SOME REMARKS ON A PAPER OF MOLNAR AND SINGH

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Received: 3/6/26, Revised: 4/27/26, Accepted: 7/21/26, Published: 7/29/26

Abstract

In this paper, we study further solutions of a Diophantine equation recently introduced by Molnar and Singh.

1. Introduction

In a recent paper, Molnar and Singh [2], motivated by the famous Lehmer's totient conjecture [1], investigated the solutions of a Diophantine equation of the form

$$k\prod_{j=1}^{r}(x_j-1)=\left(\prod_{j=1}^{r}x_j\right)-1,\quad k\in\mathbb{Z}_{>0} \tag{1}$$

in positive integers x_j, $j=1,\dots,r$. They proved the finiteness of integer solutions of Equation (1) for any integer $r\geq 2$. Moreover, they showed that for $2\leq r\leq 6$, Equation (1) has exactly 76 solutions, 31 with odd $x_1,\dots,x_r\geq 3$ and 45 with even $x_1,\dots,x_r\geq 2$.

Before stating the main results, we briefly recall some terminology. With a slight modification, we follow the notation introduced by Molnar and Singh. A finite multiset of positive integers $F=\{x_1,\dots,x_r\}$ is said to be a *spoof factorization* F. In a minor abuse of notation, we write $x_1\cdots x_r$ for the spoof factorization $\{x_1,\dots,x_r\}$.

Let $\mathcal{F}$ be the set of all spoof factorizations. The *spoof evaluation function* $\tilde{\epsilon}:\mathcal{F}\to\mathbb{Z}$ is defined by

$$\tilde{\epsilon}:\{x_1,\dots,x_r\}\mapsto\prod_{i=1}^{r}x_i.$$

We also introduce the *spoof unitary totient function* $\tilde{\varphi^*}:\mathcal{F}\to\mathbb{Z}$ as

$$\tilde{\varphi^*}:x_1\cdots x_r\mapsto(x_1-1)\cdots(x_r-1)=\tilde{\epsilon}(x_1\cdots x_r)\prod_{i=1}^{r}\left(1-\frac{1}{x_i}\right).$$

DOI: 10.5281/zenodo.21678654
[1]This research was supported by ADA University Faculty Research and Development Funds.

Recall that the unitary totient function φ^* is defined as $\varphi^*(n) = \prod_{i=1}^{r}(p_i^{\alpha_i} - 1)$ for the prime factorization $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$.

Now we can recast Equation (1) in terms of this new terminology as follows. We say that a spoof factorization $F \in \mathcal{F}$ is a *unitary k-Lehmer factorization* if

$$k\tilde{\varphi^*}(F) = \tilde{\epsilon}(F) - 1, \tag{2}$$

where F has at least two factors.

Our first result shows that there exists an explicit upper bound on unitary k-Lehmer factorizations with a fixed number of spoof factors.

Theorem 1. *If F is a unitary k-Lehmer factorization with r spoof factors then*

$$\tilde{\epsilon}(F) \leq 2^{2^r} - 2^{2^{r-1}}.$$

Note that the above bound is not tight. For $r = 3$, the bound gives $\tilde{\epsilon}(F) \leq 240$, while the largest value of $\tilde{\epsilon}(F)$ for a unitary k-Lehmer factorization with exactly three spoof factors is 225 (it is attained at $F = 3 \cdot 5 \cdot 15$).

We write $[x]^s$ for the spoof factorization $x \cdots x$ with x repeated s times. For $s \geq 2$ an integer, the spoof factorization $[2]^s$ is a unitary $(2^s - 1)$-Lehmer factorization. It is easy to check that $[3]^2$ and $[3]^4$ are unitary 2-Lehmer factorizations. The next theorem shows that these are the only examples of unitary k-Lehmer factorizations with repeated spoof factors.

Theorem 2. *If a spoof factorization $[x]^s$ is unitary k-Lehmer with $k \geq 2$ then either $x = 2$ and $s \geq 2$ or $x = 3$ and $s = 2, 4$.*

Our last result is an analogue of Lehmer's lemma [1, Lemma 2].

Theorem 3. *If F is a spoof factorization satisfying $k\tilde{\varphi^*}(F) = \tilde{\epsilon}(F) + \alpha$, $\alpha = \pm 1$, $k \geq 2$ and if x and y are integers for which*

$$k\tilde{\varphi^*}(F \cdot x \cdot y) = \tilde{\epsilon}(F \cdot x \cdot y) - 1, \tag{3}$$

then

$$(\alpha x - \tilde{\epsilon}(F) - \alpha)(\alpha y - \tilde{\epsilon}(F) - \alpha) - ((\tilde{\epsilon}(F))^2 + \alpha\tilde{\epsilon}(F) - \alpha) = 0. \tag{4}$$

We now give some examples to illustrate the application of Theorem 3.

Example 1. Let $\alpha = 1$ and $F = 3 \cdot 5$. Clearly, F satisfies $2\tilde{\varphi^*}(F) = \tilde{\epsilon}(F) + 1$. Let $3 \cdot 5 \cdot x \cdot y$ be a unitary 2-Lehmer factorization with $x \leq y$. Then, by Equation (3) we get

$$(x - 16)(y - 16) = 239.$$

Since 239 is a prime, $x - 16 = 1$ and $y - 16 = 239$. Therefore, $3 \cdot 5 \cdot 17 \cdot 255$ is a unitary 2-Lehmer factorization. More generally, $F(s) = \prod_{i=0}^{s-1}(2^{2^i} + 1)$ satisfies

$2\tilde{\varphi^*}(F) = \tilde{\epsilon}(F)+1$ and by Equation (3) $F(s)\cdot x\cdot y$ is a unitary 2-Lehmer factorization if $x = 2^{2^s} + d_1$ and $y = 2^{2^s} + d_2$ for any factorization $2^{2^{s+1}} - 2^{2^s} - 1 = d_1 d_2$. This is the same family as in the Example 5 of Molnar and Singh.

Example 2. Let $\alpha = 1$ and $F = 5 \cdot 5 \cdot 5 \cdot 43$. Then F satisfies $2\tilde{\varphi^*}(F) = \tilde{\epsilon}(F) + 1$ and $F(s) \cdot x \cdot y$ is a unitary 2-Lehmer factorization if x and y satisfy

$$(x - 5376)(y - 5376) = 28895999.$$

Since $28895999 = 11 \cdot 31 \cdot 101 \cdot 839$, we get the corresponding 7 sporadic 2-Lehmer factorizations from Table 1 of Molnar and Singh.

Example 3. Using Theorem 3 we can also generate some Lehmer factorizations with negative bases. For example, $F = 3 \cdot 3$ is a unitary 2-Lehmer factorization. Then $3 \cdot 3 \cdot x \cdot y$ is also a unitary 2-Lehmer factorization if

$$(x + 8)(y + 8) = 73$$

which gives $3 \cdot 3 \cdot (-7) \cdot 65$. This example is merely for illustrative purposes. A systematic search for solutions of Equations (1) when negative x_i's are allowed requires some additional analysis and we leave it to the interested reader.

2. Preliminaries

In this section we shall collect some preliminary results. The first lemma due to Nielsen [3, Lemma 1.4] plays an important role in the proof of Theorem 1.

Lemma 1. *Let $r, a, b \in \mathbb{Z}_{>0}$ and let $x_1, \ldots, x_r$ be integers such that $1 < x_1 \leq \ldots \leq x_r$. If*

$$\prod_{j=1}^{r} \left(1 - \frac{1}{x_j}\right) \leq \frac{a}{b} < \prod_{j=1}^{r-1} \left(1 - \frac{1}{x_j}\right), \tag{5}$$

then

$$a \prod_{j=1}^{r} x_j \leq (a+1)^{2^r} - (a+1)^{2^{r-1}}. \tag{6}$$

For a prime p and non-zero integer a, denote by $v_p(a)$ the p-adic value of a:

$$v_p(a) = e \quad \text{when} \quad p^e \mid a, \quad \text{but} \quad p^{e+1} \nmid a.$$

The next lemma lists some properties of the p-adic valuation.

Lemma 2. *(i) $v_p(ab) = v_p(a) + v_p(b)$.*
(ii) If p is an odd prime and $p \mid a - b$, then

$$v_p\left(\frac{a^n - b^n}{a - b}\right) = v_p(n).$$

(iii) If $4|a - b$, then $v_2\left(\frac{a^n - b^n}{a-b}\right) = v_2(n)$.

3. Proofs

Proof of Theorem 1. Let $F = x_1 \cdots x_r$ be a unitary k-Lehmer factorization with $x_1 \leq \ldots \leq x_r$. Then

$$\prod_{i=1}^{r}\left(1-\frac{1}{x_i}\right) = \frac{\tilde{\varphi^*}(F)}{\tilde{\epsilon}(F)} < \frac{\tilde{\varphi^*}(F)}{\tilde{\epsilon}(F)-1}.$$

Moreover,

$$\frac{\frac{\tilde{\varphi^*}(F)}{\tilde{\epsilon}(F)-1}}{\prod_{i=1}^{r-1}\left(1-\frac{1}{x_i}\right)} = \frac{\tilde{\epsilon}(F)\prod_{i=1}^{r}\left(1-\frac{1}{x_i}\right)}{(\tilde{\epsilon}(F)-1)\prod_{i=1}^{r-1}\left(1-\frac{1}{x_i}\right)} = \frac{1-\frac{1}{x_r}}{1-\frac{1}{\tilde{\epsilon}(F)}} < 1.$$

Thus,

$$\prod_{i=1}^{r}\left(1-\frac{1}{x_i}\right) < \frac{\tilde{\varphi^*}(F)}{\tilde{\epsilon}(F)-1} < \prod_{i=1}^{r-1}\left(1-\frac{1}{x_i}\right).$$

Hence, the inequality (5) is satisfied for $a = 1$ and $b = \frac{\tilde{\epsilon}(F)-1}{\tilde{\varphi^*}(F)}$. From (6) we get

$$\tilde{\epsilon}(F) = \prod_{i=1}^{r} x_i \leq 2^{2^r} - 2^{2^{r-1}}.$$

□

Proof of Theorem 2. Let $[x]^s$ be a unitary k-Lehmer factorization. Equivalently, x and $s \geq 2$ satisfy the congruence

$$x^s \equiv 1 \pmod{(x-1)^s}.$$

When $x = 2$, it is satisfied for all $s \geq 2$. When $x = 3$, we get the congruence $3^s \equiv 1 \pmod{2^s}$. Note that $\mathrm{ord}_{2^s}(3) = 2^{s-2}$ for $s \geq 3$ (see [4]). Then s is divisible by 2^{s-2}, which exceeds s for $s \geq 5$, a contradiction. Hence, $s = 2$ or $s = 4$.

Now let $x \geq 4$. If p is an odd prime divisor of $x-1$, then by Lemma 2 (i)-(ii) we have $v_p(x^s-1) = v_p(x-1)+v_p(s)$. For divisibility, we require $v_p(x^s-1) \geq sv_p(x-1)$. Thus $v_p(s) \geq (s-1)v_p(x-1) \geq s-1$ (since $v_p(x-1) \geq 1$). But $p^{s-1} > s$ for $p \geq 3$, $s \geq 2$ and therefore $v_p(s) < s-1$ for all $s \geq 2$ and $p \geq 3$. Hence, $x-1 = 2^k$ with $k \geq 2$. By Lemma 2 (iii) and the divisibility condition, we get $k + v_2(s) \geq ks$, or $v_2(s) \geq k(s-1)$, which has no solutions for $k \geq 2$ and $s \geq 2$. □

Proof of Theorem 3. Multiplying out the left-hand side of (4), regrouping and adding and subtracting $\alpha\tilde{\epsilon}(F)xy$ we obtain

$$\begin{aligned}\alpha^2xy - \alpha\tilde{\epsilon}(F)y - \alpha^2y - \alpha\tilde{\epsilon}(F)x + (\tilde{\epsilon}(F))^2 + \alpha\tilde{\epsilon}(F) - \alpha^2x + \alpha\tilde{\epsilon}(F) + \alpha^2 - (\tilde{\epsilon}(F))^2 \\ - \alpha\tilde{\epsilon}(F) + \alpha + \alpha\tilde{\epsilon}(F)xy - \alpha\tilde{\epsilon}(F)xy \\ = \alpha\{(\tilde{\epsilon}(F) + \alpha)(x-1)(y-1) - \tilde{\epsilon}(F)xy + 1\} \\ = \alpha\{k\tilde{\varphi}^*(F \cdot x \cdot y) - \tilde{\epsilon}(F)xy + 1\}.\end{aligned}$$

However, from (3), this equals 0, and hence the theorem follows. □

Acknowledgements. The author thanks the anonymous referee for his/her helpful comments.

References

[1] D. H. Lehmer, On Euler's totient function, *Bull. Amer. Math, Soc.* **38** (1932), 745–751.

[2] G. Molnar and G. Singh, Spoof Lehmer factorizations, *Integers* **26** (2026), #A33.

[3] P. Nielsen, Odd perfect numbers, Diophantine equations, and upper bounds, *Math. Comp.* **84** (2015), 2549–2567.

[4] I. Niven, H. S. Zuckerman and H. L. Montgomery, An Introduction to the Theory of Numbers, John Wiley & Sons, Inc., New York, 1991.