#A101 INTEGERS 26 (2026)

POWER SUM INEQUALITIES FOR MONOTONE SEQUENCES WITH APPLICATIONS TO FIBONACCI NUMBERS

Sanja Kovač
Faculty of Geotechnical Engineering, University of Zagreb, Varaždin, Croatia
skovac@gfv.hr

Josip Pečarić
Croatian Academy of Sciences and Arts, Zagreb, Croatia
jopecaric@gmail.com

Received: 2/4/26, Accepted: 7/22/26, Published: 8/24/26

Abstract

We establish new power sum inequalities for monotone sequences arising in discrete mathematics and number theory. Our results provide sharp bounds for weighted and unweighted power sums and extend several classical inequalities of Cauchy–Schwarz and Hölder type. As principal applications, we obtain new inequalities for Fibonacci numbers and related linear recurrence sequences, including Lucas numbers. In particular, we derive explicit two-sided estimates for sums of powers of Fibonacci-type sequences that improve or generalize several known results in the literature. The approach relies on monotonicity arguments combined with discrete inequality techniques, making the results well-suited for applications in number theory and discrete mathematics.

1. Introduction

The classical Cauchy–Schwarz inequality plays a central role in analysis and discrete mathematics. For real vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$, it states that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Equality holds if and only if **a** and **b** are linearly dependent. In recent decades, considerable attention has been devoted to converse and refined forms of this inequality, especially in discrete settings where monotonicity and ordering of sequences play an essential role. A notable contribution in this direction is due to Alzer [1], who obtained a discrete analogue of a converse inequality originally proposed by Zagier

DOI: 10.5281/zenodo.22081870

[12]. For positive monotone sequences $a_1 \geq \cdots \geq a_n > 0$ and $b_1 \geq \cdots \geq b_n > 0$, Alzer showed that

$$\frac{\sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2}{\max\left\{a_1 \sum_{i=1}^n b_i,\, b_1 \sum_{i=1}^n a_i\right\}} \leq \sum_{i=1}^n a_i b_i.$$

Equality holds if and only if $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$.

Pečarić [8] subsequently extended this result to a weighted form, for w_i positive numbers, as

$$\frac{\sum_{i=1}^n w_i a_i^2 \cdot \sum_{i=1}^n w_i b_i^2}{\max\left\{a_1 \sum_{i=1}^n w_i b_i,\, b_1 \sum_{i=1}^n w_i a_i\right\}} \leq \sum_{i=1}^n w_i a_i b_i,$$

which extends the unweighted case by incorporating a system of positive weights. This generalization enables the derivation of more flexible and sharper inequalities in applications where natural weightings arise, especially when analyzing number-theoretic sequences. Such converse inequalities are particularly well-suited for studying power sums of monotone sequences. Given a positive sequence $x = (x_1, \ldots, x_n)$ and a real parameter $\alpha > 0$, the power sum

$$S_n^{[\alpha]}(x) = \sum_{k=1}^n x_k^\alpha$$

is a fundamental object in inequalities, number theory, and discrete analysis. Understanding the relationships between different power sums under monotonicity constraints leads to nontrivial bounds that are both sharp and widely applicable.

An important source of applications is provided by Fibonacci and Lucas sequences, which are among the most studied linear recurrence sequences in number theory. The Fibonacci numbers $(F_n)_{n\geq 0}$, defined by $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$, exhibit strong monotonicity and log-convexity properties that make them particularly suitable for the application of power-sum inequalities. Many classical identities and inequalities for Fibonacci numbers can be viewed as special cases of more general results for monotone sequences. Fibonacci numbers can be obtained using the following formula:

$$F_n = \frac{\phi^n - (\phi - \sqrt{5})^n}{\sqrt{5}}, \quad n \in \mathbb{N},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio.

Recent work has demonstrated that Cauchy–Schwarz type inequalities yield powerful bounds for Fibonacci sums. For example, Alzer and Luca proved the following theorem [2].

Theorem 1 ([2]). *Let $r, s \in \mathbb{R}$ with $r + s \geq 4$. Then, for $n \geq 1$,*

$$\sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^s \geq (F_n F_{n+1})^2. \tag{1}$$

The sign of equality is valid in (1) if and only if $n = 1, 2$, or $n \geq 3$ and $r = s = 2$.

This result highlights how inequalities for monotone sequences can produce sharp estimates for sums of powers of Fibonacci numbers.

The aim of the present paper is to develop a systematic framework of power sum inequalities for monotone sequences based on converse Cauchy type inequalities, and to apply these results to Fibonacci numbers, Lucas numbers, and general linear recurrence sequences of order two. We obtain explicit upper and lower bounds for sums of powers of such sequences, extending and improving several known inequalities in the literature. Unlike existing results based on converse forms of the Cauchy–Schwarz inequality, the inequalities obtained in this paper provide explicit two-sided bounds for general power sums of monotone sequences under minimal structural assumptions. The weighted versions presented here cannot be deduced from previously known inequalities and yield a unified framework applicable to a broad class of sequences. In particular, the applications to linear recurrence sequences of order two extend earlier results that were mainly restricted to the Fibonacci setting.

Our approach is elementary and is based on monotonicity arguments combined with discrete inequality techniques, which makes the results well-suited for applications in number theory and discrete mathematics.

2. Main Result

We will need the converse Cauchy inequality for nondecreasing sequences:

$$\frac{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}{\max\left\{a_n \sum_{i=1}^{n} b_i,\, b_n \sum_{i=1}^{n} a_i\right\}} \leq \sum_{i=1}^{n} a_i b_i, \tag{2}$$

where $0 < a_1 \leq a_2 \leq \ldots \leq a_n$ and $0 < b_1 \leq b_2 \leq \ldots \leq b_n$. If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$ such that $0 < x_1 \leq x_2 \leq \ldots \leq x_n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n$ such that $0 < y_1 \leq y_2 \leq \ldots \leq y_n$, then we can write Inequality (2) as

$$\frac{S_n^{[2]}(\mathbf{x}) \cdot S_n^{[2]}(\mathbf{y})}{\max\left\{x_n S_n^{[1]}(\mathbf{y}), y_n S_n^{[1]}(\mathbf{x})\right\}} \leq \sum_{i=1}^{n} x_i y_i. \tag{3}$$

We will also use the following well-known monotonicity property of power sums [7], p.4:

Proposition 1. *If $\alpha > \beta > 0$, then*

$$(S_n^{[\alpha]}(\mathbf{x}))^{\frac{1}{\alpha}} \leq (S_n^{[\beta]}(\mathbf{x}))^{\frac{1}{\beta}}.$$

We also recall Jensen's inequality: if $\varphi : I \to \mathbb{R}$ is convex function, $x_1, \ldots, x_n \in I$, and $\lambda_k \geq 0$ with $\sum_{k=1}^n \lambda_k = 1$, then

$$\varphi\left(\sum_{k=1}^n \lambda_k x_k\right) \leq \sum_{k=1}^n \lambda_k \varphi(x_k).$$

In particular,

$$\varphi\left(\frac{1}{n}\sum_{k=1}^n x_k\right) \leq \frac{1}{n}\sum_{k=1}^n \varphi(x_k).$$

Theorem 2. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$ such that $0 < x_1 \leq x_2 \leq \ldots \leq x_n$, $r, s \in \mathbb{R}_+$, and $\alpha = \frac{r+s}{2}$.*

(i) If $\alpha \geq \beta > 0$, then we have

$$\frac{S_n^{[r]}(\mathbf{x}) S_n^{[s]}(\mathbf{x})}{\max\{x_n^{r/2} S_n^{[\frac{s}{2}]}(\mathbf{x}), x_n^{s/2} S_n^{[\frac{r}{2}]}(\mathbf{x})\}} \leq S_n^{[\alpha]}(\mathbf{x}) \leq (S_n^{[\beta]}(\mathbf{x}))^{\frac{\alpha}{\beta}}.$$

(ii) If $0 < \alpha < \beta$, then we have

$$\frac{S_n^{[r]}(\mathbf{x}) S_n^{[s]}(\mathbf{x})}{\max\{x_n^{r/2} S_n^{[\frac{s}{2}]}(\mathbf{x}), x_n^{s/2} S_n^{[\frac{r}{2}]}(\mathbf{x})\}} \leq S_n^{[\alpha]}(\mathbf{x}) \leq n \cdot (\frac{S_n^{[\beta]}(\mathbf{x})}{n})^{\frac{\alpha}{\beta}}.$$

If, in addition, $x_1 \geq 1$, then

$$\frac{S_n^{[r]}(\mathbf{x}) S_n^{[s]}(\mathbf{x})}{\max\{x_n^{r/2} S_n^{[\frac{s}{2}]}(\mathbf{x}), x_n^{s/2} S_n^{[\frac{r}{2}]}(\mathbf{x})\}} \leq S_n^{[\alpha]}(\mathbf{x}) \leq n \cdot (\frac{S_n^{[\beta]}(\mathbf{x})}{n})^{\frac{\alpha}{\beta}} \leq S_n^{[\beta]}(\mathbf{x}). \quad (4)$$

Proof. To prove (i), we use Inequality (2) for $a_i = x_i^{\frac{r}{2}}$ and $b_i = x_i^{\frac{s}{2}}$, while the second inequality follows from the Proposition 1.

Now, let us prove (ii). The first inequality follows from Inequality (2) for $a_i = x_i^{\frac{r}{2}}$ and $b_i = x_i^{\frac{s}{2}}$.

Since $\beta/\alpha > 1$, the function

$$\varphi(t) = t^{\beta/\alpha}, \qquad t > 0,$$

is convex. Applying Jensen's inequality to the numbers $x_1^\alpha, \ldots, x_n^\alpha$, we obtain:

$$S_n^{[\beta]}(\mathbf{x}) = \sum_{k=1}^n x_k^\beta = \sum_{k=1}^n (x_k^\alpha)^{\frac{\beta}{\alpha}} \geq n \cdot (\frac{\sum_{k=1}^n x_k^\alpha}{n})^{\frac{\beta}{\alpha}} = n^{1-\frac{\beta}{\alpha}} \cdot (S_n^{[\alpha]}(\mathbf{x}))^{\frac{\beta}{\alpha}}.$$

If $x_1 \geq 1$, then the function $t \to (\frac{(S_n^{[\beta]})(\mathbf{x})}{n})^t$ is monotone increasing, so

$$(\frac{S_n^{[\beta]}(\mathbf{x})}{n})^{\frac{\alpha}{\beta}} \leq (\frac{S_n^{[\beta]}(\mathbf{x})}{n})^1.$$

Hence, the last inequality is valid. $\square$

Now we apply these results for Fibonacci numbers.

Theorem 3. *Let $r, s \in \mathbb{R}_+$. Then for $n \geq 1$ we have:*

(i) If $\frac{r+s}{2} \geq 2$

$$\frac{\sum_{k=1}^n F_k^r \sum_{k=1}^n F_k^s}{F_n^{\frac{s}{2}} \sum_{k=1}^n F_k^{\frac{r}{2}}} \leq \sum_{k=1}^n F_k^{\frac{r+s}{2}} \leq (F_n F_{n+1})^{\frac{r+s}{4}}.$$

(ii) If $0 < \frac{r+s}{2} < 2$,

$$\frac{\sum_{k=1}^n F_k^r \sum_{k=1}^n F_k^s}{F_n^{\frac{s}{2}} \sum_{k=1}^n F_k^{\frac{r}{2}}} \leq \sum_{k=1}^n F_k^{\frac{r+s}{2}} \leq n \cdot (\frac{F_n F_{n+1}}{n})^{\frac{r+s}{4}} \leq F_n F_{n+1}.$$

Proof. We apply Theorem 2 for $x_i = F_i$, $\alpha = \frac{r+s}{2}$, and $\beta = 2$:

$$\frac{\sum_{k=1}^n F_k^r \sum_{k=1}^n F_k^s}{\max\{F_n^{\frac{r}{2}} \sum_{k=1}^n F_k^{\frac{s}{2}}, F_n^{\frac{s}{2}} \sum_{k=1}^n F_k^{\frac{r}{2}}\}} \leq \sum_{k=1}^n F_k^{\frac{r+s}{2}}.$$

Without loss of generality, we can assume that $r \leq s$. We rewrite

$$F_n^{\frac{s}{2}} \sum_{k=1}^n F_k^{\frac{r}{2}} = F_n^{\frac{s-r}{2}} \sum_{k=1}^n (F_n F_k)^{\frac{r}{2}}$$

and

$$F_n^{\frac{r}{2}} \sum_{k=1}^n F_k^{\frac{s}{2}} = \sum_{k=1}^n (F_n F_k)^{\frac{r}{2}} \cdot F_k^{\frac{s-r}{2}}.$$

Since $\frac{s-r}{2} \geq 0$ and $F_k \leq F_n$, we have

$$F_k^{\frac{s-r}{2}} \leq F_n^{\frac{s-r}{2}}.$$

Therefore,

$$F_n^{\frac{r}{2}} \sum_{k=1}^n F_k^{\frac{s}{2}} = \sum_{k=1}^n (F_n F_k)^{\frac{r}{2}} \cdot F_k^{\frac{s-r}{2}} \leq F_n^{\frac{s-r}{2}} \sum_{k=1}^n (F_n F_k)^{\frac{r}{2}} = F_n^{\frac{s}{2}} \sum_{k=1}^n F_k^{\frac{r}{2}}.$$

So we have

$$\max\{F_n^{\frac{r}{2}} \sum_{k=1}^n F_k^{\frac{s}{2}}, F_n^{\frac{s}{2}} \sum_{k=1}^n F_k^{\frac{r}{2}}\} = F_n^{\frac{s}{2}} \sum_{k=1}^n F_k^{\frac{r}{2}}.$$

The rest of the proof follows by applying Fibonacci numbers into Theorem 2. $\square$

Remark 1. The inequalities obtained in Theorems 2 and 3 provide two-sided estimates for power sums of monotone sequences and their applications to Fibonacci-type sequences, offering several improvements over recent related works. The specific inequalities for power sums of Fibonacci numbers in [2] and [3] are extended here to arbitrary positive exponents $\alpha > 0$, with both lower and upper bounds. In particular, the lower bounds obtained in Theorems 2 and 3 are genuinely new and cannot be deduced from classical Cauchy–Schwarz or Hölder inequalities. Compared to the power mean inequalities for Fibonacci sequences in [5] which primarily yield one-sided bounds derived from classical inequalities, our approach supplies explicit lower bounds via the converse Cauchy inequality, while the upper bounds coincide or improve upon theirs in the case $\beta = 2$ (utilizing the exact identity $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$). Furthermore, the forthcoming work [4] on Fibonacci numbers and Hölder's inequality is complemented by our converse-type estimates, which unify and generalize a broader class of power sum inequalities under minimal monotonicity assumptions. These two-sided bounds, combined with the catalogue of applications in Section 3, thus offer sharper and more flexible tools for estimating sums of powers in linear recurrence sequences of order two.

3. Application to Special Sequences

In this section, we shall use some identities involving Fibonacci numbers to derive new inequalities.

Lemma 1. *The following identities hold [6]:*

a) For $i = 1, \ldots, n$, $x_i = F_i$, $S_n^{[1]}(\mathbf{x}) = F_{n+2} - 1$.

b) For $i = 1, \ldots, n$, $x_i = F_{2i-1}$, $S_n^{[1]}(\mathbf{x}) = F_{2n}$.

c) For $i = 1, \ldots, n$, $x_i = F_{2i}$, $S_n^{[1]}(\mathbf{x}) = F_{2n+1} - 1$.

d) For $i = 1, \ldots, n$, $x_i = iF_i$, $S_n^{[1]}(\mathbf{x}) = nF_{n+2} - F_{n+3} + 2$.

e) For $i = 1, \ldots, n$, $x_i = F_i F_{3i}$, $S_n^{[1]}(\mathbf{x}) = F_n F_{n+1} F_{2n+1}$.

f) For $i = 1, \ldots, n$, $x_i = F_{4i-2}$, $S_n^{[1]}(\mathbf{x}) = F_{2n}^2$.

g) For $i = 1, \ldots, n$, $x_i = \binom{n}{i} F_i$, $S_n^{[1]}(\mathbf{x}) = F_{2n}$.

h) For $i = 1, \ldots, 2n+1$, $x_i = F_i \sqrt{\binom{2n+1}{i}}$, $S_{2n+1}^{[2]}(\mathbf{x}) = 5^n F_{2n+1}$.

Example 1. Let $x_i = F_i$ and $\alpha = r = s = 1$. Considering Lemma 1 (a) we get $S_n^{[1]}(\mathbf{x}) = F_{n+2} - 1$. For $\beta = \frac{1}{2}$ and after applying Theorem 2 (i), we get

$$\frac{(F_{n+2}-1)^2}{\sqrt{F_n}\sum_{k=1}^{n}\sqrt{F_k}} \leq F_{n+2} - 1 \leq \left(\sum_{k=1}^{n}\sqrt{F_k}\right)^2.$$

For $\beta = 2$, after applying Theorem 2 (ii), we get

$$\frac{(F_{n+2}-1)^2}{\sqrt{F_n}\sum_{k=1}^{n}\sqrt{F_k}} \leq F_{n+2} - 1 \leq \sqrt{nF_nF_{n+1}} \leq F_nF_{n+1}.$$

Example 2. For $x_i = F_{2i-1}$ and $\alpha = r = s = 1$, we have $S_n^{[1]}(\mathbf{x}) = F_{2n}$. Applying Theorem 2 for $\beta = 1/2$ and $\beta = 2$, we have

$$\frac{(F_{2n})^2}{\sqrt{F_{2n-1}}\sum_{k=1}^{n}\sqrt{F_{2k-1}}} \leq F_{2n} \leq \left(\sum_{k=1}^{n}\sqrt{F_{2k-1}}\right)^2$$

and

$$\frac{(F_{2n})^2}{\sqrt{F_{2n-1}}\sum_{k=1}^{n}\sqrt{F_{2k-1}}} \leq F_{2n} \leq \sqrt{n\sum_{k=1}^{n}F_{2k-1}^2} \leq \sum_{k=1}^{n}F_{2k-1}^2,$$

respectively.

Example 3. For $x_i = F_{2i}$ and $\alpha = r = s = 1$, we have $S_n^{[1]}(\mathbf{x}) = F_{2n+1} - 1$. After applying Theorem 2 for $\beta = 1/2$ and $\beta = 2$, we get

$$\frac{(F_{2n+1}-1)^2}{\sqrt{F_{2n}}\sum_{k=1}^{n}\sqrt{F_{2k}}} \leq F_{2n+1} - 1 \leq \left(\sum_{k=1}^{n}\sqrt{F_{2k}}\right)^2$$

and

$$\frac{(F_{2n+1}-1)^2}{\sqrt{F_{2n}}\sum_{k=1}^{n}\sqrt{F_{2k}}} \leq F_{2n+1} - 1 \leq \sqrt{n\sum_{k=1}^{n}F_{2k}^2} \leq \sum_{k=1}^{n}F_{2k}^2,$$

respectively.

Example 4. For $x_i = iF_i$ and $\alpha = r = s = 1$ we have $S_n^{[1]}(\mathbf{x}) = nF_{n+2} - F_{n+3} + 2$. Here, we apply Theorem 2 and Lemma 1 (d) for $\beta = 1/2$ and $\beta = 2$ to obtain

$$\frac{(nF_{n+2}-F_{n+3}+2)^2}{\sqrt{nF_n}\sum_{k=1}^{n}\sqrt{kF_k}} \leq nF_{n+2} - F_{n+3} + 2 \leq \left(\sum_{k=1}^{n}\sqrt{kF_k}\right)^2$$

and

$$\frac{(nF_{n+2}-F_{n+3}+2)^2}{\sqrt{nF_n}\sum_{k=1}^{n}\sqrt{kF_k}} \leq nF_{n+2} - F_{n+3} + 2 \leq \sqrt{n\cdot\sum_{k=1}^{n}(kF_k)^2},$$

respectively.

Example 5. For $x_i = i$ and $y_i = F_i$, $i = 1, \dots, n$, by Inequality (3) and Lemma 1 (d), we have

$$\frac{n(n+1)(2n+1)}{6} F_n F_{n+1} \leq \max\{n(F_{n+2}-1), \frac{n(n+1)}{2} F_n\} \cdot (nF_{n+2} - F_{n+3} + 2).$$

Example 6. For $x_i = F_i F_{3i}$ and $\alpha = r = s = 1$ we have $S_n^{[1]}(\mathbf{x}) = F_n F_{n+1} F_{2n+1}$. We apply Theorem 2 and Lemma 1 (e) for $\beta = 1/2$ and $\beta = 2$ to get

$$\frac{(F_n F_{n+1} F_{2n+1})^2}{\sqrt{F_n F_{3n}} \sum_{k=1}^n \sqrt{F_k F_{3k}}} \leq F_n F_{n+1} F_{2n+1} \leq \left(\sum_{k=1}^n \sqrt{F_k F_{3k}} \right)^2$$

and

$$\frac{(F_n F_{n+1} F_{2n+1})^2}{\sqrt{F_n F_{3n}} \sum_{k=1}^n \sqrt{F_k F_{3k}}} \leq F_n F_{n+1} F_{2n+1} \leq \sqrt{n \cdot \sum_{k=1}^n (F_k F_{3k})^2},$$

respectively.

Example 7. For $x_i = F_{4i-2}$ and $\alpha = r = s = 1$, we have $S_n^{[1]}(\mathbf{x}) = F_{2n}^2$. Applying Theorem 2 for $\beta = 1/2$ and $\beta = 2$, we have

$$\frac{F_{2n}^4}{\sqrt{F_{4n-2}} \sum_{k=1}^n \sqrt{F_{4k-2}}} \leq F_{2n}^2 \leq \left(\sum_{k=1}^n \sqrt{F_{4k-2}} \right)^2$$

and

$$\frac{F_{2n}^4}{\sqrt{F_{4n-2}} \sum_{k=1}^n \sqrt{F_{4k-2}}} \leq F_{2n}^2 \leq \sqrt{n \sum_{k=1}^n F_{4k-2}^2} \leq \sum_{k=1}^n F_{4k-2}^2,$$

respectively.

Example 8. For $x_i = \binom{n}{i} F_i$ and $\alpha = r = s = 1$ we have $S_n^{[1]}(\mathbf{x}) = F_{2n}$. For $\beta = \frac{1}{2}$, we have after applying Theorem 2 (i) and Lemma 1 (g):

$$\frac{F_{2n}^2}{\sqrt{F_n} \sum_{k=1}^n \sqrt{\binom{n}{k} F_k}} \leq F_{2n} \leq \left(\sum_{k=1}^n \sqrt{\binom{n}{k} F_k} \right)^2.$$

On the other hand, for $\beta = 2$, we have after applying Theorem 2 (ii) and Lemma 1 (g):

$$\frac{F_{2n}^2}{\sqrt{F_n} \sum_{k=1}^n \sqrt{\binom{n}{k} F_k}} \leq F_{2n} \leq \sqrt{n \cdot \sum_{k=1}^n \left(\binom{n}{k} F_k \right)^2}.$$

Example 9. We consider the sequence defined by

$$x_i = F_i \cdot F_{i+2}, \quad i = 1, 2, \ldots, n,$$

where F_i denotes the i-th Fibonacci number. Note that the Fibonacci sequence $\{F_i\}$ is strictly increasing for $i \geq 2$. Therefore, the sequence $\{x_i\} = \{F_i F_{i+2}\}$ is also strictly increasing and satisfies the conditions of Theorem 2. We apply Theorem 2 in the case:

$$\alpha = 1, \quad \beta \leq 1.$$

Since $\alpha \geq \beta$, we have:

$$\frac{S_n^{[r]}(\mathbf{x}) \cdot S_n^{[s]}(\mathbf{x})}{\max\left\{x_n^{r/2} S_n^{[s/2]}(\mathbf{x}),\ x_n^{s/2} S_n^{[r/2]}(\mathbf{x})\right\}} \leq S_n^{[\alpha]}(\mathbf{x}) \leq \left(S_n^{[\beta]}(\mathbf{x})\right)^{\alpha/\beta}.$$

From [4] we have

$$S_n(\mathbf{x}) = F_{n+1}F_{n+2} - \frac{1 + (-1)^n}{2}.$$

Without loss of generality, we can assume that $r \leq s$. We rewrite

$$(F_nF_{n+2})^{\frac{s}{2}} \sum_{k=1}^{n} (F_kF_{k+2})^{\frac{r}{2}} = (F_nF_{n+2})^{\frac{s-r}{2}} \sum_{k=1}^{n} (F_nF_{n+2}F_kF_{k+2})^{\frac{r}{2}}$$

and

$$(F_nF_{n+2})^{\frac{r}{2}} \sum_{k=1}^{n} (F_kF_{k+2})^{\frac{s}{2}} = \sum_{k=1}^{n} (F_nF_{n+2}F_kF_{k+2})^{\frac{r}{2}} \cdot (F_kF_{k+2})^{\frac{s-r}{2}}.$$

Since $\frac{s-r}{2} \geq 0$ and $1 \leq F_kF_{k+2} \leq F_nF_{n+2}$, we have

$$(F_kF_{k+2})^{\frac{s-r}{2}} \leq (F_nF_{n+2})^{\frac{s-r}{2}}.$$

Therefore,

$$\begin{aligned} &\max\{(F_nF_{n+2})^{\frac{r}{2}} \sum_{k=1}^{n} (F_kF_{k+2})^{\frac{s}{2}}, (F_nF_{n+2})^{\frac{s}{2}} \sum_{k=1}^{n} (F_kF_{k+2})^{\frac{r}{2}}\} \\ &= (F_nF_{n+2})^{\frac{s}{2}} \sum_{k=1}^{n} (F_kF_{k+2})^{\frac{r}{2}}. \end{aligned}$$

Now we have

$$\frac{\sum_{i=1}^{n} (F_iF_{i+2})^r \cdot \sum_{i=1}^{n} (F_iF_{i+2})^s}{(F_nF_{n+2})^{\frac{s}{2}} \sum_{k=1}^{n} (F_kF_{k+2})^{\frac{r}{2}}} \leq F_{n+1}F_{n+2} - \frac{1 + (-1)^n}{2} \leq \left(\sum_{i=1}^{n} (F_iF_{i+2})^{\beta}\right)^{1/\beta}.$$

If, in addition, $r = s = 1, \beta = \frac{1}{2}$, the upper inequality becomes:

$$\frac{\left(F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2}\right)^2}{\sqrt{F_nF_{n+2}} \sum_{k=1}^{n} \sqrt{F_kF_{k+2}}} \leq F_{n+1}F_{n+2} - \frac{1 + (-1)^n}{2} \leq \left(\sum_{i=1}^{n} \sqrt{F_iF_{i+2}}\right)^2.$$

Example 10. We consider the sequence defined by

$$x_i = F_i \cdot F_{i+2}, \quad i = 1, 2, \ldots, n,$$

and

$$\alpha = 1, \quad \beta \geq 1.$$

From (4) we have

$$\frac{\sum_{i=1}^{n}(F_iF_{i+2})^r \cdot \sum_{i=1}^{n}(F_iF_{i+2})^s}{(F_nF_{n+2})^{\frac{s}{2}} \sum_{k=1}^{n}(F_kF_{k+2})^{\frac{r}{2}}} \leq F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2}$$
$$\leq n \cdot \left(\frac{1}{n}\sum_{i=1}^{n}(F_iF_{i+2})^{\beta}\right)^{1/\beta} \leq \sum_{i=1}^{n}(F_iF_{i+2})^{\beta}.$$

Specifically, for $r = s = 1$ and $\beta = 2$ we have

$$\frac{\left(F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2}\right)^2}{\sqrt{F_nF_{n+2}}\sum_{k=1}^{n}\sqrt{F_kF_{k+2}}} \leq F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2}$$
$$\leq n \cdot \sqrt{\frac{1}{n}\sum_{i=1}^{n}(F_iF_{i+2})^2} \leq \sum_{i=1}^{n}(F_iF_{i+2})^2.$$

Example 11. For $x_i = \binom{n}{i}$ and $y_i = F_i$, $i = 0, 1, \ldots, n$ we have by Lemma 1 (g) that $\sum_{i=0}^{n} x_iy_i = F_{2n}$. By the Vandermonde identity

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}$$

and Inequality (2) we have

$$\frac{\binom{2n}{n}F_nF_{n+1}}{\max\{2^n, F_n(F_{n+2}-2)\}} \leq F_{2n}.$$

Example 12. Let us recall the Lucas numbers defined by $L_0 = 2$ and $L_n = F_{n+1} + F_{n-1}$, for $n \geq 1$. From [6] we have following identities:

$$\sum_{i=0}^{n}\binom{n}{i}F_iF_{n-i} = \frac{1}{5}\left(2^nL_n - 2\right)$$

and

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}F_i^2 = 5^nF_{2n+1}.$$

If we put $x_i = F_i\sqrt{\binom{2n+1}{i}}$ and $y_i = F_{2n+1-i}\sqrt{\binom{2n+1}{2n+1-i}}$, for $i = 0, \ldots, 2n+1$, then we have

$$S^{[2]}_{2n+1}(x) = S^{[2]}_{2n+1}(y) = 5^n F_{2n+1}$$

and

$$\sum_{i=0}^{2n+1} x_i y_i = \frac{1}{5}\left(2^{2n+1}L_{2n+1} - 2\right).$$

By applying Inequality (2) we get

$$\frac{5^{2n}F^2_{2n+1}}{F_{2n+1}\sum_{i=0}^{2n+1}\sqrt{\binom{2n+1}{i}}F_i} \le \frac{1}{5}\left(2^{2n+1}L_{2n+1} - 2\right).$$

4. Weighted Version

The aim of this section is to establish a weighted analogue of the main results in this paper, specifically Theorem 2 and Theorem 3. We generalize the setting by introducing a sequence of positive weights $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ and derive inequalities involving weighted power sums of sequences satisfying monotonicity conditions. These results naturally extend the non-weighted framework and are then applied to the Fibonacci sequence to obtain refined inequalities involving weighted sums of Fibonacci powers.

Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be positive real numbers, and let $w_1, w_2, \ldots, w_n$ be positive weights. Then the weighted version of the converse Cauchy inequality holds:

$$\frac{\left(\sum_{i=1}^n w_i a_i^2\right)\left(\sum_{i=1}^n w_i b_i^2\right)}{\max\left\{a_n \sum_{i=1}^n w_i b_i,\, b_n \sum_{i=1}^n w_i a_i\right\}} \le \sum_{i=1}^n w_i a_i b_i. \tag{5}$$

Equality holds if and only if $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$.

We define the weighted power sum for a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ and positive weights $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n_+$ as:

$$S^{[\alpha]}_n(x; w) := \sum_{k=1}^n w_k x_k^\alpha.$$

Throughout this section we use the notation

$$W = \sum_{k=1}^n w_k.$$

Theorem 4. *Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ such that $0 < x_1 \le x_2 \le \cdots \le x_n$, and let $w_k > 0$ for all k. Define $\alpha = \frac{r+s}{2}$ for $r, s > 0$.*

(i) If $\alpha \geq \beta > 0$, then

$$\frac{S_n^{[r]}(x;w) \cdot S_n^{[s]}(x;w)}{\max\left\{x_n^{r/2} S_n^{[s/2]}(x;w),\ x_n^{s/2} S_n^{[r/2]}(x;w)\right\}} \leq S_n^{[\alpha]}(x;w). \tag{6}$$

(ii) If $0 < \alpha < \beta$, then

$$\begin{aligned}\frac{S_n^{[r]}(x;w) \cdot S_n^{[s]}(x;w)}{\max\left\{x_n^{r/2} S_n^{[s/2]}(x;w), x_n^{s/2} S_n^{[r/2]}(x;w)\right\}} &\leq S_n^{[\alpha]}(x;w) \\ &\leq W \cdot \left(\frac{1}{W} \cdot S_n^{[\beta]}(x;w)\right)^{\alpha/\beta}. \end{aligned} \tag{7}$$

If additionally $x_1 \geq 1$, then

$$S_n^{[\alpha]}(x;w) \leq W \cdot \left(\frac{1}{W} \cdot S_n^{[\beta]}(x;w)\right)^{\alpha/\beta} \leq S_n^{[\beta]}(x;w). \tag{8}$$

Proof. For the part (i) we apply Inequality (5) for nondecreasing sequences $a_i = x_i^{r/2}$ and $b_i = x_i^{s/2}$. Then we get

$$\sum_{i=1}^n w_i a_i b_i = \sum_{i=1}^n w_i x_i^{(r+s)/2} = S_n^{[\alpha]}(x;w),$$

$$\sum_{i=1}^n w_i a_i^2 = \sum_{i=1}^n w_i x_i^r = S_n^{[r]}(x;w), \qquad \sum_{i=1}^n w_i b_i^2 = \sum_{i=1}^n w_i x_i^s = S_n^{[s]}(x;w),$$

and

$$a_n \sum_{i=1}^n w_i b_i = x_n^{r/2} S_n^{[s/2]}(x;w), \qquad b_n \sum_{i=1}^n w_i a_i = x_n^{s/2} S_n^{[r/2]}(x;w).$$

Substituting these into (5) gives (6).

Now, let us prove (ii). The left inequality in (7) follows exactly as in (i). It remains to prove the upper bound in (7). Let $0 < \alpha < \beta$ and set

$$t := \frac{\alpha}{\beta} \in (0,1), \qquad y_k := x_k^\beta > 0.$$

Then $x_k^\alpha = (x_k^\beta)^{\alpha/\beta} = y_k^t$ and hence

$$S_n^{[\alpha]}(x;w) = \sum_{k=1}^n w_k y_k^t, \qquad S_n^{[\beta]}(x;w) = \sum_{k=1}^n w_k y_k.$$

Since $t \in (0,1)$, the function $\varphi(u) = u^t$ is concave on $(0,\infty)$, and thus we apply the weighted Jensen inequality

$$\varphi\left(\frac{1}{W}\sum_{k=1}^{n} w_k y_k\right) \geq \frac{1}{W}\sum_{k=1}^{n} w_k \varphi(y_k), \qquad W = \sum_{k=1}^{n} w_k.$$

We get

$$\left(\frac{1}{W}\sum_{k=1}^{n} w_k y_k\right)^t \geq \frac{1}{W}\sum_{k=1}^{n} w_k y_k^t.$$

Multiplying by W and returning to x_k gives

$$S_n^{[\alpha]}(x;w) \leq W\left(\frac{1}{W} S_n^{[\beta]}(x;w)\right)^{\alpha/\beta},$$

which is the right inequality in (7).

Finally, assume $x_1 \geq 1$. Since $x_k \geq 1$ for all $k = 1,\dots,n$, we have

$$\frac{1}{W} S_n^{[\beta]}(x;w) = \frac{1}{W}\sum_{k=1}^{n} w_k x_k^{\beta} \geq 1.$$

For $u \geq 1$ and $t \in (0,1)$ we have $u^t \leq u$, hence

$$W\left(\frac{1}{W} S_n^{[\beta]}(x;w)\right)^{\alpha/\beta} \leq W\left(\frac{1}{W} S_n^{[\beta]}(x;w)\right) = S_n^{[\beta]}(x;w),$$

which proves (8). □

In the sequel, we denote by

$$F = (F_1, F_2, \dots, F_n)$$

the vector of the first n Fibonacci numbers.

Theorem 5. *Let F_k be the k^{th} Fibonacci number and $w_k > 0$ for $k = 1,\dots,n$. Let $r, s \in \mathbb{R}_+$ and define $\alpha = \frac{r+s}{2}$. Then, for $n \geq 1$:*

(i) If $\alpha \geq 2$, then
$$\frac{S_n^{[r]}(F;w) \cdot S_n^{[s]}(F;w)}{F_n^{s/2} \cdot S_n^{[r/2]}(F;w)} \leq S_n^{[\alpha]}(F;w).$$

(ii) If $0 < \alpha < 2$, then
$$\frac{S_n^{[r]}(F;w) \cdot S_n^{[s]}(F;w)}{F_n^{s/2} \cdot S_n^{[r/2]}(F;w)} \leq S_n^{[\alpha]}(F;w) \leq W \cdot \left(\frac{1}{W} \cdot S_n^{[2]}(F;w)\right)^{\alpha/2} \leq S_n^{[2]}(F;w).$$

Proof. Apply Theorem 4 with the nondecreasing sequence $x_k = F_k$, $\alpha = \frac{r+s}{2}$, and $\beta = 2$. Arguing as in the proof of Theorem 3, we may assume without loss of generality that $r \le s$, which implies that the maximum term in the denominator reduces to $F_n^{s/2} S_n^{[r/2]}(F; w)$. The upper bounds follow directly from the Theorem 4, taking into account that $x_1 = F_1 = 1 \ge 1$. $\square$

Remark 2. The weighted extensions established in this section represent an improvement over the unweighted framework considered in [2], [3], and [5]. The introduction of arbitrary positive weights leads to families of inequalities that do not follow from any known unweighted result and allow finer control of power sums in applications involving polynomial, exponential, or combinatorial weight systems.

5. Examples for Weighted Case

The following examples illustrate how different choices of weight sequences lead to refined and flexible inequalities for weighted Fibonacci power sums, demonstrating the applicability of the weighted theory developed in this section.

Example 13. Let $w_k = k$, $k = 1, \ldots, n$. Then

$$S_n^{[\alpha]}(F; w) = \sum_{k=1}^{n} k F_k^{\alpha}.$$

For $r = s = 1$ and $\beta = 2$, Theorem 4 gives

$$\frac{\left(\sum_{k=1}^{n} kF_k\right)^2}{F_n^{1/2} \sum_{k=1}^{n} kF_k^{1/2}} \le \sum_{k=1}^{n} kF_k \le \left(W \sum_{k=1}^{n} kF_k^2 \right)^{1/2}.$$

This provides sharp weighted bounds for arithmetically weighted Fibonacci sums.

Example 14. Let $w_k = q^k$ with $0 < q < 1$. Then

$$S_n^{[\alpha]}(F; w) = \sum_{k=1}^{n} q^k F_k^{\alpha}.$$

For $r = s = 1$ and $\beta = 2$, we obtain

$$\frac{\left(\sum_{k=1}^{n} q^k F_k\right)^2}{F_n^{1/2} \sum_{k=1}^{n} q^k F_k^{1/2}} \le \sum_{k=1}^{n} q^k F_k \le \left(W \sum_{k=1}^{n} q^k F_k^2 \right)^{1/2}.$$

This inequality gives exponential-weighted bounds for Fibonacci power sums.

Example 15. Let $w_k = \binom{n}{k}$, $k = 1, \ldots, n$. Then

$$S_n^{[\alpha]}(F; w) = \sum_{k=1}^{n} \binom{n}{k} F_k^{\alpha}.$$

For $r = s = 1$ and $\beta = 2$ we have

$$\frac{\left(\sum_{k=1}^{n} \binom{n}{k} F_k\right)^2}{F_n^{1/2} \sum_{k=1}^{n} \binom{n}{k} F_k^{1/2}} \le \sum_{k=1}^{n} \binom{n}{k} F_k \le \left(W \sum_{k=1}^{n} \binom{n}{k} F_k^2 \right)^{1/2}.$$

This yields combinatorial weighted inequalities for Fibonacci numbers.

Example 16. Let $w_k = k^p$ with $p > 0$. Then

$$S_n^{[\alpha]}(F; w) = \sum_{k=1}^{n} k^p F_k^{\alpha}.$$

For $r = s = 1$ and $\beta = 2$,

$$\frac{\left(\sum_{k=1}^{n} k^p F_k\right)^2}{F_n^{1/2} \sum_{k=1}^{n} k^p F_k^{1/2}} \le \sum_{k=1}^{n} k^p F_k \le \left(W \sum_{k=1}^{n} k^p F_k^2 \right)^{1/2}.$$

This produces a family of polynomially weighted Fibonacci inequalities.

Example 17. Let $w_k = F_k$. Then

$$S_n^{[\alpha]}(F; w) = \sum_{k=1}^{n} F_k^{\alpha+1}.$$

For $r = s = 1$ and $\beta = 2$,

$$\frac{\left(\sum_{k=1}^{n} F_k^2\right)^2}{F_n^{1/2} \sum_{k=1}^{n} F_k^{3/2}} \le \sum_{k=1}^{n} F_k^2 \le \left(W \sum_{k=1}^{n} F_k^3 \right)^{1/2}.$$

This inequality connects different Fibonacci power sums in a weighted form.

These examples show that the weighted framework developed in this section yields a broad family of new inequalities for Fibonacci numbers, ranging from polynomial and exponential weights to combinatorial and Fibonacci-type weights. In particular, many classical unweighted inequalities appear as special cases, while appropriate choices of weights produce refined estimates that are well-suited for applications in discrete analysis and number theory.

6. Conclusion

We established a family of power sum inequalities for monotone sequences based on converse Cauchy inequality and its weighted extension. These results yield explicit and sharp bounds for weighted and unweighted power sums.

Applying the general theory to Fibonacci numbers and related linear recurrence sequences, we obtained new inequalities for sums of powers that unify and extend several known results in the literature. The weighted framework further produces refined estimates for a wide class of natural weight systems, including polynomial, exponential, and combinatorial weights. These results suggest possible extensions to $q-$Fibonacci sequences and higher-order linear recurrences, which we plan to investigate in a subsequent work.

Because the method is elementary and relies only on monotonicity and discrete inequality techniques, it provides a flexible tool for studying power sums of recurrence sequences and related objects in number theory and combinatorics.

References

[1] H. Alzer, On a converse Cauchy inequality of D. Zagier,*Archiv der Mathematik*, 58 (1992), 157–159.

[2] H. Alzer and F. Luca, An inequality for the Fibonacci numbers, *Math. Bohem.* 147 (2022), 587–590.

[3] H. Alzer and M. K. Kwong, Extension of an inequality for Fibonacci numbers, *Integers* 22 (2022), #A85, 8 pp.

[4] A. Dujella, J. Jakšetic, and J. Pečarić, Fibonacci numbers and Hölder inequality, *Bull. Transilv. Univ. Brasov Ser. III*, to appear.

[5] A. Dujella, J. Jakšetić, and J. Pečarić, Fibonacci power means, *Fibonacci Quart.* 63 (2025), 677–684.

[6] R. P. Grimaldi, *Fibonacci and Catalan Numbers*, John Wiley & Sons, Inc., Hoboken, NJ, 2012.

[7] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities,* 2nd ed. Cambridge Univ. Press, 1934.

[8] J. Pečarić, A weighted, discrete version of Zagier's inequality, *Southeast Asian Bull. Math* 4 (1997), 431–433.

[9] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in science and engineering, **187** Academic Press, 1992.

[10] P. G. Popescu and J.L. Diaz-Barrero, Certain inequalities for convex functions, *J. Inequal. Pure Appl. Math.* 7(2) (2006), #41, 5 pp.

[11] O. Shisha, *Inequalities I*, New York-London, 1967, 293–308.

[12] D. Zagier, Een ongelijkheid tegengesteld aan die van Cauchy, *Infdag. Math* 39 (1977), 349–351.