 0$,

$$p_n \le p_{n+1} + \cdots + p_{n+k}, \quad n = 1, 2, \ldots$$

by (7), and

$$0 \le x - \left(\frac{c_k}{q^k} + \frac{c_{2k}}{q^{2k}} + \frac{c_{3k}}{q^{3k}} + \cdots\right) \le p_1 + p_2 + \cdots$$

by (6).

(b) See the original papers for the ergodic theoretical proofs, and Sidorov [61] for several related theorems. $\square$

The assumption $q < G$ in the above theorem is essential:

Proposition 7. *(Erdős, Horváth, Joó [24]) If $q = G$, then $x = 1$ has countably many distinct expansions: a periodic one*

$$1 = \frac{1}{q} + \frac{1}{q^3} + \frac{1}{q^5} + \frac{1}{q^7} + \cdots,$$

and for each $N = 0, 1, \ldots$ the two expansions

$$1 = \left(\sum_{i=1}^{N} \frac{1}{q^{2i-1}}\right) + \frac{1}{q^{2N+1}} + \frac{1}{q^{2N+2}} = \left(\sum_{i=1}^{N} \frac{1}{q^{2i-1}}\right) + \left(\sum_{i=2N+2}^{\infty} \frac{1}{q^i}\right).$$

Remark 8. Sidorov and Vershik [64] also proved that in base G the numbers $x = nG \mod 1$ have $\aleph_0$ expansions, while the other elements of J_G have $2^{\aleph_0}$ expansions.

A simple proof of Proposition 7 is based on the following elementary lemma[3]:

Lemma 9. *Consider an expansion* (1) *and a positive integer n. There exists another expansion*

$$x = \frac{d_1}{q} + \frac{d_2}{q^2} + \frac{d_3}{q^3} + \cdots \tag{8}$$

of x satisfying $c_i = d_i$ for all $i < n$ and $c_n \neq d_n$ if and only if

$$d_n - c_n \le \frac{c_{n+1}}{q} + \frac{c_{n+2}}{q^2} + \cdots \quad \text{if} \quad d_n > c_n \tag{9}$$

and

$$c_n - d_n \le \frac{m - c_{n+1}}{q} + \frac{m - c_{n+2}}{q^2} + \cdots \quad \text{if} \quad d_n < c_n. \tag{10}$$

Moreover, if equality holds in (9) *or* (10)*, then the expansion* (8) *is unique: $d_i = 0$ for all $i > n$ in the first case, and $d_i = m$ for all $i > n$ in the second case.*

Proof. If there exists such an expansion (8), then we deduce from the equality

$$\frac{d_1}{q} + \frac{d_2}{q^2} + \frac{d_3}{q^3} + \cdots = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots \tag{11}$$

that

$$d_n - c_n = \frac{c_{n+1} - d_{n+1}}{q} + \frac{c_{n+2} - d_{n+1}}{q^2} + \cdots \le \frac{c_{n+1}}{q} + \frac{c_{n+2}}{q^2} + \cdots$$

and

$$c_n - d_n = \frac{d_{n+1} - c_{n+1}}{q} + \frac{d_{n+2} - c_{n+1}}{q^2} + \cdots \le \frac{m - c_{n+1}}{q} + \frac{m - c_{n+2}}{q^2} + \cdots ,$$

proving the necessity of the conditions (9)-(10).

Conversely, if (9) is satisfied for some $n \ge 1$ and $d_n \in \{c_n + 1, \ldots, m\}$, or if (10) is satisfied for some $n \ge 1$ and $d_n \in \{0, \ldots, c_n - 1\}$, then

$$c_n - d_n + \frac{c_{n+1}}{q} + \frac{c_{n+2}}{q^2} + \cdots \tag{12}$$

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[3]This lemma has not been formulated before. As we will see, its application simplifies a number of proofs.

belongs to J_q. Applying Theorem 2 we get an expansion

$$\frac{d_{n+1}}{q} + \frac{d_{n+2}}{q^2} + \cdots = c_n - d_n + \frac{c_{n+1}}{q} + \frac{c_{n+2}}{q^2} + \cdots$$

which implies (11) and hence (8).

If equality holds in (9) or (10), then the expression (12) is equal to 0 or to

$$\frac{m}{q} + \frac{m}{q^2} + \cdots ,$$

so that we have necessarily $d_i = 0$ for all $i > n$ or $d_i = m$ for all $i > n$. □

Proof of Proposition 7. We recall from (5) that $(c_i) := (10)^\infty$ is an expansion of $x = 1$. Hence

$$1 = \frac{c_{n+1}}{q} + \frac{c_{n+2}}{q^2} + \cdots \quad \text{if} \quad c_n = 0$$

and

$$1 = \frac{1 - c_{n+1}}{q} + \frac{1 - c_{n+2}}{q^2} + \cdots \quad \text{if} \quad c_n = 1.$$

Applying Lemma 9 it follows that for each $n = 1, 2, \dots$ there is exactly one expansion (d_i) of $x = 1$ such that $d_n \neq c_n$ and $d_i = c_i$ for all $i < n$: $d_n = 1$ and $d_i = 0$ for all $i > n$ if $c_n = 0$, and $d_n = 0$ and $d_i = 1$ for all $i > n$ if $c_n = 1$. □

In the rest of this section we investigate the exceptional cases to Theorem 6 (b).

Theorem 10. *We have*

(a) *(Erdős, Horváth, Joó [24]) Let $(c_i) = 1(10)^\infty$. If $1 < q < 2$ is defined by the equation*

$$1 = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots , \tag{13}$$

then (c_i) is the unique expansion of $x = 1$ in base q.

(b) *(Erdős, Joó [25]) Let N be a positive integer and $(c_i) = 1^9(0^91)^{N-1}(0^41)^\infty$. If $1 < q < 2$ is defined by (13), then there are exactly N distinct expansions of $x = 1$ in base q.*

(c) *([27]) Let N be a positive integer and $(c_i) = (1001\ 0000)^N(1001)^\infty$. For each $j = 1, \dots, N$ there exists $1 < q < 2$ such that the number*

$$x := \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots$$

has exactly j expansions in base q.

Proof. (a) None of the conditions (9) and (10) of Lemma 9 is satisfied for any n.

(b) One can readily check that

- condition (9) is not satisfied for any n;
- condition (10) is satisfied with equality for $n = 9 + 10i$, $i = 0, \ldots, N-2$;
- condition (10) is not satisfied for any other n.

We conclude by applying Lemma 9.

(c) See the original paper. □

Remarks 11.

(a) More results of this kind are given in [48].

(b) The existence of noninteger bases in which $x = 1$ has only one expansion was discovered by Erdős, Horváth and Joó [24]. Theorem 10 (a) is a special case of their results.

(c) It was proved in [47] that there is a smallest such *univoque base* $q' \approx 1.787$. It is the positive solution of the equation

$$1 = \frac{\tau_1}{q} + \frac{\tau_2}{q^2} + \frac{\tau_3}{q^3} + \cdots \tag{14}$$

where $\tau_1, \tau_2, \ldots$ is the truncated Thue–Morse sequence. We recall that the Thue–Morse sequence $\tau_0, \tau_1, \tau_2, \ldots$ is defined by the recursive formulae

$$\tau_0 := 0 \quad \text{and} \quad \tau_\ell \ldots \tau_{2\ell-1} := \overline{\tau_0 \ldots \tau_{\ell-1}}, \quad \ell = 1, 2, 4, 8, \ldots, \tag{15}$$

where we use the notation $\overline{\tau} := 1 - \tau$.

(d) Allouche and Cosnard [2] proved that q' is transcendental.

(e) The second half of the present review is devoted to unique expansions.

The constant q' also appears in the description of the size of the exceptional set:

Theorem 12. *(Sidorov [60]) Let $1 < q \le 2$. The set of numbers $x \in J_q$ having less than a continuum of distinct expansions is*

- *the two-point set of the endpoints of J_q if $q < G$;*
- *countably infinite if $G \le q < q'$;*
- *a continuum of Hausdorff dimension 0 if $q = q'$;*
- *a continuum of Hausdorff dimension strictly between 0 and 1 if $q' < q < 2$;*
- *the complementer of a countable set in $[0, 1]$ if $q = 2$.*

3. Universal Expansions

We start with Borel's theorem on "normal numbers":

Theorem 13. *(Borel [6]) Fix an integer $q \ge 2$. For almost all expansions of the form* (1),

(a) *each digit has frequency $1/q$;*

(b) *more generally, every finite block of digits of length k has frequency $1/q^k$.*

Proof. Since the digits are independent, we may apply the law of large numbers; see, e.g., Kac [37] or Dajani and Kraaikamp [13]. □

When extending this theorem to noninteger bases, we need to specify which kind of expansions are considered among the infinitely many possibilities.

Theorem 14. *(Rényi [58]) Consider the β-expansions of the numbers $x \in J_q$ in some base $q > 1$. The greedy expansion of almost every $x \in J_q$ has the following properties:*

(a) *each digit $c \in \{0, 1, \dots, m\}$ has a positive frequency;*

(b) *each finite block of digits has a frequency.*

For example, the digits 0 *and* 1 *have frequencies* $(5+\sqrt{5})/10$ *and* $(5-\sqrt{5})/10$ *in base G, respectively.*

Proof. The digits are not independent any more, but we may still apply Birkhoff's ergodic theorem instead; see, e.g., Dajani and Kraaikamp [13]. □

Remarks 15.

(a) In base G the block 011 has frequency zero because it never occurs in greedy expansions: it can be replaced by 100.

(b) More generally, forbidden finite blocks exist in all *noninteger* bases. Indeed, since $m > q - 1$, there exists a positive integer k such that

$$1 \le \frac{m}{q} + \cdots + \frac{m}{q^k}.$$

If $c_n \dots c_{n+k} = 0m^k$ for some expansion (1), then the inequalities

$$\begin{aligned}\frac{c_1}{q} + \cdots + \frac{c_{n-1}}{q^{n-1}} + \frac{1}{q^n} &\le \frac{c_1}{q} + \cdots + \frac{c_{n-1}}{q^{n-1}} + \frac{0}{q^n} + \frac{m}{q^{n+1}} + \cdots + \frac{m}{q^{n+k}}\\ &= \frac{c_1}{q} + \cdots + \frac{c_{n+k}}{q^{n+k}}\\ &\le x\end{aligned}$$

show that $c_n = 0$ cannot be defined by the greedy algorithm. Hence the block $0m^k$ cannot occur in any β-expansion.

In view of the last remark we may ask whether there exist expansions in noninteger bases which contain all finite blocks of digits.

Definition 16. An expansion (c_i) is *universal* if it contains all finite blocks of digits.

Example 17. In base G the number $x = 1$ has no universal expansion by Proposition 7 because periodic or ultimately periodic expansions are never universal.

There is an interesting connection between universal expansions and Diophantine approximation. In order to state this result we fix a base $q > 1$ and we consider the numbers of the form

$$y = c_0 + c_1 q + \cdots + c_n q^n, \quad n = 0, 1, \ldots$$

with $c_i \in \{0, 1, \ldots, m\}$. They can be arranged into a strictly increasing sequence

$$y_0 < y_1 < y_2 < \cdots,$$

tending to infinity.

Examples 18.

(a) For $q = 2, 3, \ldots$ we have $y_k = k$ for all k.

(b) For $1 < q \le G$ the sequence (y_i) begins with $0, 1, q, q^2$.

(c) For $G \le q \le 2$ the sequence (y_i) begins with $0, 1, q, 1 + q$.

Theorem 19. *([26], [30]) If $y_{k+1} - y_k \to 0$ for some $q > 1$, then every interior point of J_q has a universal expansion in base q.*

For the proof we need the following

Lemma 20. *Assume that $y_{k+1} - y_k \to 0$ for some $q > 1$. Given $0 < x' \le 1$ and an arbitrary finite block of digits $a_1 \ldots a_N$ there exists another block of digits $b_1 \ldots b_{n+N}$, ending with $a_1 \ldots a_N$ and satisfying the inequalities*

$$0 < x' - \sum_{i=1}^{n+N} \frac{b_i}{q^i} < \frac{1}{q^{n+N}}.$$

Proof. Set $A = \sum_{i=1}^{N} a_i q^{-i}$ and choose a large n such that $q^n x' > A$. Then

$$y_k < q^n x' - A \le y_{k+1}$$

for some k. Choosing n large enough we also have $0 < y_{k+1} - y_k < q^{-N}$ and therefore

$$0 < q^n x' - y_k - A < \frac{1}{q^N},$$

i.e.,

$$0 < x' - \frac{y_k + A}{q^n} < \frac{1}{q^{n+N}}.$$

Since $x' \le 1$, we have $y_k < q^n$ and therefore

$$\frac{y_k + A}{q^n} = \sum_{i=1}^{n+N} \frac{b_i}{q^i}$$

with $b_{n+i} = a_i$ for $i = 1, \ldots, N$. □

Proof of Theorem 19. Let $B_1, B_2, \ldots$ be an enumeration of all finite blocks of digits. Given $0 < x < m/(q-1)$ arbitrarily, we construct a sequence (c_i) of digits and a sequence $n_0 < n_1 < n_2 < \cdots$ of indices such that for each $k = 1, 2, \ldots$, the initial sequence $c_1 \ldots c_{n_k}$ ends with B_k, and

$$0 < x - \sum_{i=1}^{n_k} \frac{c_i}{q^i} < \frac{1}{q^{n_k}}. \tag{16}$$

Then letting $k \to \infty$ in (16) we obtain that (c_i) an expansion of x, and it contains all possible finite blocks of digits by construction.

First we choose a finite block of digits $c_1 \ldots c_{n_0}$ satisfying (16) for $k = 0$.[4] Proceeding by induction, if $c_1 \ldots c_{n_k}$ has already been defined for some $k \ge 0$, then we apply the lemma with

$$x' := x - \sum_{i=1}^{n_k} \frac{c_i}{q^i} \quad \text{and} \quad a_1 \ldots a_n := B_{k+1}.$$

We obtain a sequence $b_1 \ldots b_{n_{k+1}}$ ending with B_{k+1} and satisfying

$$0 < x' - \sum_{i=1}^{n_{k+1}} \frac{b_i}{q^i} < \frac{1}{q^{n_{k+1}}}.$$

Since $x' < q^{-n_k}$, we have $b_i = 0$ for all $i \le n_k$. Therefore, setting $c_i := b_i$ for all $n_k < i \le n_{k+1}$ and using (16) we obtain (16) with $k+1$ in place of k. □

Now we give some sufficient conditions ensuring the relation $y_{k+1} - y_k \to 0$.

Theorem 21. *The relation $y_{k+1} - y_k \to 0$ holds in the following cases:*

[4]For example, if $b_{n_0} < m$ for some index n_0, where (b_i) is the greedy expansion of x, then we may choose $c_1 \ldots c_{n_0} = b_1 \ldots b_{n_0}$.

(a) *([28])* $q = \sqrt{2}$*;*

(b) *([28])* $1 < q < \sqrt{2}$ *is transcendental;*

(c) *([30],[1])* $1 < q \le 2^{1/3} \approx 1.2599$*.*

Consequently, in all these bases, every interior point of J_q *has a universal expansion in base* q*.*

Remarks 22.

(a) Part (b) was proved in [28] under the weaker condition that $1 < q < \sqrt{2}$ and q^2 is not a root of any polynomial with coefficients $\in \{-1, 0, 1\}$. More general results were obtained recently by Sidorov and Solomyak [62].

(b) Part (c) was proved in [30] for all $1 < q \le 2^{1/4} \approx 1.1892$ with the possible exception of the square root of the second Pisot number. The exceptional case was solved in collaboration with S. Akiyama during the present workshop; see [1].

Proof of Theorem 21. (a) Fix $\delta > 0$ and choose an integer $N > 1/\delta$. By the pigeonhole principle there exist two integers $0 \le k < \ell \le N$ such that the fractional part of $\ell\sqrt{2} - k\sqrt{2}$ is in $(0, 1/N)$ or in $(1/N, 1)$. Taking integer multiples of $(\ell - k)\sqrt{2}$ it follows that there exists a finite sequence of integers $k_1 < \cdots < k_N$ such that every interval of length δ contains at least one number having the same fractional part as one of the numbers $k_i\sqrt{2}$, $1 \le i \le N$.

It follows that every interval $(x, x+\delta)$, $x > k_N\sqrt{2}$, contains at least one y_k. Indeed, let $x < x' < x + \delta$ and $1 \le i \le N$ such that x' and $k_i\sqrt{2}$ have the same fractional part. Then $\ell := x' - k_i\sqrt{2}$ is a positive integer and hence $x' = \ell + k_i\sqrt{2}$ is in the sequence (y_k).

For proofs of (b) and (c), we refer to the original papers. $\square$

The above strategy does not work for $q \ge G$:

Proposition 23. *([26], [28]) If* $q \ge G$*, then* $y_{k+1} - y_k = 1$ *for infinitely many indices* k*, so that* $y_{k+1} - y_k \not\to 0$*.*

Proof. It suffices to show that none of the open intervals

$$I_n := (q^2 + \cdots + q^{2n}, 1 + q^2 + \cdots + q^{2n}), \quad n = 0, 1, \ldots$$

contains any y_k. This is true for $I_0 = (0, 1)$ because $y_0 = 0$ and $y_1 = 1$.

Assume on the contrary that some of the intervals I_n contains some y_k. Let n be the smallest such integer (then $n \ge 1$). We have thus

$$q^2 + \cdots + q^{2n} < y_k < 1 + q^2 + \cdots + q^{2n}.$$

Since $q \geq G$, using (5) we see that

$$q^{2n+1} > 1 + q^2 + \cdots + q^{2n} \quad \text{and} \quad 1 + q + q^2 + \cdots + q^{2n-1} \leq q^2 + \cdots + q^{2n};$$

hence y_k has the form $y_k = c_0 + c_1 q + \cdots + c_{2n-1} q^{2n-1} + q^{2n}$ with suitable coefficients $c_i \in \{0, 1\}$. But then

$$c_0 + c_1 q + \cdots + c_{2n-1} q^{2n-1} = y_k - q^{2n} \in I_{n-1},$$

contradicting the minimality of n. □

Nevertheless, an ergodic theoretical approach yielded the following result:

Theorem 24. *(Sidorov [60]) Let $1 < q < 2$. Almost every $x \in J_q$ has a universal expansion in base q.*

4. Spectra of Polynomials

Theorem 19 shows the usefulness of the sequence (y_k) in the study of expansions. In this section we investigate these sequences more closely.

Fix a base $q > 1$. For each positive integer[5] m, let

$$y_0^m < y_1^m < y_2^m < \cdots$$

be the strictly increasing sequence of the numbers of the form

$$y = c_0 + c_1 q + \cdots + c_n q^n, \quad n = 0, 1, \ldots$$

with coefficients $c_i \in \{0, 1, \ldots, m\}$.

Proposition 25. *([26], [30]) We have*

(a) *If $q \leq m+1$, then $y_{k+1}^m - y_k^m \leq 1$ for all k.*

(b) *If $q \geq m+1$, then $y_{k+1}^m - y_k^m \geq 1$ for all k.*

Proof. Set $x_k := c_0 + c_1 q + c_2 q^2 + \cdots$ where $k = c_0 + c_1(m+1) + c_2(m+1)^2 + \cdots$ is the representation of $k = 0, 1, \ldots$ in base $m+1$. Since $x_0 = 0$, and (x_k) and (y_k) run over the same set, it suffices to show that $x_{k+1} - x_k \leq 1$ for all k.

Let

$$x_{k+1} = c_0' + c_1' q + c_2' q^2 + \cdots .$$

[5] In this section m denotes an arbitrary positive integer, not necessarily the greatest integer $< q$.

If j is the smallest index for which $c'_j > c_j$, then

$$x_{k+1} - x_k = q^j - m(q^{j-1} + \cdots + q + 1) = \frac{q^j(q-1-m)+m}{q-1}.$$

If $q \le m+1$, then

$$\frac{q^j(q-1-m)+m}{q-1} \le \frac{q^0(q-1-m)+m}{q-1} = 1;$$

if $q \ge m+1$, then

$$\frac{q^j(q-1-m)+m}{q-1} \ge \frac{q^0(q-1-m)+m}{q-1} = 1. \qquad \square$$

Next we are going to investigate the quantities

$$\ell^m(q) := \liminf(y^m_{k+1} - y^m_k) \quad \text{and} \quad L^m(q) := \limsup(y^m_{k+1} - y^m_k).$$

Remarks 26.

(a) We have $\ell^1(q) \ge \ell^2(q) \ge \cdots \ge 0$ and $L^1(q) \ge L^2(q) \ge \cdots \ge 0$.

(b) We have

$$\ell^m(q) = \inf(y^m_{k+1} - y^m_k). \tag{17}$$

The inequality $\ge$ being obvious, it suffices to prove that $\ell^m(q) \le y^m_{k+1} - y^m_k$ for each fixed index k. For every integer n satisfying $q^n > y^m_{k+1}$ there exist two indices $s > r > k$ such that $q^n + y^m_k = y^m_r$ and $q^n + y^m_{k+1} = y^m_s$. Hence

$$y^m_{r+1} - y^m_r \le y^m_s - y^m_r = y^m_{k+1} - y^m_k.$$

Since $n \to \infty$ implies that $r \to \infty$, we conclude that $\ell^m(q) \le y^m_{k+1} - y^m_k$.

The behavior of the sequence (y^m_k) is intimately related to an algebraic property of the base q.

Definition 27. A *Pisot number* is an algebraic integer > 1 all of whose conjugates have modulus < 1.

Examples 28.

(a) The rational integers $2, 3, \ldots$ are Pisot numbers.

(b) The Golden ratio is a Pisot number because it is an algebraic integer (its minimal polynomial is $x^2 - x - 1$) and its conjugate $(1-\sqrt{5})/2$ belongs to $(-1, 0)$.

Proposition 29. *The following hold:*

(a) *(Drobot [21], Drobot and McDonald [22]) If $q < m + 1$ does not satisfy any algebraic equation with integer coefficients a_i satisfying $|a_i| \le m$, then $\ell^m(q) = 0$.*

(b) *(Garsia [34]) If q is a Pisot number, then $\ell^m(q) > 0$ for all m.*

Proof (see [28]). (a) For each fixed $n = 1, 2, \dots$, the expression $c_0 + c_1q + \dots + c_{n-1}q^{n-1}$, where the digits run over the set $\{0, 1, \dots, m\}$, gives $(m + 1)^n$ different elements of the sequence (y_k^m) in the interval $[0, 1 + q + \dots + q^{n-1}]$, so that

$$\inf(y_{k+1}^m - y_k^m) \le \frac{1 + q + \dots + q^{n-1}}{(m + 1)^n - 1} = \frac{1}{q - 1} \cdot \frac{q^n - 1}{(m + 1)^n - 1}.$$

Letting $n \to \infty$ we get $\inf(y_{k+1}^m - y_k^m) = 0$. We conclude by applying (17).

(b) Denoting by $q_1, \dots, q_d$ the algebraic conjugates of q, the sums of powers $q^n + q_1^n + \dots + q_d^n$ are integers by Viète's formula for all $n = 1, 2, \dots$. Since $|q_j| < 1$ for all j, it follows that

$$\sum_{n=0}^{\infty} \text{dist } (q^n, \mathbb{Z}) \le \sum_{n=0}^{\infty} (|q_1|^n + \dots + |q_j|^n) < \infty.$$

We may thus choose a positive integer N such that

$$\sum_{n=N}^{\infty} \text{dist } (q^n, \mathbb{Z}) < \frac{1}{(q + 1)m}.$$

Then

$$\text{dist } (q^N y_n^m - q^N y_k^m, \mathbb{Z}) < \frac{1}{q + 1} \quad \text{for all } n \text{ and } k. \tag{18}$$

Assume on the contrary that $\ell^m(q) = 0$ for some m. Then we have also $q^N \ell^m(q) = 0$, so that there exist two indices n', k' such that $0 < |q^N y_{n'}^m - q^N y_{k'}^m| < 1/(q + 1)$. There exists a positive integer M such that

$$\frac{1}{q + 1} \le |q^{N+M} y_{n'}^m - q^{N+M} y_{k'}^m| < \frac{q}{q + 1} = 1 - \frac{1}{q + 1}.$$

Since $q^M y_{n'}^m = y_n^m$ and $q^M y_{k'}^m = y_k^m$ for suitable indices n and k, this contradicts (18). □

In the rest of this section we state some deeper results; we refer to the original papers for proof. The following two theorems improve several earlier results of Erdős, Joó and Schnitzer [29], Bugeaud [9] and of [30]:

Theorem 30. *([1]) If $q > 1$ is not a Pisot number, then $\ell^{2k}(q) = L^{3k}(q) = 0$ for all integers $k > q - 1$.*

For $1 < q < 2$ we have stronger results[6]:

Theorem 31. *([1]) The following hold:*

(i) *If $1 < q \le \sqrt[3]{2} \approx 1.2599$ is not a Pisot number, then $L^1(q) = 0$.*

(ii) *If $1 < q \le \sqrt{2} \approx 1.4142$ is not a Pisot number, then $\ell^1(q) = L^2(q) = 0$.*

(iii) *If $1 < q < 2$ is not a Pisot number, then $\ell^2(q) = L^3(q) = 0$.*

Corollary 32. *Let $q > 1$.*

(a) *If q is integer, then $\ell^m(q) = 1$ for all m.*

(b) *If q is not integer, then $\ell^m(q) \to 0$ as $m \to \infty$.*

Proof. (a) This follows from Proposition 25 because $y^m_{k+1} - y^m_k = 1$ for all k and m.

(b) This is true by Theorem 30 and Remark 26 (a) if q is not Pisot. If q is a noninteger Pisot number, then it is irrational. Choose two sequences $k_i, n_i \to \infty$ of nonzero integers satisfying $k_i - n_i q \to 0$. Since $k_i - n_i q \ne 0$ is the difference of two elements of the sequence (y^m_k) if $m \ge \max\{k_i, n_i\}$, applying (17) it follows that $\ell^m(q) \le |k_i - n_i q|$ if $m \ge \max\{k_i, n_i\}$. We conclude by letting $i \to \infty$. □

Extensive numerical experiments revealed a regular character of the sequences $(\ell^m(q))$ for many Pisot numbers. For example, the initial sequence for the Golden ratio $q = G$ is given by the following table:

$$\begin{aligned}
&\ell^m(q) = |q-1| \approx 0.6180 \quad \text{for} \quad m = 1\\
&\ell^m(q) = |2q-3| \approx 0.2361 \quad \text{for} \quad m = 2\\
&\ell^m(q) = |3q-5| \approx 0.1459 \quad \text{for} \quad m = 3, 4\\
&\ell^m(q) = |5q-8| \approx 0.0902 \quad \text{for} \quad m = 5, 6\\
&\ell^m(q) = |8q-13| \approx 0.0557 \quad \text{for} \quad m = 7, \dots, 11\\
&\dots\\
&\ell^m(q) = |377q-610| \approx 0.0012 \quad \text{for} \quad m = 322, \dots, 521\\
&\ell^m(q) = |610q-987| \approx 0.0007 \quad \text{for} \quad m = 522, \dots, 842.
\end{aligned}$$

This is contained in the following theorem where we use the Fibonacci sequence

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \dots.$$

[6]Only part (ii) is new here: Part (i) is a reformulation of Theorem 21 (c), while Part (iii) is the case $k = 1$ of the preceding theorem.

Theorem 33. *([53]) If $q = G$ and m is a positive integer, then*

$$\ell^m(q) = |F_k q - F_{k+1}|$$

where k is the smallest integer satisfying $q^{k-1} \geq m$.

Remarks 34.

(a) Using the theory of continued fractions, the theorem was extended to a class of quadratic Pisot numbers by Borwein and Hare [8], and by Komatsu [43].

(b) Borwein and Hare [7] and Feng and Wen [32] devised an efficient algorithm for the determination of $\ell^m(q)$ for any specific value of m and q.

5. Lexicographic Characterizations

In order to generalize Rényi's Theorem 14 on the distribution of digits to arbitrary bases, Parry [55] gave a *lexicographic* characterization of the β-expansions. This became an excellent tool in investigating the combinatorial and topological nature of such expansions.

In order to formulate the results in an elegant way, we slightly modify, following Daróczy and Kátai [16], the greedy expansions.

Definition 35. (Daróczy and Kátai [16]; [5]) Given $q > 1$ and $x \in J_q$ we define the sequence $(a_i) = (a_i(q,x))$ by induction as follows.[7] For $x = 0$ we set $(a_i) := 0^\infty$. For $x > 0$, if $a_1, \ldots, a_{n-1}$ have already been defined (no assumption if $n = 1$), then let a_n be the largest integer $\leq m$ satisfying

$$\frac{a_1}{q} + \cdots + \frac{a_n}{q^n} < x.$$

Remarks 36. Fix $q > 1$, $x \in J_q$, and write for brevity (a_i), (b_i) instead of $(a_i(q,x))$, $(b_i(q,x))$, and (α_i), (β_i) instead of $(a_i(q,1))$, $(b_i(q,1))$.

(a) If (b_i) has a last nonzero digit b_k, then $(a_i) = b_1 \ldots b_{k-1} b_k^-(\alpha_i)$ with $b_k^- = b_k - 1$. Otherwise we have $(a_i) = (b_i)$.

(b) It follows from (a) that (a_i) is an expansion of x in base q. It is called the *quasi-greedy expansion* of x in base q.

(c) It follows (b) and from the definition that the quasi-greedy expansion $(a_i(q,x))$ is the lexicographically largest *infinite* expansion of x in base q. Here and in the sequel a sequence or an expansion (c_i) is called *infinite*, if it does not have

[7] Henceforth the letter m denotes again the largest integer $< q$.

a last nonzero digit c_k, i.e., either it has infinitely many nonzero digits or $(c_i) = 0^\infty$.[8]

(d) It follows from (a) that if (β_i) has a last nonzero digit β_k, then (α_i) is purely periodic, and its smallest period is $\beta_1 \ldots \beta_{k-1}\beta_k^-$ with $\beta_k^- = \beta_k - 1$. Otherwise $(\alpha_i) = (\beta_i)$.

Examples 37.

(a) For $q = 2, 3, \ldots$ we have $(\alpha_i) = (\beta_i) = m^\infty$.

(b) If q is a *Multinacci number*, i.e., the positive solution of $q^n = q^{n-1} + \cdots + q + 1$ for some $n = 2, 3, \ldots$, then $(\alpha_i) = (1^{n-1}0)^\infty$ and $(\beta_i) = 1^n 0^\infty$.

(c) In particular, for $q = G$ we have $(\alpha_i) = (10)^\infty$ and $(\beta_i) = 110^\infty$.

Using the quasi-greedy expansion $(\alpha_i) := (a_i(q, 1))$ of $x = 1$ we can give elegant lexicographic characterizations of the greedy and quasi-greedy expansions.

Theorem 38. *The following hold:*

(a) *(Parry [55], Daróczy and Kátai [16]) Fix $q > 1$. A sequence (b_i) on the alphabet $\{0, 1, \ldots, m\}$ is the greedy expansion of some $x \in J_q$ if and only if*

$$(b_{n+i}) < (\alpha_i) \quad \textit{whenever} \quad b_n < m.$$

(b) *([26], [49]) A sequence (β_i) on the alphabet $\{0, 1, \ldots, m\}$ is the greedy expansion of $x = 1$ in some base $q > 1$ if and only if*

$$(\beta_{n+i}) < (\beta_i) \quad \textit{whenever} \quad \beta_n < m.$$

In this case we have necessarily $m < q \le m + 1$.

(c) *([5]) Fix $q > 1$. An infinite sequence (a_i) on the alphabet $\{0, 1, \ldots, m\}$ is the quasi-greedy expansion of some $x \in J_q$ if and only if*

$$(a_{n+i}) \le (\alpha_i) \quad \textit{whenever} \quad a_n < m.$$

(d) *([5]) An infinite sequence (α_i) on the alphabet $\{0, 1, \ldots, m\}$ is the quasi-greedy expansion of $x = 1$ in some base $q > 1$ if and only if*

$$(\alpha_{n+i}) \le (\alpha_i) \quad \textit{whenever} \quad \alpha_n < m. \tag{19}$$

In this case we have necessarily $m < q \le m + 1$.

Remark 39. Parry's theorem describes the forbidden blocks in greedy expansions. For example, it provides a simple new proof of the results in Remark 15.

[8]By considering the zero sequence to be infinite we simplify many statements in the present theory.

6. Univoque Bases

We start with an easy consequence of Parry's theorem. As before we denote by $(\alpha_i) := (a_i(q,1))$ the quasi-greedy expansion of $x = 1$ in base q.

Corollary 40. *Fix $q > 1$ and $x \in J_q$. An expansion* (1) *is the unique possible expansion of x if and only if the following two conditions are satisfied:*

$$(c_{n+i}) < (\alpha_i) \quad \textit{whenever} \quad c_n < m; \tag{20}$$

$$(m - c_{n+i}) < (\alpha_i) \quad \textit{whenever} \quad c_n > 0. \tag{21}$$

Proof. If (c_i) is the unique expansion of x, then it coincides with the β-expansion, so that (20) is satisfied by Parry's theorem. Furthermore, since (c_i) is an expansion of x if and only if $(m-c_i)$ is an expansion of $(m/(q-1))-x$, the expansion $(m-c_i)$ of $(m/(q-1))-x$ is also unique; this yields (21).

Conversely, since by construction the β-expansion of x is the lexicographically largest expansion of x, the conditions (20)-(21) imply that (c_i) is at the same time the lexicographically largest and smallest expansion of x; hence it is the only one. □

Examples 41.

(a) If q is integer, then $(\alpha_i) = m^\infty$ and the conditions (20)-(21) simply mean that, apart from the trivial expansions 0^∞ and m^∞, an expansion is unique if and only if it does not end with 0^∞ or m^∞.

(b) If $q = G$, then $(\alpha_i) = (10)^\infty$, and the conditions (20)-(21) are satisfied only by the trivial expansions 0^∞ and m^∞.

(c) If $q < q'$, then $(\alpha_i(q)) < (\alpha_i(q'))$ lexicographically, and therefore the conditions (20)-(21) are more strict for smaller bases. Hence the conclusion of (b) remains valid for all bases $1 < q < G$, too.

The rest of this review is mainly devoted to unique expansions. For the proofs we usually refer to the original papers.

In this section, mostly following [50], we investigate the bases in which the expansion of $x = 1$ is unique.

Definition 42. We write $q \in \mathcal{U}$ if $q > 1$ and $x = 1$ has a unique expansion in base q. The elements of $\mathcal{U}$ are called *univoque bases.*

Examples 43.

(a) The integers $q = 2, 3, \dots$ belong to $\mathcal{U}$ because the only expansion of $x = 1$ is m^∞.

(b) We have already encountered noninteger univoque bases in Remark 11.

The *univoque set* $\mathcal{U}$ has interesting properties. Parts (a), (b) and (c) of the next theorem are due to Erdős, Horváth and Joó [24]; Part (d) was obtained by Daróczy and Kátai [16].

Theorem 44. *We have*

(a) *$\mathcal{U}$ has zero Lebesgue measure;*

(b) *$\mathcal{U}$ has the power of continuum;*

(c) *$\mathcal{U}$ is of the first category;*

(d) *$\mathcal{U}$ has Hausdorff dimension one.*

We have the following variant of Corollary 40:

Theorem 45. *([26]) An expansion*

$$1 = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots$$

is unique if and only if

$$(c_{n+i}) < c_1 c_2 \ldots \quad \textit{whenever} \quad c_n < m \tag{22}$$

and

$$(m - c_{n+i}) < c_1 c_2 \ldots \quad \textit{whenever} \quad c_n > 0. \tag{23}$$

Example 46. Using Theorem 45 we may reprove the results of Remark 11.

The following results were obtained with the help of Theorem 45:

Theorem 47. *The following hold:*

(a) *([47]) There exists a smallest univoque base $q' \approx 1.787$ (see (14)-(15)).*

(b) *([54]) The formula*

$$(c_i) := \tau_1 \ldots \tau_{\ell-1} \left(\overline{\tau_0 \tau_1 \ldots \tau_{\ell-1}}\right)^{\infty}, \quad \ell = 2, 4, 8, 16, \ldots,$$

where $\tau_0 \tau_1 \ldots$ is the Thue–Morse sequence and $\overline{\tau} = 1 - \tau$, defines a decreasing sequence of univoque bases, converging to $q' \approx 1.787$ (see (14)-(15)).

For example, for $n = 1, 2, 3$ we get $(c_i) = 1(10)^{\infty}$, $(c_i) = 110(1001)^{\infty}$ and $(c_i) = 110\ 1001\ (1001\ 0110)^{\infty}$.

Next we describe the topological properties of the univoque set. We recall that a Cantor set is a nonempty closed set having no interior or isolated points.

Theorem 48. *([50]) The following hold:*

(a) *A base $q > 1$ belongs to the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$ if and only if the sequence (α_i) of Definition 35 satisfies the following lexicographic condition:*

$$(m - \alpha_{n+i}) < (\alpha_i) \quad \textit{whenever} \quad \alpha_n > 0; \tag{24}$$

(b) *$\mathcal{U}$ is closed from above: if $q_n \in \mathcal{U}$ and $q_n \searrow q$, then $q \in \mathcal{U}$;*

(c) *$\overline{\mathcal{U}} \setminus \mathcal{U}$ is countable and dense in $\overline{\mathcal{U}}$;*

(d) *$\overline{\mathcal{U}}$ is a Cantor set of zero Lebesgue measure;*

(e) *$\mathcal{U}$ is nowhere dense.*

Idea of the proof. The innocently-looking Part (a) is the key result. Although similar to Theorem 45 on the characterization of $\mathcal{U}$, its proof is sensibly more intricate. (Observe the asymetry between the conditions (19) and (24).) Parts (b), (c) follow from (a), and (d), (e) are easy consequences of (c) and of Theorem 44 (a) and (c). □

Examples 49.

(a) The Multinacci numbers (see Example 37) belong to $\overline{\mathcal{U}} \setminus \mathcal{U}$ for $n = 3, 4, \ldots$ because $(c_i) := (\alpha_i) = (1^{n-1}0)^\infty$ satisfies (24) but not (22).

(b) The Golden ratio is not in $\overline{\mathcal{U}}$ because $(\alpha_i) = (10)^\infty$ does not satisfy (24).

We remove the asymmetry between (19) and (24) by introducing the following

Definition 50. We write $q \in \mathcal{V}$ if

$$(m - \alpha_{n+i}) \le (\alpha_i) \quad \text{whenever} \quad \alpha_n > 0. \tag{25}$$

Theorem 51. *([50])*

(a) *We have $\mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V}$.*

(b) *$\mathcal{V}$ is a closed set and its smallest element is the Golden ratio.*

(c) *$\mathcal{V} \setminus \overline{\mathcal{U}}$ is a discrete (hence countable), dense subset of $\mathcal{V}$.*

Remarks 52.

(a) The proofs of Theorems 48 and 51 show that the greedy expansion of $x = 1$ is finite in each base $q \in \mathcal{V} \setminus \mathcal{U}$. Hence the elements of $\mathcal{V} \setminus \mathcal{U}$ are algebraic integers. A theorem of Parry [55] implies that all algebraic conjugates of these bases have modulus < 2.

(b) If $q \in \mathcal{V} \setminus \mathcal{U}$, than $x = 1$ has exactly $\aleph_0$ expansions. All these expansions are given explicitly in [50]: the lists are different for $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$ and for $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$.

The following theorem exhibits an analogy between $\overline{\mathcal{U}}$ and the triadic Cantor set.

Theorem 53. *([19]) Let us write $(1, \infty) \setminus \overline{\mathcal{U}} = \cup^*(p_1, p_2)$ with pairwise disjoint open intervals.*

(a) *The closed intervals $[p_1, p_2]$ are also pairwise disjoint.*

(b) *The set of left endpoints p_1 is $\{1, 2, \dots\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$.*

(c) *The set of right endpoints p_2 is a countable dense subset $\mathcal{U}^*$ of $\mathcal{U}$.*

(d) *In each (p_1, p_2), the elements of $\mathcal{V}$ form a strictly increasing sequence tending to p_2.*

Examples 54. For each fixed interval (p_1, p_2) let us denote by $v_1 < v_2 < \cdots$ the increasing sequence of the elements of $\mathcal{V} \cap (p_1, p_2)$. We describe them more precisely. Let m denote here the integer part of p_1.

(a) If $p_1 = m$ is a positive integer, then starting with $c_1c_2 = m1$ we define a sequence (c_i) by the recurrence relations

$$c_{\ell+1} \dots c_{2\ell-1} := \overline{c_1 \dots c_{\ell-1}} \quad \text{and} \quad c_{2\ell} := \overline{c_\ell} + 1, \quad \ell = 2, 4, 8, \dots,$$

with the notation $\overline{c_i} := m - c_i$. Then the greedy expansion of $x = 1$ in base v_n is $c_1 \dots c_{2^n}$, $n = 1, 2, \dots$.

One may check that $(c_i) = (\tau_i + (m-1)\tau_{2i-1})$ where (τ_i) is the Thue–Morse sequence.

For instance, for $m = 1$ we have $(p_1, p_2) = (1, q')$ with $q' \approx 1.787$ defined in Remark 11 (b), and $(c_i) = (\tau_i)$ is the Thue–Morse sequence. The greedy expansions of $x = 1$ in bases $v_1(-G), v_2, v_3$ are given by 11, 1101 and 1101 0011, respectively.

(b) If p_1 is not a positive integer, then starting with the finite greedy expansion $c_1 \dots c_k$ of $x = 1$ in base p_1 where $c_k > 0$ (observe that $c_1 = m$), we define a Thue–Morse type sequence sequence (c_i) by the recurrence relations

$$c_{\ell+1} \dots c_{2\ell-1} := \overline{c_1 \dots c_{\ell-1}} \quad \text{and} \quad c_{2\ell} := \overline{c_\ell} + 1, \quad \ell = 2k, 4k, 8k, \dots,$$

with the notation $\overline{c_i} := m - c_i$. Then the greedy expansion of $x = 1$ in base v_n is $c_1 \ldots c_{2^n k}$.The unique expansion of $x = 1$ in base p_2 is (c_i).

(c) We describe the finite greedy expansions $c_1 \ldots c_k$ occurring in the construction of (b). It follows from Theorems 45, 48 (a) and Remark 36 (d) that a finite sequence $c_1 \ldots c_k$ with $c_1 \geq c_k > 0$ is the finite greedy expansion of $x = 1$ in some base $p_1 \in \overline{\mathcal{U}} \setminus \mathcal{U}$ if and only if the periodic sequence $(\alpha_i) := (c_1 \ldots c_{k-1} c_k^-)^\infty$ with $c_k^- := c_k - 1$ satisfies (19) and (24) with $m = c_1$.

7. Univoque Sets

In this section, following mostly [19] and [20], we investigate the unique expansions in a given base.

Definition 55. For each $q > 1$ we denote by $\mathcal{U}_q$ the set of numbers $x \in J_q$ whose expansion is unique in base q.

Remarks 56.

(a) If q is integer, then $\mathcal{U}_q$ is a non-closed dense set of full Lebesgue measure in $J_q = [0, 1]$ because $[0, 1] \setminus \mathcal{U}_q$ is countable.

(b) If q is not integer, then $\mathcal{U}_q$ is a Lebesgue null set by Theorem 6 (b).

Next we investigate the Hausdorff dimension $\dim_H \ \mathcal{U}_q$ of $\mathcal{U}_q$. Daróczy, Kátai and Kallós [15], [42], [40], [41] developed a strategy for the computation of $\dim_H \ \mathcal{U}_q$ for any given q. Here we state only some weaker results:

Theorem 57. *(Glendinning–Sidorov [35]; [20]) Let* $q > 1$.

(a) $\dim_H \ \mathcal{U}_q = 1$ *if q is integer.*

(b) $\dim_H \ \mathcal{U}_q < 1$ *if q is not integer.*

(c) *If* $q \nearrow 2$*, then* $\dim_H \ \mathcal{U}_q \to 1$.

Proof [9] *of (b) for* $1 < q < 2$. Fix $q > 1$, and denote by K the largest positive integer satisfying

$$\frac{1}{q} + \cdots + \frac{1}{q^K} < 1$$

so that $(\alpha_i(q))$ begins with $1^K 0$. Let us denote by $\mathcal{F}'_q$ the set of sequences of the form $1^{n_1} 0^{n_2} 1^{n_3} 0^{n_4} \ldots$ where all exponents $n_1, n_2, \ldots$ belong to the $\{1, \ldots, K\}$, and set

$$\mathcal{F}_q := \Big\{ \sum_{i=1}^{\infty} \frac{c_i}{q^i} \ : \ (c_i) \in \mathcal{F}'_q \Big\}.$$

[9] This proof has not been published before.

It follows from Corollary 40 that every $x \in \mathcal{U}_q \setminus \{0, 1/(q-1)\}$ has the form

$$x = \frac{y}{q^m} \quad \text{or} \quad x = \frac{1}{q} + \cdots + \frac{1}{q^m} + \frac{y}{q^m}$$

for some nonnegative integer m and for some $y \in \mathcal{F}_q$. This shows that $\mathcal{U}_q$ may be covered by countably many sets, similar to $\mathcal{F}_q$. Since the union of countable many sets of Hausdorff dimension s is still of Hausdorff dimension s, it suffices to prove that $\dim_H \ \mathcal{F}_q < 1$.

Let us introduce the similarities $S_{j,k} : J_q \to J_q$ by

$$S_{j,k}(x) := \frac{1}{q} + \cdots + \frac{1}{q^j} + \frac{x}{q^{j+k}}, \quad j, k = 1, \ldots, K, \quad x \in J_q.$$

It follows from the definition of $\mathcal{F}_q$ that

$$\mathcal{F}_q = \bigcup_{j,k=1}^{K} S_{j,k}(\mathcal{F}_q)$$

and hence that its closure $\overline{\mathcal{F}_q}$ is the (nonempty compact) invariant set of this system of similarities. Applying Proposition 9.6 in [31] we conclude that

$$\dim_H \ \mathcal{F}_q \le \dim_H \ \overline{\mathcal{F}_q} \le s$$

where s is the solution of the equation

$$\sum_{j=1}^{K} \sum_{k=1}^{K} q^{-(j+k)s} = 1.$$

Since

$$\sum_{j=1}^{K} \sum_{k=1}^{K} q^{-(j+k)} = \left(\frac{1}{q} + \cdots + \frac{1}{q^K} \right)^2 < 1,$$

we have $s < 1$. □

In order to describe the size of $\mathcal{U}_q$ we need, beside the constants $G \approx 1.618$ and $q' \approx 1.787$, the smallest element $q'' \approx 2.536$ of $\mathcal{U} \cap (2, 3)$, determined in [49].[10]

Theorem 58. *Let $q > 1$ be a real number.*

(a) *If $q \in (1, G)$, then $\mathcal{U}_q$ consists merely of the endpoints of J_q.*

(b) *If $q \in (G, q') \cup (2, q'')$, then $|\mathcal{U}_q| = \aleph_0$.*

(c) *If $q \in [q', 2] \cup [q'', \infty)$, then $|\mathcal{U}_q| = 2^{\aleph_0}$.*

[10] It is interesting to compare the following result with Theorem 12.

Remark 59.

(a) Part (a) follows from Theorem 6 (a).

(b) Parts (b) and (c) were proved by Glendinning and Sidorov [35] for $1 < q \le 2$ and in [19] for all $q > 1$.

(c) Much more precise results were proved by de Vries [18].

The topological properties of $\mathcal{U}_q$ depend essentially on whether q belongs to $\mathbb{N}$, $\mathcal{U}$, $\overline{\mathcal{U}}$ or $\mathcal{V}$. For example we have the following unexpected result:

Theorem 60. *([19]) $\mathcal{U}_q$ is closed if and only if $q \notin \overline{\mathcal{U}}$.*

In order to describe the closure $\overline{\mathcal{U}}_q$ of $\mathcal{U}_q$ for $q \in \overline{\mathcal{U}}$, we introduce an analogue $\mathcal{V}_q$ of the set $\mathcal{V}$ of the preceding section.[11]

Definition 61. Let $q > 1$, $x \in J_q$, and consider the quasi-greedy expansions $(a_i) = (a_i(q,x))$ and $(\alpha_i) := (a_i(q,1))$. We write $x \in \mathcal{V}_q$ if

$$(m - a_{n+i}) \le (\alpha_i) \quad \text{whenever} \quad a_n > 0.$$

Analogously to the inclusions $\mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V}$, we have $\mathcal{U}_q \subset \overline{\mathcal{U}}_q \subset \mathcal{V}_q$ for each $q > 1$. However, while the three sets $\mathcal{U}$, $\overline{\mathcal{U}}$ and $\mathcal{V}$ are different, among $\mathcal{U}_q$, $\overline{\mathcal{U}}_q$ and $\mathcal{V}_q$ at least two always coincide. The following theorem clarifies the situation.

Theorem 62. *([19]) Let $q > 1$.*

(a) *The set $\mathcal{V}_q$ is always closed, and $\mathcal{U}_q \subset \overline{\mathcal{U}}_q \subset \mathcal{V}_q$.*

(b) *If $q \in \overline{\mathcal{U}}$, then $\mathcal{U}_q \subsetneq \overline{\mathcal{U}_q} = \mathcal{V}_q$.*

(c) *If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then $\mathcal{U}_q = \overline{\mathcal{U}}_q \subsetneq \mathcal{V}_q$.*

(d) *If $q \in (1, \infty) \setminus \mathcal{V}$, then $\mathcal{U}_q = \overline{\mathcal{U}}_q = \mathcal{V}_q$.*

Remarks 63. Let $q \in \mathcal{V}$, so that $\mathcal{U}_q \subsetneq \mathcal{V}_q$. We recall some further results from [19].

(a) Analogously to Theorem 51 (c), $\mathcal{V}_q \setminus \mathcal{U}_q$ is a countable dense subset of $\mathcal{V}_q$.

(b) If $q \in \mathcal{U}$, then each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly two expansions.

(c) If $q \in \mathcal{V} \setminus \mathcal{U}$, then each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly $\aleph_0$ expansions.

All the expansions in (b), (c) are given implicitly in [19] and explicitly in [45]. See also [4] for the changes in integer bases q if the digit q is also allowed.

[11] See Definition 50.

Next we discuss the Cantor property of $\mathcal{U}_q$ and $\overline{\mathcal{U}_q}$:

Theorem 64. *([19]) Let $q > 1$.*

(a) *If q is integer, then neither $\mathcal{U}_q$ nor $\overline{\mathcal{U}_q}$ is a Cantor set.*

(b) *If $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then $\mathcal{U}_q$ is not a Cantor set, but $\overline{\mathcal{U}_q}$ is a Cantor set.*

(c) *If $p_1 < q \le v_1$, where (p_1, p_2) is one of the open intervals in Theorem 53, except $(1, q')$ and $(2, q'')$, and v_1 is the smallest element of $\mathcal{V} \cap (p_1, p_2)$, then $\mathcal{U}_q$ is a Cantor set. Otherwise $\mathcal{U}_q$ is not a Cantor set.*

We finish this section by investigating the *two-dimensional* univoque set

$$\mathbf{U} = \{(x, q) \in \mathbb{R} \times (1, \infty) \ : \ x \in \mathcal{U}_q\}$$

and its closure $\overline{\mathbf{U}}$.

Theorem 65. *([20]) We have*

(a) $\mathbf{U}$ *is not closed,* $\overline{\mathbf{U}}$ *is a Cantor set;*

(b) $\mathbf{U}$ *and* $\overline{\mathbf{U}}$ *are two-dimensional Lebesgue null sets;*

(c) $\mathbf{U}$ *and* $\overline{\mathbf{U}}$ *have Hausdorff dimension two.*

8. General Alphabets

Many results exposed until now may be extended to general *finite alphabets* $A = \{a_1 < \cdots < a_n\}$ of real numbers.

Definition 66. By an *expansion* of a real number x in some base $q > 1$ on the alphabet A we mean a sequence (c_i) of elements of A satisfying the equality (1).

Remarks 67.

(a) In order to have an expansion, x must belong to $J_{A,q} := [a_1/(q-1), a_n/(q-1)]$.

(b) (Pedicini [56]) Conversely, every $x \in J_{A,q}$ has at least one expansion in base q if and only if

$$q \le 1 + \frac{a_n - a_1}{\max\{a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}\}}.$$

(c) The expansions of the endpoints of $J_{A,q}$ are always unique.

(d) The dynamical properties of such expansions have been investigated by Dajani and Kalle [11].

The Golden ratio admits the following analogue (see Theorem 58 (a), (b)):

Proposition 68. *([46]) For each finite alphabet A of real numbers there exists a number $G_A > 1$ such that*

(a) *for $1 < q < G_A$ only the endpoints of $J_{A,q}$ have unique expansions in base q;*

(b) *for $q > G_A$ there are also other numbers x having unique expansions in base q.*

The *critical base* G_A has been determined for all ternary alphabets. In order to state this result we assume by a scaling argument that $A = \{0, 1, n\}$ with $n \geq 2$.

Theorem 69. *([46]) Let us denote by G_n the critical base for the alphabet $A = \{0, 1, n\}$ with $n \geq 2$.*

(a) *The function $n \mapsto G_n$ is continuous and*

$$2 \leq G_n \leq P_n := 1 + \sqrt{\frac{n}{n-1}}$$

for all n.

(b) *We have $G_n = 2$ if and only if $n = 2^k$ for some positive integer k.*

(c) *The set $C := \{n \geq 2 \ : \ G_n = P_n\}$ is a Cantor set; its smallest element is $1 + x \approx 2.3247$ where x is the first Pisot number, i.e., the positive root of the equation $x^3 = x + 1$;*

(d) *Each connected component (n_d, N_d) of $[2, \infty) \setminus C$ has a point ν_d such that $n \mapsto G_n$ is strictly decreasing in $[n_d, \nu_d]$ and strictly increasing in $[\nu_d, N_d]$.*

Remark 70. We refer to [46] for the explicit determination of G_n, n_d, N_d, ν_d and for the determination of those n for which there exist nontrivial univoque sequences in the critical base G_n, too.

9. Open Problems

We end this paper with a list of some open questions.

1. ([28], [44]) Is it true that $\ell^1(q) = 0$ for all non-Pisot numbers $1 < q < 2$?

2. ([28], [44]) Is it true that $L^1(q) = 0$ for all non-Pisot numbers $1 < q < G$?[12]

3. ([26]) Theorem 38 (b) gives an intrinsic characterization of the set of sequences which are greedy expansions of $x = 1$ in some base q. Find a similar intrinsic characterization of the set of sequences which are *lazy* expansions of $x = 1$ in some base q.[13] A sufficient but not necessary condition is given in [27], Proposition 2.5.

4. Similarly, Theorem 38 (d) gives an intrinsic characterization of the set of sequences which are quasi-greedy expansions of $x = 1$ in some base q. Find a similar intrinsic characterization of the set of sequences which are *quasi-lazy* expansions of $x = 1$ in some base q.[14]

5. ([48]) Investigate (and if possible, characterize) the set of bases $q > 1$ in which $x = 1$ has exactly two expansions. [15]

6. ([44]) More generally, investigate for each $2 \le N \le \aleph_0$ the set of bases $q > 1$ in which $x = 1$ has exactly N expansions.

7. Given $2 \le N \le \aleph_0$ and a noninteger number $q > 1$, investigate the set of numbers x having exactly N expansions in base q.[16]

8. ([44]) Do there exist rational but noninteger univoque bases?[17]

9. Extend the results of this review to the case of nonpositive (negative or complex) bases.[18]

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[12] No $q > \sqrt{2}$ is known for which $L^1(q) = 0$.

[13] The lazy expansion of x is by definition the lexicographically smallest expansion of x.

[14] The quasi-lazy expansion of x is by definition $(m - c_i)$ where (c_i) is the quasi-greedy expansion of $\frac{m}{q-1} - x$.

[15] Denoting by q_0 the base in which $111(100)^\infty$ is an expansion of $x = 1$, no such base is known below q_0, and q_0 is shown to be an accumulation point of such bases in [48].

[16] Sidorov [61] proved that the positive solution $q \approx 1.71064$ of $q^4 = 2q^2 + q + 1$ is the smallest base in which there exists $x \in J_q$ having exactly two expansions.

[17] Univoque Pisot bases exist: see Allouche, Frougny and Hare [3].

[18] A few papers are available on this subject: Daróczy and Kátai [14], Ito and Sadahiro [36], Frougny and Lai [33], Dajani and Kalle [12], and [51], [52].

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