ON ABSOLUTE LUCAS PSEUDOPRIMES

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Received: 7/16/25, Revised: 4/20/26, Accepted: 7/23/26, Published: 8/24/26

Abstract

A Carmichael number is *primary* if the sum of its base-p digits is equal to p for all its prime factors p. Using the work of Chernik on universal forms, i.e., polynomials that yield Carmichael numbers whenever their linear factors are all primes, Wagstaff showed that the prime k-tuples conjecture implies the existence of infinitely many primary Carmichael numbers. Williams found analogs of Carmichael numbers in the context of Lucas sequences that share the same discriminant D. These are called *absolute Lucas D-pseudoprimes*. We define the notions of a *primary* absolute Lucas D-pseudoprime and of their *signature*, develop Chernik-like universal forms and prove that the prime k-tuples conjecture implies the existence of infinitely many primary three-prime-factor absolute Lucas 5-pseudoprimes for all eight possible signatures. Moreover, definitions and theorems about "highly" Carmichael numbers are given, i.e., numbers that are absolute Lucas D-pseudoprimes in at least two essentially distinct ways.

1. Introduction

Fermat discovered that, for a prime number p,

$$a^{p-1} \equiv 1 \pmod{p}, \tag{1}$$

for all integers a not divisible by p. It is often stated in the equivalent way as $a^p \equiv a \pmod{p}$ for all integers a.

A *pseudoprime* to the base a is a positive composite integer n that acts as a prime with respect to the Fermat congruence (1). It satisfies $a^{n-1} \equiv 1 \pmod{n}$. Hardy and Wright [6, Theorem 89, p. 72] gave a simple elementary proof of the existence of infinitely many pseudoprimes to a given base a, $a \geq 2$. For $a = 2$, the least pseudoprime is well-known to be $341 = 11 \cdot 31$.

DOI: 10.5281/zenodo.22081900

A *Carmichael number* n is a positive composite number that satisfies the Fermat congruence $a^{n-1} \equiv 1 \pmod{n}$ for all integers a prime to n. It can then be proved that $a^n \equiv a \pmod{n}$ for all bases a. They are sometimes referred to as *absolute pseudoprimes*. Before Carmichael [4] discovered such numbers existed ($561 = 3 \cdot 11 \cdot 17$ being the smallest one), Korselt [10] had given a simple necessary and sufficient criterion for n to be an absolute pseudoprime, namely

$$n \text{ is squarefree, and } p - 1 \mid n - 1, \tag{2}$$

for all prime factors p of n. This is called the *Korselt criterion*.

Chernick [5] showed how to generate polynomials $P(x)$ such that if each linear factor of $P(x)$ is a prime number for $x = t$, then $P(t)$ is a Carmichael number. The best known example is $P(x) = (6x + 1)(12x + 1)(18x + 1)$. For instance, $P(1)$ is $1729 = 7 \cdot 13 \cdot 19$, the third smallest Carmichael number after 561 and 1105.

Chernik's paper certainly grounded the belief there should exist an infinitude of Carmichael numbers, a result proved in 1994 [1]. We recommend the pleasant and instructive survey of Pomerance [12] that tells the various historical developments on Carmichael numbers up until the milestone proof of their infinitude.

A *fundamental Lucas sequence* $U = U(P, Q)$ with parameters P and Q is a second order linear recurring sequence that satisfies

$$U_{n+2} = PU_{n+1} - QU_n,$$

for all $n \geq 0$, with $U_0 = 0$ and $U_1 = 1$. The parameters P and Q are nonzero integers. The congruence

$$U_{p-\epsilon_p} \equiv 0 \pmod{p} \tag{3}$$

holds for all primes p, $p \nmid Q$, where ϵ_p is the Legendre character $(D \mid p)$ and $D = P^2 - 4Q$. (The Fermat congruence (1) corresponds to the case $U = U(a+1, a)$.)

For another example of (3), the Lucas sequence $U(1, -1)$, better known as the Fibonacci sequence and denoted $F = (F_n)_{n \geq 0}$, satisfies $7 \mid F_8 = 21$, or $11 \mid F_{10} = 55$.

A *U-Lucas pseudoprime* is an odd composite integer n, prime to QD, such that $U_{n-\epsilon_n} \equiv 0 \pmod{n}$, where ϵ_n is the Jacobi symbol $(D \mid n)$. Emma Lehmer wrote a short note [11], where she proves, mimicking the above-mentioned proof of Hardy and Wright, that there are infinitely many F-Lucas pseudoprimes. The least F-pseudoprime in Lehmer's infinite list is the integer $n = F_{14} = 377 = 13 \cdot 29$.

Williams [17] pushed this idea further and considered all Lucas sequences $U(P, Q)$, $\gcd(P, Q) = 1$, which share the same discriminant D. An *absolute Lucas D-pseudoprime*, or a *D-Carmichael* number as we conveniently call them here, is an odd composite integer n, prime to QD, that satisfies

$$U_{n-\epsilon_n}(P, Q) \equiv 0 \pmod{n}, \tag{4}$$

for all choices of coprime P and Q with $P^2 - 4Q = D$.[1]

[1] The restriction that P and Q be coprime could probably harmlessly be removed

For instance, $n = 377$ is not a 5-Carmichael number. We have $\epsilon_n = (5 \mid 13 \cdot 29) = (13 \mid 5) \cdot (29 \mid 5) = (-1) \cdot 1 = -1$ so we need to compute $U_{378} \pmod{377}$. It is a U-Lucas pseudoprime with respect to $U(1,-1)$, $U(3,1)$, and $U(5,5)$, for instance. But if $U = U(7,11)$, then $U_{378} \equiv -13 \not\equiv 0 \pmod{377}$.

Williams [17, Theorem 4] showed that an absolute Lucas D-pseudoprime satisfies a Korselt-like criterion.

Theorem 1 ([17])**.** *An odd positive integer n is a D-Carmichael number if, and only if,*

$$n \text{ is squarefree, and } p - \epsilon_p \text{ divides } n - \epsilon_n, \tag{5}$$

for all prime factors p of n.

Thus, in the previous example, $13+1$ divides $377+1$, but $29-1$ does not divide 378. Hence, with criterion (5) and the prime factorization of 377, we do not need to find P and Q with $D = 5$ for which $n = 377$ fails to satisfy (4), to see that 377 is not an absolute Lucas 5-pseudoprime.

In [9], Kellner and Sondow introduced the notion of a *primary* Carmichael number.

Definition 1. A Carmichael number n is primary if the sum, $s_p(n)$, of its p-ary digits is equal to p, for all the prime factors p of n.

The least two Carmichael numbers 561 and 1105 are not primary. Indeed, $561 = 1\cdot 3+2\cdot 3^2+2\cdot 3^3+2\cdot 3^5$ so that $s_3(561) = 7 \neq 3$ and $1105 = 1\cdot 5+4\cdot 5^2+3\cdot 5^3+1\cdot 5^4$ so that $s_5(1105) = 9 \neq 5$. However, 1729 is primary and, thus, the least primary Carmichael number. The condition $s_p(n) = p$ may seem to be strong. But an integer $n \geq 2$ is a Carmichael number if, and only if, n is squarefree and $s_p(n) \equiv 1 \pmod{p-1}$, for all primes p that divide n. (Clearly $n \equiv s_p(n) \pmod{p-1}$ for all integers $n > 0$. Thus, $n \equiv 1 \pmod{p-1}$ iff $s_p(n) \equiv 1 \pmod{p-1}$.) Since a Carmichael number n is composite, the least value $s_p(n)$ may take is p.

We state next the prime k-tuples conjecture, proposed in 1923 by Hardy and Littlewood.

Conjecture 1. (The prime k-tuples conjecture) Suppose $P(x)$ is a polynomial of the form $\prod_{i=1}^{r}(a_i x+b_i)$, where $a_i \geq 1$ and b_i are integers. This conjecture says that if $P(x)$ modulo p is never identically zero for any prime p, then there are infinitely many positive integers n such that $a_i n + b_i$ is a prime number for all $i = 1, 2, \ldots, r$.

A stronger conjecture, the *H-hypothesis*, which we also refer to as the *generalized k-tuples conjecture*, which, if true, has been shown to have many consequences, was considered in [13].

Conjecture 2. (The H-hypothesis) If $P(x) = \prod_{i=1}^{r} Q_i(x)$, where the $Q_i(x)$'s are integer-coefficient, irreducible polynomials over $\mathbb{Z}[x]$, with positive dominant coefficients, is never identically zero for any prime p, then the $Q_i(n)$, $1 \leq i \leq r$, are all prime numbers for infinitely many positive integers n.

There is a quantitative version, due to Bateman and Horn [3], of the generalized k-tuples conjecture. We will use the version given in [14, p. 120].

Conjecture 3. (Quantitative version of the H-hypothesis) Let $P(x) = \prod_{i=1}^{r} Q_i(x)$, where the $Q_i(x)$'s are integer-coefficient, irreducible polynomials over $\mathbb{Z}[x]$, and $\rho(p)$ denotes the number of zeros of $P(x)$ modulo the prime p. The number $\pi_P(x)$ of positive integers $n \leq x$ such that $|Q_i(n)|$ is prime for all i, $1 \leq i \leq r$, is asymptotically equivalent to

$$C_P\, x \prod_{i=1}^{r} \frac{1}{\log |Q_i(x)|}, \text{ as } x \to \infty,$$

where $C_P := \prod_p (1 - \rho(p)/p)(1 - 1/p)^{-r}$. (The product defining C_P is known to converge.)

As already mentioned, Chernik [5] constructed polynomials $P(x)$ with $r \geq 3$, called *universal forms*, with the property that whenever each factor $a_i n + b_i$ is a prime number, then $P(n)$ is a Carmichael number.

Using Chernik's universal forms, Wagstaff [15, Theorem 1] shows, among other results, the following theorem.

Theorem 2 ([15])**.** *The prime k-tuples conjecture implies the existence of infinitely many primary Carmichael numbers with three prime factors.*

Wagstaff [15] also defined the *degree* of a Carmichael number.

Definition 2. Suppose n is a Carmichael number. Then, for each prime factor p of n, there is a $d_p \geq 1$ such that $s_p(n) = 1 + d_p(p-1)$. The *degree* d of the Carmichael number n is the maximum of the integers d_p, over all prime factors p of n. Primary Carmichael numbers correspond to degree one. If $d = 2$, then n is said to be *secondary*.

One of the intentions of this paper is to establish an analogue of Theorem 2 for 5-Carmichael numbers. To this end, we need to introduce new definitions.

Given an integer D, $D \equiv 0$ or $1 \pmod 4$, we define the *signature* σ of a squarefree integer $n = p_1 p_2 \cdots p_r$, $r \geq 2$, where $p_1 < p_2 < \cdots < p_r$ are prime numbers, as the r-uple $(\epsilon_{p_1}, \epsilon_{p_2}, \ldots, \epsilon_{p_r})$, where $\epsilon_{p_i} = (D \,|\, p_i)$. The *length* of σ is the integer r, and the *sign* of σ is the product $\epsilon_\sigma = \prod_{i=1}^{r} \epsilon_{p_i}$. If n is D-Carmichael, then the signature of n is an r-uple of ± 1's. Its sign is ϵ_n. However, we often write signatures as a string of $+$ and $-$ symbols, where $+$ stands for 1 and $-$ for -1.

To define the degree of an absolute Lucas D-pseudoprime n, we first need to define an appropriate extended sum-of-digit function. Suppose $\sum_{i=1}^{r} c_i p^i$ is the p-ary expansion of n. Then the digit-function s_p is defined as

$$s_p(n) := \sum_{i=1}^{r} c_i \epsilon_p^i, \tag{6}$$

where $\epsilon_p = (D \mid p)$. Note that this function is an extension of the usual sum-of-digit function when applied to Carmichael numbers. Indeed, Carmichael numbers may be viewed as absolute Lucas 1-pseudoprimes, and for a 1-Carmichael number, all ϵ_p and ϵ_n are equal to 1.

We define an extended *degree* notion for an absolute Lucas D-pseudoprime n. Note that $n \equiv s_p(n) \pmod{p - \epsilon_p}$ and that, by (5), $n \equiv \epsilon_n \pmod{p - \epsilon_p}$.

Definition 3. Suppose n is a D-Carmichael number. Then, for each prime factor p of n, there is an integer d_p such that $s_p(n) = \epsilon_n + d_p(p - \epsilon_p)$. The *degree* d of the D-Carmichael number n is the maximum of the $|d_p|$'s, over all prime factors p of n. If $d \leq 1$, then n is said to be a *primary* D-Carmichael number. If $d = 2$, then n is said to be *secondary*.

For practical purposes, it is worth noting that a D-Carmichael number n is primary if and only if $|s_p(n)| \leq p + 2$, for all prime factors p of n.

For example, consider $m = 6601 = 7 \cdot 23 \cdot 41$, the least of all 5-Carmichael numbers with three or more prime factors. Its signature is $(-1, -1, 1)$. We have

$$6601 = 5 \cdot 7 + 7^2 + 5 \cdot 7^3 + 2 \cdot 7^4 = 11 \cdot 23 + 12 \cdot 23^2 = 38 \cdot 41 + 3 \cdot 41^2.$$

Thus, $s_7(m) = -7$, $s_{23}(m) = 1$ and $s_{41}(m) = 41$ so that m is a primary 5-Carmichael number. The number $n = m \cdot 17 \cdot 353$ is a 5-Carmichael number of signature $(-1, -1, -1, 1, -1)$. It is not primary since $n = 37 \cdot 41 + 30 \cdot 41^2 + 14 \cdot 41^4$ and, for $p = 41$, we find that $s_p(n) = 37 + 30 + 14 = 81 = 2p - 1 = \epsilon_n + 2(p - \epsilon_p)$. One can check that $|s_p(n)| \leq p + 2$ for all primes p, $p \mid n$, $p \neq 41$. Thus, n is a secondary 5-Carmichael number.

If $r = 2$, then there are four possible signatures. However, Williams [17, p. 139-40] had proved that

Proposition 1 ([17])**.** *The two-prime-factor number p_1p_2, $p_1 < p_2$, is D-Carmichael if, and only if, $p_2 = p_1 + 2$, $\epsilon_{p_1} = -1$, and $\epsilon_{p_2} = 1$.*

This implies the only length-two signature that can occur for D-Carmichael numbers is $(-1, 1)$.

Define a *universal D-form* with respect to a signature σ as a product $P(x)$ of irreducible polynomials in $\mathbb{Z}[x]$, where $P(x)$ is not identically zero modulo any prime, such that if all irreducible factors of $P(x)$ take on a prime value for some $n \geq 0$, then $P(n)$ is a D-Carmichael number with signature σ.

Several authors have computed data and made observations on primary Carmichael numbers which we state here. This paper includes, for comparison, corresponding data for primary D-Carmichael numbers.

Wagstaff [15, Table 1] computed the proportion of primary Carmichael numbers among all Carmichael numbers below 10^ℓ, $3 \leq \ell \leq 18$. Perhaps unsurprisingly this proportion is initially sizeable. For $\ell = 4$, their proportion is 28.57 percent, and

decreases with ℓ down to 2.29 percent when $\ell = 18$. Carmichael numbers must have at least three prime factors, and among the 35,586 Carmichael numbers less than 10^{18} that have three prime factors, 87.4 percent are primary ([15, Table 2]). In fact, Kellner [8] suggested the proportion of primary among 3-factor Carmichael numbers may tend to 1 as $\ell \to \infty$. There are 1,401,644 Carmichael numbers less than 10^{18}. Wagstaff observes that no primary Carmichael number less than 10^{18} has more than five prime factors.

The paper is made up of three sections besides the introduction. Section 2 is concerned with 2- and 3-factor D-Carmichael numbers. Given a D-Carmichael number n of signature σ, $n = pq$, or $n = pqr$, where $p < q < r$ are primes, which we see as a seed number, we seek to construct universal D-forms, which, by the prime k-tuples conjecture "generate" infinitely many D-Carmichael numbers all of signature σ, with n being the smallest of them. We give further theorems for $D = 5$, where we find universal D-forms that generate only primary 5-Carmichael numbers, and infinitely many if we assume the prime k-tuples conjecture. Theorem 4 concerns 2-factor D-Carmichael numbers (all of them are necessarily primary), Theorem 5 exhibits universal D-forms for the 3-factor case assuming $p > 3$, and Theorem 6, together with Table 1, is an analogue of Wagstaff's Theorem 2, expliciting eight universal 5-forms, one for all eight possible signatures. Also, there are exactly six D-Carmichael numbers of the shape $3qr$ as shown in Theorem 7.

Note that 5-Carmichael numbers of signature $+++$ are ordinary Carmichael numbers. Thus, the prime k-tuples conjecture implies there are infinitely many positive integers n, with $\omega(n) = 3$, which are simultaneously Carmichael and 5-Carmichael.

Section 2 also contains a number of tables, data, and associated remarks. The distribution of k-factor 5-Carmichael numbers according to signature, for $k = 3$ and 4, is given in Tables 2 and 3. Additional data for their distribution when $k = 5$ is given in Remark 3. Data on the proportion of primary 5-Carmichael numbers is included. Also, we computed and recorded the distribution of 3-factor D-Carmichael numbers according to signature for $D = 13$, -7, and -11 in Table 4. In all the above, the tabulation of D-Carmichael numbers ($< 2^{64}$) made by Helmreich and Webster [16] for $D = 5$, 13, -7, and -11 was used.

Section 3 is a short section with three theorems. Theorem 8 shows that the polynomial

$$P(x) = (30x + 17)(30x + 19)(180x + 107)(180x + 109),$$

is a universal 5-form such that when the four factors p, q, r, s are primes, then pq is a 5-Carmichael number, pqr and pqs are both primary 5-Carmichael numbers, and $pqrs$ is a secondary 5-Carmichael number. Theorem 9 is a result analogous to Theorem 8, but for $D = 13$, while Theorem 10 exhibits a four-factor universal form with respect to both $D = 5$ and $D = 13$ (with identical signatures for the two

discriminants).

In Section 4, we discuss "highly" Carmichael numbers, i.e., D-Carmichael numbers with respect to distinct signatures; we limit ourselves to length-3 signatures. Theorem 11 shows the H-hypothesis implies there are infinitely many 5-Carmichael numbers of signature $+--$ that are simultaneously (-11)-Carmichael numbers of signature $-+-$. Theorems 12 and 13 are unconditional and show there are no D-Carmichael numbers with respect to two signatures of opposite signs. Many open questions are also pointed out.

We should say, in general, that this paper opens many more questions than it proves theorems.

We end this introduction with a final remark in the form of a relevant definition and a theorem. Theorem 1 suggests the right analogue of an absolute pseudoprime in the context of Lucas sequences is a D-Carmichael number. It appears that signature is the right invariant rather than the discriminant. Indeed, if n is a D-Carmichael number with respect to the signature σ, then n is a Δ-Carmichael number with respect to all Δ's of signature σ. We define a *σ-Carmichael* number.

Definition 4. Let $\sigma = (\epsilon_1, \ldots, \epsilon_r)$, where $\epsilon_i = \pm 1$ for all i, $1 \le i \le r$. A positive integer $n = p_1 p_2 \cdots p_r$, $p_1 < p_2 < \cdots < p_r$, is a σ-Carmichael number if and only if n is a D-Carmichael number, for all discriminants D such that $(D \,|\, p_i) = \epsilon_i$, $1 \le i \le r$.

The following Korselt criterion is then easily established using Theorem 1.

Theorem 3. *Let $\sigma = (\epsilon_1, \ldots, \epsilon_r)$, where $\epsilon_i = \pm 1$, $1 \le i \le r$. An odd positive integer n is a σ-Carmichael number if, and only if,*

$$n \text{ is squarefree, and } p - \epsilon_p \text{ divides } n - \epsilon_\sigma, \tag{7}$$

for all prime factors p of n, where $\epsilon_\sigma = \prod_{i=1}^{r} \epsilon_i$.

2. Determining D-Carmichael Numbers

We begin with a lemma on Legendre symbols.

Lemma 1. *Suppose p is an odd prime and $D \equiv 0$ or $1 \pmod 4$ is an integer prime to p. If $p' = p + Dx$ is an odd prime number for some integer x, then $(D \,|\, p') = (D \,|\, p)$.*

Proof. If $D \equiv 1 \pmod 4$, then $(D \,|\, p') = (D \,|\, p)$ because Jacobi symbols satisfy the usual quadratic reciprocity law. If $D = 2 \cdot 4^k \cdot d$, where d is odd, $k \ge 1$, then $p' \equiv p \pmod 8$ so that $(2 \mid p') = (2 \mid p)$. Denoting $(-1)^{\frac{p'-1}{2} \cdot \frac{d-1}{2}}$ by η, we see that

$\eta = (-1)^{\frac{p-1}{2}\cdot\frac{d-1}{2}}$. Thus,

$$\begin{aligned}(D \mid p') &= (2d \mid p') = (2 \mid p')(d \mid p') = (2 \mid p)\cdot\eta\cdot(p' \mid d)\\ &= (2 \mid p)\cdot\eta\cdot(p \mid d) = (2 \mid p)(d \mid p) = (D \mid p).\end{aligned}$$

Finally, if $D = 4^k \cdot d$, $k \geq 1$, d odd, then $p' \equiv p \pmod 4$ and $p' \equiv p \pmod d$. Hence, $(D \mid p') = (d \mid p') = (d \mid p) = (D \mid p)$. □

Theorem 4. *Suppose p and q are prime numbers such that pq is D-Carmichael for some integer D. Then $P(x) = (p+Dx)(q+Dx)$ is a universal D-form of signature $(-1,1)$, $q = p+2$, and, whenever $p+Dx$ and $q+Dx$ are prime numbers, $P(x)$ is a primary D-Carmichael number. Thus, the prime k-tuples conjecture implies the existence of infinitely many primary D-Carmichael numbers of signature $(-1,1)$, all of them a product of twin primes. The set of two-prime-factor 5-Carmichael numbers is the set of integers of the form $(10x+7)(10x+9)$, where $x \geq 1$ and $10x+7$ and $10x+9$ are prime numbers.*

Proof. By Proposition 1, pq is D-Carmichael implies its signature is $(-1,1)$ and $q = p+2$. Since $P(0) \equiv 1 \pmod 2$ and $\gcd(D, pq) = 1$, we see that $P(x) = (p+Dx)(q+Dx)$ is not identically zero modulo any prime. Suppose $u = p+Dx$ and $v = q+Dx = u+2$ are both primes. By Lemma 1, the signature of uv is $(-1,1)$. Moreover,

$$uv + 1 \equiv -v + 1 = -(v-1) = -(u+1) \equiv 0 \pmod{u+1}.$$

Hence, $P(x)$ is a universal D-form of signature $(-1,1)$. It remains to check that $P(x) = uv$ is primary. Since $P(x) = (v-2)v = u(u+2) = 2u+u^2$, we obtain $s_u(P(x)) = -2+1 = -1$ and $s_v(P(x)) = v-2$.

In the case $D = 5$, pq is 5-Carmichael if, and only if, $(5 \mid p) = -1$, $(5 \mid q) = 1$, and $q = p+2$, since these conditions imply that $p+1 (= q-1)$ divides $pq+1$. Thus, pq is 5-Carmichael if, and only if, $p \equiv 2, 3 \pmod 5$, $q \equiv 1, 4 \pmod 5$, and $q = p+2$, which occurs if, and only if, $p \equiv 2 \pmod 5$ and $q = p+2$. This last condition is equivalent to $p \equiv 7 \pmod{10}$ and $q = p+2$, which yields our claim. □

If $C(x) = \#\{1 \leq n \leq x;\ 10n+7 \text{ and } 10n+9 \text{ are primes}\}$, then we emphasize that the quantitative version of the H-hypothesis (Conj. 3) gives the asymptotic estimate

$$C(x) \sim x \prod_p \frac{1-\frac{\rho(p)}{p}}{(1-\frac{1}{p})^2} \cdot \frac{1}{\log(10x+7)(10x+9)},$$

as $x \to \infty$, where $\rho(p)$ is the number of zeros of $(10x+7)(10x+9)$ modulo p. Since $\rho(2) = \rho(5) = 0$ and $\rho(p) = 2$ for all the other primes, we see that

$$C(x) \sim \frac{20}{3} C_2 \frac{x}{\log^2(10x)}, \tag{8}$$

as $x \to \infty$, where $C_2 \approx 0.66022832\ldots$ is the famous twin prime constant, i.e., $\prod_{p\geq 3} \frac{1-\frac{2}{p}}{(1-\frac{1}{p})^2}$. Calculations using Pari gp gave $C(10^9) = 9\,138\,015$. The right-hand side of (8) with $x = 10^9$, yields $\frac{2\cdot 10^8 \cdot C_2}{3\log^2(10)}$, which rounds up to $8\,301\,786$.

Remark 1. In the case $D = 5$, the universal D-form of Theorem 4 yields all 5-Carmichael numbers with two factors. Indeed, the least pair of primes that satisfies the conditions $p \equiv 7 \pmod{10}$ and $q = p + 2$ is $p = 17$ and $q = 19$. Thus, $P(x) = (17 + 5x)(19 + 5x)$, but for $P(x)$ to be a product of two primes, x must be even. Thus, we may consider instead $(p+10x)(q+10x)$, $x \geq 0$, or $(7+10x)(9+10x)$, $x \geq 1$. However, this is generally not the case. For $D = 13$, there are three possible pairs $(n, r = n + 2)$, where n is a nonresidue of 13 and r a residue, namely $(2, 4)$, $(7, 9)$, and $(8, 10)$. These pairs lead to universal 13-forms that yield distinct sets of primes. They are $(41 + 26x)(43 + 26x)$, $(59 + 26x)(61 + 26x)$, and $(281 + 26x)(283 + 26x)$.

We next prove a Chernik-like theorem for generating D-Carmichael numbers with three prime factors—similar theorems can be proved for D-Carmichael numbers with more prime factors. Given a D-Carmichael number $p_1p_2p_3$ of signature σ, the theorem produces a D-universal form for the signature σ that stems from the number $p_1p_2p_3$, and includes it as its least member. The theorem is preceded by two lemmas.

Lemma 2. *Let $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$ and c_1, c_2, c_3 be positive pairwise coprime integers. Let g be a nonzero integer and consider*

$$n := (c_1g + \epsilon_1)(c_2g + \epsilon_2)(c_3g + \epsilon_3).$$

Then $c_ig \mid n - \epsilon_n$, where $\epsilon_n = \epsilon_1\epsilon_2\epsilon_3$, for all $i = 1, 2, 3$ if, and only if,

$$g(\epsilon_3c_1c_2 + \epsilon_2c_1c_3 + \epsilon_1c_2c_3) + \epsilon_n(\epsilon_1c_1 + \epsilon_2c_2 + \epsilon_3c_3) \equiv 0 \pmod{c_1c_2c_3}. \quad (9)$$

Proof. That c_ig divides $n - \epsilon_n$, $i = 1, 2, 3$, translates into the single congruence

$$(c_1g + \epsilon_1)(c_2g + \epsilon_2)(c_3g + \epsilon_3) \equiv \epsilon_n \pmod{gc_1c_2c_3}.$$

But the above congruence is equivalent to

$$g^2(c_1c_2\epsilon_3 + c_1c_3\epsilon_2 + c_2c_3\epsilon_1) + g(c_1\epsilon_2\epsilon_3 + c_2\epsilon_1\epsilon_3 + c_3\epsilon_1\epsilon_2) \equiv 0 \pmod{gc_1c_2c_3},$$

which gives the claim by dividing through by g, or multiplying through by g for the converse. □

Lemma 3. *Let c_1, c_2, c_3, g, and n be as in Lemma 2. Suppose h is a nonzero integer such that $h \equiv g \pmod{c_1c_2c_3}$. Let $m := (c_1h + \epsilon_1)(c_2h + \epsilon_2)(c_3h + \epsilon_3)$. Then $c_ih \mid m - \epsilon_n$ for $i = 1, 2, 3$.*

Proof. Replace g by h in (9) and use Lemma 2 to obtain $c_i h \mid m - \epsilon_n$ for $i = 1, 2, 3$. □

Theorem 5. *Let $\sigma = (\epsilon_1, \epsilon_2, \epsilon_3)$ be a signature. Suppose $3 < p_1 < p_2 < p_3$ are prime numbers such that $p_1 p_2 p_3$ is a D-Carmichael number with signature σ. Put $g = \gcd(p_1 - \epsilon_1, p_2 - \epsilon_2, p_3 - \epsilon_3)$, $gc_1 = p_1 - \epsilon_1$, $gc_2 = p_2 - \epsilon_2$, $gc_3 = p_3 - \epsilon_3$, and assume the c_i's are pairwise coprime. Then*

$$P(x) = (Dc_1^2 c_2 c_3 x + p_1)(Dc_1 c_2^2 c_3 x + p_2)(Dc_1 c_2 c_3^2 x + p_3)$$

is a universal D-form with respect to the signature σ.

Proof. Put $n = p_1 p_2 p_3$ and $\epsilon_n = \epsilon_1 \epsilon_2 \epsilon_3$. If $6 \mid x$, then $P(x) \equiv 1 \pmod 2$ and $P(x) \equiv \pm 1 \pmod 3$. Thus, if $P(x)$ is identically zero modulo some prime, it must be some p_i, and p_i must divide $c_1 c_2 c_3$. If $p_1 \mid c_1 c_2 c_3$, then $p_1 \mid c_2 c_3$. If $p_1 \mid c_j$, $j = 2$ or 3, then $p_1 \mid p_j - \epsilon_j$. Since $p_j - \epsilon_j$ divides $n - \epsilon_n$, we obtain the contradictory facts $p_1 \mid n - \epsilon_n$ and $p_1 \mid n$. Similar arguments apply for $i = 2$ and 3. Hence, there is no prime modulo which $P(x)$ is identically zero.

Suppose that, for some x, $q_i := c_i D c_1 c_2 c_3 x + p_i$ is a prime number, $i = 1, 2, 3$. Then, by Lemma 1,

$$(D \mid q_i) = (D \mid p_i) = \epsilon_i. \tag{10}$$

Put $h = Dc_1 c_2 c_3 x + g$. Then $q_i = c_i h + (D \mid q_i)$, $i = 1, 2, 3$, and $h \equiv g \pmod{c_1 c_2 c_3}$. By Lemma 3, $q_i - (D \mid q_i)$ divides $P(x) - \epsilon_n$, $i = 1, 2, 3$, so that $P(x)$ is a D-Carmichael number of signature $(\epsilon_1, \epsilon_2, \epsilon_3)$. □

Remark 2. If $\gcd(D, c_1 c_2 c_3) = \ell > 1$ in Theorem 5, then we can reduce the coefficients of $P(x)$ by replacing D with D/ℓ.

Theorem 6. *Given a signature σ of length 3, the prime k-tuples conjecture implies the existence of infinitely many primary 5-Carmichael numbers of signature σ.*

We tag along Theorem 6 a table, Table 1, of polynomials that yield primary 5-Carmichael numbers for all eight signatures.

Proof of Theorem 6. We show that the eight universal 5-forms of Table 1, one for each signature, generate only primary 5-Carmichael numbers whenever the three linear factors of $P(x)$ are prime numbers, for all $x \geq 0$, or all $x \geq 1$ in Case 2. The prime k-tuples conjecture implies there are infinitely many such x's, and therefore infinitely many primary 5-Carmichael numbers with three prime factors of all eight signatures. We proceed case by case.

`Case 1:` $P_1(x) = (30x+1)(60x+1)(90x+1)$; $\sigma = +++$. We know from Chernik's work that the polynomial

$$C(x) = (6x + 1)(12x + 1)(18x + 1)$$

σ	a universal form with resp. to σ	examples	x
$+++$	$(30x+1)(60x+1)(90x+1)$	$211 \cdot 421 \cdot 541$	7
$++-$	$(43890x+199) \times (921690x+4159) \times$ $(18346020x+82763)$	$199 \cdot 4159 \cdot 82763$	0
		$44089 \cdot 925849 \cdot 18428783$	1
		$219649 \cdot 4612609 \cdot 91812863$	5
$+-+$	$(840x+19)(2520x+53)(47040x+1009)$	$19 \cdot 53 \cdot 1009$	0
		$4219 \cdot 12653 \cdot 236209$	5
		$10099 \cdot 30293 \cdot 565489$	12
$+--$	$(5600x+11)(7840x+13)(35840x+47)$	$11 \cdot 13 \cdot 47$	0
		$365411 \cdot 511573 \cdot 1753967$	87
		$504011 \cdot 705613 \cdot 2419247$	120
$-++$	$(30x+17)(30x+19)(180x+109)$	$17 \cdot 19 \cdot 109$	0
		$137 \cdot 139 \cdot 829$	4
		$347 \cdot 349 \cdot 2089$	11
$-+-$	$(15x+2)(15x+4)(45x+8)$	$17 \cdot 19 \cdot 53$	1
		$197 \cdot 199 \cdot 593$	13
$--+$	$(15x+7)(45x+23)(75x+41)$	$7 \cdot 23 \cdot 41$	0
		$37 \cdot 113 \cdot 191$	2
		$97 \cdot 293 \cdot 491$	6
$---$	$(210x+13)(1260x+83)(1470x+97)$	$13 \cdot 83 \cdot 97$	0
		$643 \cdot 3863 \cdot 4507$	3

Table 1: Universal forms yielding primary 5-Carmichael numbers

produces Carmichael numbers whenever the three linear factors are primes. We have $P_1(x) = C(5x)$. The prime k-tuples conjecture predicts $P_1(x)$ to be a product of three primes infinitely often. In this event, $P_1(x)$ is a 5-Carmichael number of signature $+++$ because $(5 \mid 30x+1) = (5 \mid 60x+1) = (5 \mid 90x+1) = 1$. These 5-Carmichael numbers are Carmichael numbers and they were shown to all be primary in [15, Cororallary 1].

The seven remaining cases yield 5-Carmichael numbers of a given signature because they are constructed using Theorem 5. Theorem 5 requires one instance of a 5-Carmichael number of the given signature to apply. In each case, we chose a 5-Carmichael with three prime factors of the given signature from the tabulation made by Webster [16] and constructed a universal form using Theorem 5 after checking that the corresponding c_i's are pairwise coprime. For instance, in Case 5, $35207 = 17 \cdot 19 \cdot 109$, $g = 18$, $c_1 = c_2 = 1$, and $c_3 = 6$. To show $P(x)$ only yields primary 5-Carmichael numbers, we now write $P(x) = N = pqr$, where $p < q < r$ are primes. Then, we find the p-adic expansion of qr, from which we get $s_p(N)$, and do the same for the q-adic and the r-adic expansions of, respectively, pr and pq. Finding some of these expansions took a little guessing, but once we have these

expansions, they are easily verified, especially using some software.

Case 2: $P_2(x) = (43890x+199)(921690x+4159)(18346020x+82763)$; $\sigma = ++-$. Then if $x \geq 1$, we find that

$$\begin{aligned} qr &= 8380 + (p - 17159)p + 8777p^2, \\ pr &= (48070x + 197) + (873620x + 3960)q, \\ pq &= (2205x + 11) + (2205x + 10)r, \end{aligned}$$

so that $s_p(N) = p - 2$, $s_q(N) = q - 2$, $s_r(N) = -1$. For $x = 0$, the seed number $199 \cdot 4159 \cdot 82763$ is a secondary 5-Carmichael number. The above expressions for pr and pq remain valid, but $qr = 22 + 196p + 134p^2 + 43p^3$ and $s_p(N) = 395 = 2p - 3 = -1 + 2(p - \epsilon_p)$.

Case 3: $P_3(x) = (840x + 19)(2520x + 53)(47040x + 1009)$; $\sigma = +-+$. Then

$$\begin{aligned} qr &= 220 + (840x - 370)p + 167p^2, \\ pr &= (560x + 38) + (560x + 43)q + 6q^2, \\ pq &= (46995x + 1007) + (45x)r, \end{aligned}$$

so that $s_p(N) = 220 + (840x - 370) + 167 = p - 2$, $s_q(N) = 0 - (560x + 38) + (560x + 43) - 6 = -1$, and $s_r(N) = 0 + (46995x + 1007) + 45x = r - 2$.

Case 4: $P_4(x) = (4200x + 11)(5880x + 13)(20160x + 47)$; $\sigma = +--$. Then, for all $x \geq 1$, we have

$$\begin{aligned} qr &= (1176x + 17) + (3024x - 12)p + 6p^2, \\ pr &= (2640x + 10) + (2640x + 13)q + 2q^2, \\ pq &= (1225x + 2) + (1225x + 3)r, \end{aligned}$$

so that $s_p(N) = p$, $s_q(N) = 1$, and $s_r(N) = 1$. For $x = 0$, we find that the first two identities above take a different form: $qr = 13 \cdot 47 = 6 + 5 \cdot 11^2$ and $11 \cdot 47 = 10 + 3 \cdot 13^2$ so that $s_p(N) = 6 + 5 = p$ and $s_q(N) = -10 - 3 = -q = 1 + (-1)(q + 1)$. Nevertheless, $11 \cdot 13 \cdot 47$ is a primary 5-Carmichael number.

Case 5: $P_5(x) = (30x + 17)(30x + 19)(180x + 109)$; $\sigma = -++$. Then, for all $x \geq 1$, we find that

$$\begin{aligned} qr &= 14 + 19p + 6p^2, \\ pr &= 10 + (q - 17)q + 5q^2, \\ pq &= (175x + 105) + (5x + 2)r, \end{aligned}$$

so that $s_p(N) = -14 + 19 - 6 = -1$, $s_q(N) = q - 2$, and $s_r(N) = 180x + 107 = r - 2$. For $x = 0$, we find that $qr = 19 \cdot 109 = 14 + 2 \cdot 17 + 7 \cdot 17^2$, so that $s_p(N) = -14 + 2 - 7 = -p - 2$.

Case 6: $P_6(x) = (15x+2)(15x+4)(45x+8)$; $\sigma = --+-$. Then

$$\begin{aligned} qr &= 4 + 8p + 3p^2, \\ pr &= 8 + (q-10)q + 2q^2, \\ pq &= 5x + (5x+1)r, \end{aligned}$$

so that $s_p(N) = -4 + 8 - 3 = 1$, $s_q(N) = q$, and $s_r(N) = 1$.

Case 7: $P_7(x) = (15x+7)(45x+23)(75x+41)$; $\sigma = --+-$. Then

$$\begin{aligned} qr &= 12 + 28p + 15p^2, \\ pr &= (25x+11) + (25x+12)q, \\ pq &= (66x+38) + (9x+3)r, \end{aligned}$$

so that $s_p(N) = -12 + 28 - 15 = 1$, $s_q(N) = 1$, and $s_r(N) = 75x + 41 = r$. If $x = 0$, then $p = 7$ and $12 + 28p + 15p^2$ is not the 7-adic expansion of $23 \cdot 41$. But $23 \cdot 41 = 5 + 7 + 5 \cdot 7^2 + 2 \cdot 7^3$ so that $s_p(N) = -5 + 1 - 5 + 2$ and is still 1.

Case 8: $P_8(x) = (210x+13)(1260x+83)(1470x+97)$; $\sigma = ---$. Then

$$\begin{aligned} qr &= 30 + 71p + 42p^2, \\ pr &= (245x+16) + (245x+15)q, \\ pq &= (180x+12) + (180x+11)r, \end{aligned}$$

so that $s_p(N) = s_q(N) = s_r(N) = -1$. The first identity $qr = 30 + 71p + 42p^2$ is not the p-adic expansion of qr when $x = 0$ because the coefficients 30, 71, and 42 are larger than $p = 13$. Nevertheless, $83 \cdot 97 = 8051 = 4 + 8 \cdot 13 + 3 \cdot 13^2 + 3 \cdot 13^3$ so that $s_{13}(N) = 0 - 4 + 8 - 8 + 3 = -1$. □

Helmreich and Webster [7, Table 1, p. 15] found 86,227 5-Carmichael numbers with three prime factors less than 2^{64}. We calculated their distribution among the eight possible signatures; see Table 2. The rarest signature appears to be $++-$. The least 5-Carmichael number of signature $++-$ is $199 \cdot 4159 \cdot 82763 = 68{,}498{,}052{,}083$. It is the 580th number in the list of 3-factor 5-Carmichael numbers tabulated by Helmreich and Webster [16].

We found that 81,597 of the 86,227 three-factor 5-Carmichael numbers less than 2^{64} are primary, a proportion of 94.63%. If $C_k^D(x)$ denotes the number of D-Carmichael numbers with k prime factors that do not exceed x, and $P_k^D(x)$ the number of primary D-Carmichael numbers no larger than x with k factors, we then observed that the proportion of primary with three factors $P_3^5(x)/C_3^5(x)$ seems to increase as x increases within the range $[1, 2^{64}]$. Recall that for Carmichael numbers Kellner [8, Conjecture 5.3] conjectured that $P_3^1(x)/C_3^1(x)$ tends to 1 as $x \to \infty$.

There are 70,917 four-factor 5-Carmichael numbers less than 2^{64}. We found that 18,858 of them are primary, i.e., 26.59%. Table 3 gives the distribution of these 70,917 numbers according to signature.

signature	$0 < x < 2^{64}$	percentage
$+++$	32,227	37.37%
$++-$	1,146	1.33%
$+-+$	1,823	2.11%
$+--$	7,840	9.09%
$-++$	4,335	5.03%
$-+-$	16,722	19.39%
$--+$	15,197	17.62%
$---$	6,937	8.05%

Table 2: Signature distribution of three-prime-factor 5-Carmichael numbers $< 2^{64}$

signature	$0 < x < 2^{64}$	percentage	signature	$0 < x < 2^{64}$	percentage
$++++$	26,896	37.93%	$-+++$	340	0.48%
$+++-$	189	0.27%	$-++-$	6,421	9.05%
$++-+$	243	0.34%	$-+-+$	7,919	11.17%
$+-++$	301	0.42%	$-+--$	1,433	2.02%
$++--$	3,044	4.29%	$--++$	9,271	13.07%
$+-+-$	3,880	5.47%	$--+-$	1,420	2%
$+--+$	4,599	6.49%	$---+$	1,258	1.77%
$+---$	1,189	1.68%	$----$	2,514	3.54%

Table 3: σ-distribution of 5-Carmichael numbers $< 2^{64}$ with four prime factors

Remark 3. Looking at Tables 2 and 3, there are two obvious observations. The proportion of 5-Carmichael numbers that are Carmichael numbers, i.e., that have respective signatures $+++$ or $++++$, is the largest, and lies between 37 and 38% in either case. The rarest signatures are, respectively, $++-$ and $+++-$. In addition, we mention there are 338,435 five-factor 5-Carmichael numbers $< 2^{64}$; the proportion of primary is down to 6.84%. The most common signature is $+++++$ (24.11%), the least common is $++++-$ with 162 numbers, and the second-least is $+++-+$ with 182 cases. There are still some primary six-factor 5-Carmichael numbers, but they only represent about 1.776% of all six-factor 5-Carmichael numbers $< 2^{64}$.

We complete this section with a table for the discriminants 13, -7, and -11 using the tabulation of D-Carmichael numbers found in [16].

We see that $D = 13$ behaves somewhat like $D = 5$, but the distributions for $D = -7$ and -11 show a different behavior with more uniformity among signatures. However, the proportions of primary $P_3^D(x)/C_3^D(x)$, $x = 2^{64}$, are very comparable to $D = 5$ with respective values 94.63%, 94.62%, and 94.50% for $D = 13$, -7, and -11.

signature	$D = 13$	$D = -7$	$D = -11$
$+++$	16,708	26,575	18,254
$++-$	6,420	28,687	19,557
$+-+$	8,301	28,453	20,306
$+--$	10,740	30,120	21,994
$-++$	11,615	39,097	28,198
$-+-$	14,387	37,071	27,213
$--+$	11,941	29,903	20,189
$---$	9,575	28,567	19,524
$C_3^D(2^{64})$	89,687	248,473	175,235

Table 4: Signature distribution for three-factor D-Carmichael numbers $< 2^{64}$

In Theorem 5, we had to exclude the case when $p_1 = 3$ because the resulting polynomial $P(x)$ can, in some instances, be identically zero modulo 3. We next consider D-Carmichael numbers of the form $3qr$ to complete the case of D-Carmichael numbers with three prime factors.

Lemma 4. *If $3qr$, $3 < q < r$, is a D-Carmichael number of signature (α, β, γ), then*

$$q - \beta \text{ divides } 3r - \alpha\gamma \quad \text{and} \quad r - \gamma \text{ divides } 3q - \alpha\beta.$$

Moreover,

$$\begin{cases} r = 3q + 2\gamma, & \text{if } \gamma = -\alpha\beta; \\ 2r = 3q + \gamma, & \text{if } \gamma = \alpha\beta. \end{cases}$$

Proof. By hypothesis, $q - \beta \mid 3qr - \alpha\beta\gamma$ and $r - \gamma \mid 3qr - \alpha\beta\gamma$. Since $q \equiv \beta \pmod{q-\beta}$ and $r \equiv \gamma \pmod{r-\gamma}$, we see that $q - \beta \mid 3r - \alpha\gamma$ and $r - \gamma \mid 3q - \alpha\beta$. Since $r \geq q + 2$, we have $3(r - \gamma) > 3q - \alpha\beta$. Hence, either $r - \gamma = 3q - \alpha\beta$, or $2r - 2\gamma = 3q - \alpha\beta$. If $\gamma = -\alpha\beta$ and $2r - 2\gamma = 3q - \alpha\beta$, then $2r = 3(q + \gamma)$. But $3 \mid r$ contradicts the primality of r. Therefore, $r = 3q + 2\gamma$. If $\gamma = \alpha\beta$, then $r - \gamma = 3q - \alpha\beta$ leads to $r = 3q$. Hence, $2r = 3q + \gamma$. □

Theorem 7. *The numbers* 105, 195, 255, 399, 561, *and* 2703 *are the only numbers of the form $3qr$, $3 < q < r$ primes, that are D-Carmichael numbers for some D. They can only be so with one signature, which is indicated in Table 5 below.*

Proof. `Case 1:` $\gamma = -\alpha\beta$. By Lemma 4, $r = 3q + 2\gamma$ and $q - \beta$ divides $3r - \alpha\gamma$. Thus, $q - \beta \mid 3(3q + 2\gamma) + \beta = 9q + 6\gamma + \beta$. Subtracting $9(q - \beta)$ from $9q + 6\gamma + \beta$ yields $q - \beta \mid 6\gamma + 10\beta$. If $\alpha = -1$, then $\gamma = \beta$ and $q - \gamma \mid 16\gamma$. If $\gamma = 1$, then $q = 5$ and $r = 3q + 2 = 17$, or $q = 17$ and $r = 53$. If $\gamma = -1$, then $q + 1 \mid 16$. Thus, $q = 7$ and $r = 3q + 2\gamma = 19$. If $\alpha = 1$, then $\gamma = -\beta$. Thus, $q - \beta \mid 6\gamma + 10\beta$ becomes

$3qr$	signature	least $D > 0$	largest $D < 0$
$561 = 3 \cdot 11 \cdot 17$	$+++$	1	-8
$195 = 3 \cdot 5 \cdot 13$	$++-$	109	-11
$105 = 3 \cdot 5 \cdot 7$	$-+-$	41	-4
$255 = 3 \cdot 5 \cdot 17$	$-++$	89	-4
$2703 = 3 \cdot 17 \cdot 53$	$-++$	77	-4
$399 = 3 \cdot 7 \cdot 19$	$---$	41	-4

Table 5: All D-Carmichael numbers of the form $3qr$

$q - \beta \mid 4\beta$. If $\beta = 1$, then $q - 1 \mid 4$ so that $q = 5$ and $r = 13$. If $\beta = -1$, then $q + 1 \mid 4$ and there are no solutions.

`Case 2:` $\gamma = \alpha\beta$. By Lemma 4, we have $q - \beta \mid 3r - \alpha\gamma$ and $2r = 3q + \gamma$. Thus, $2(q - \beta) \mid 3(2r) - 2\alpha\gamma = 9q + \gamma(3 - 2\alpha)$. Subtracting $8(q - \beta)$ and $10(q - \beta)$ from $9q + \gamma(3 - 2\alpha)$, gives respectively that $2(q - \beta)$ divides both $q + \gamma(3 - 2\alpha) + 8\beta$ and $-q + \gamma(3 - 2\alpha) + 10\beta$. Therefore, $2(q - \beta)$ divides $2\gamma(3 - 2\alpha) + 18\beta$, or $q - \beta \mid \gamma(3 - 2\alpha) + 9\beta$. If $\alpha = -1$, then $\beta = -\gamma$. Thus, $q - \beta \mid 4$. The only solution is $\beta = 1$, $q = 5$, and $r = (3q + \gamma)/2 = 7$. If $\alpha = 1$, then $\beta = \gamma$. We get $q - \beta \mid 10$, and the only solution is $q = 11$ for $\beta = 1$ and $r = 17$. □

Remark 4. The last four numbers in Table 5 are (-4)-Carmichael numbers. If $U = U(12, 37)$, then, since no prime factor is 37, we can check that $3 \cdot 5 \mid U_4$, $7 \mid U_8$, $17 \mid U_{16}$, etc. In fact, $U_4 = 8 \cdot 105$ so the first two divisibility relations are checked.

3. Remarkable Consequences

Theorem 8. *The prime k-tuples conjecture implies the existence of infinitely many pairs (m, n) of primary 5-Carmichael numbers of the form*

$$m = pqr = (30x + 17)(30x + 19)(180x + 107),$$
$$n = pqs = (30x + 17)(30x + 19)(180x + 109),$$

where all factors are prime numbers and $x \geq 0$. In all such instances, $N = pqrs$ is a secondary 5-Carmichael number. Thus, the prime k-tuples conjecture implies there are infinitely many four-prime-factor secondary 5-Carmichael numbers.

Proof. Consider the polynomial

$$P(x) = (30x + 17)(30x + 19)(180x + 107)(180x + 109).$$

We have $P(x) \equiv 1 \pmod 6$ for all integers x. Hence, $P(x)$ is never identically zero modulo any prime. The k-tuples conjecture implies there are infinitely many

integers $x \geq 0$ such that $p = 30x + 17$, $q = 30x + 19$, $r = 180x + 107$, and $s = 180x + 109$ are four prime numbers. For such an x, we note that $q = p + 2$, $s = r + 2$, and $r = 6p + 5$ (with $p = 30x + 17$).

We now prove that $m = pqr$ and $n = pqs$ are primary 5-Carmichael numbers. Since the signature of m is $-+-$, m is 5-Carmichael if, and only if, $p+1(= q-1)$ divides $m-1$ and $r+1$ divides $m-1$. But $r+1 = 6(p+1)$. Thus, we only need to check that $m - 1 \equiv 0 \pmod{6(p+1)}$. Observing that $6 \mid p+1$ so that $(p+1)^2 \equiv 0 \pmod{6(p+1)}$, we find that

$$\begin{aligned} m - 1 = (pq)r - 1 &= \big((p+1)^2 - 1\big)(6p+5) - 1 \\ &\equiv (0-1)(6p+6-1) - 1 \equiv 0 \pmod{6(p+1)}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} qr &= (p+2)(6p+5) = 10 + 17p + 6p^2, \\ pr &= (q-2)(6q-7) = 14 + (q-19)q + 5q^2, \\ pq &= (5x+3)r + (5x+2). \end{aligned}$$

Hence, $s_p(m) = 1$, if $p > 17$, and $s_p(m) = -p$, if $p = 17$; $s_q(m) = q$ and $s_r(m) = 1$. We conclude that m is a primary 5-Carmichael.

Now the number n has signature $-++$ and belongs to the fifth universal 5-form of Theorem 6, which was shown to only produce primary 5-Carmichael numbers.

It remains to show that $N = pqrs$ is also a 5-Carmichael number. Its signature being $-+-+$, we need to verify that $p+1$ $(= q-1)$ and $r+1$ $(= s-1)$ divide $N-1$. Since $r+1 = 6(p+1)$, it suffices to check that $N - 1 \equiv 0 \pmod{r+1}$. But

$$N-1 = (pq)(rs)-1 = \big((p+1)^2-1\big)\big((r+1)^2-1\big)-1 \equiv (-1)^2-1 = 0 \pmod{r+1}.$$

We verify the numbers N are all secondary:

$$\begin{aligned} qrs &= 70 + 179p + 144p^2 + 36p^3, \\ prs &= (q-70) + 178q + (q-144)q^2 + 35q^3, \\ pqs &= 4 + \big(\frac{p+1}{6}r + 2\big)r + \frac{p+1}{6}r^2, \\ pqr &= 2(c+2) + (s - (3c+4))s + cs^2, \end{aligned}$$

where in the last identity $c = (p-5)/6$. Thus, $s_p(N) = 1$, unless $p = 17$ when $s_p(N) = p + 2$; $s_q(N) = 2q - 1$ in all cases, even for $q = 19$, or 139; $s_r(N) = (p-11)/6 < r$ and $S_s(N) = s$, which proves all N are secondary 5-Carmichael numbers. □

Many variations on Theorem 8 can be proven for $D = 5$ and other values of D as well. We chose Theorem 8 because it is linked to Theorem 4. Theorem 8 says that

for infinitely many 5-Carmichael numbers $p(p+2)$, as described in Theorem 4, we can append two twin primes r and $s = r+2$ such that $p(p+2)r$ and $p(p+2)s$ are both primary 5-Carmichael numbers. The five smallest examples of this arithmetic curiosity, initially detected by peering at the tabulations made by Helmreich and Webster [7], occur for pqr equal to $17 \cdot 19 \cdot 107$, $137 \cdot 139 \cdot 827$, $347 \cdot 349 \cdot 2087$, $827 \cdot 829 \cdot 4967$, and $1667 \cdot 1669 \cdot 10007$.

We only provide another example similar to Theorem 8 but for $D = 13$. Again peering at the tabulation [7], we could not help but notice the numbers $137 \cdot 139 \cdot 827$ and $137 \cdot 139 \cdot 829$ in the list of three-factor 13-Carmichael numbers.

Theorem 9. *The prime k-tuples conjecture implies the existence of infinitely many pairs (m, n) of primary 13-Carmichael numbers, where m and n are of respective signatures $-+-$ and $-++$ and of the form*

$$m = pqr = (78x + 137)(78x + 139)(468x + 827),$$
$$n = pqs = (78x + 137)(78x + 139)(468x + 829),$$

for some $x \geq 0$. This occurs when all factors are prime numbers. The prime k-tuples conjecture implies $N = pqrs$ is a secondary 13-Carmichael number infinitely often, namely when the four numbers p, q, r, and s are all prime numbers.

Proof. The polynomials $(78x+137)(78x+139)(468x+827)$ and $(78x+137)(78x+139)(468x+829)$ were constructed using Theorem 5. As $137 \cdot 139 \cdot 827$ and $137 \cdot 139 \cdot 829$ are 13-Carmichael numbers of respective signatures $-+-$ and $-++$, we know by Theorem 5 that both polynomials are universal 13-forms with respective signatures $-+-$ and $-++$. When the four factors $78x + 137$, $78x + 139$, $468x + 827$, and $468x + 829$ are prime numbers, which must occur infinitely often by the prime k-tuples conjecture, then we know without any further calculations that m and n are primary, and $pqrs$ is secondary, by observing that, exactly as for the primes p, q, r, and s of Theorem 8, we have $q = p + 2$, $r = 6p + 5$, and $s = 6p + 7$. □

The first few instances correspond to $x = 0$, 13, and 19, yielding $pqr = 137 \cdot 139 \cdot 827$, $1151 \cdot 1153 \cdot 6911$, and $1619 \cdot 1621 \cdot 9719$.

Numbers can be playful and combining Theorems 8 and 9 gives the remarkable following result.

Theorem 10. *The polynomials $(390z+137)(390z+139)(2340z+827)$ and $(390z+137)(390z+139)(2340z+829)$ are both universal 7- and 13-forms with respective signatures $-+-$ and $-++$. They only yield primary 5- and 13-Carmichael numbers when their three factors are prime numbers. Moreover, the polynomial $(390z+137)(390z+139)(2340z+827)(2340z+829)$ is simultaneously a universal 5- and 13-form with common signature $-+-+$ which generates secondary 5- and 13-Carmichael numbers when the four factors are prime numbers. By the prime k-*

tuples conjecture there are infinitely many occurrences when all four factors are primes.

Proof. Theorem 8 used the universal 5-form

$$P(x) = (30x + 17)(30x + 19)(180x + 107)(180x + 109),$$

while Theorem 9 used the universal 13-form

$$Q(y) = (78y + 137)(78y + 139)(468y + 827)(468y + 829).$$

If $x = az + b$ and $y = cz + d$, where a, b, c, d are integers, a and b positive, and for all nonnegative z, $30x + 17 = 78y + 137$, then $R(z) := P(az + b) = Q(cz + d)$ will be both a 5- and a 13-universal form, since, as noticed previously, if p is the least common factor of both $P(x)$ and $Q(y)$ then the other factors are $p+2$, $6p+5$, and $6p + 7$. Solving $30x + 17 = 78y + 137$ gives $x = 4 + 13z$ and $y = 5z$ for some $z \in \mathbb{Z}$. Putting $x = 4 + 13z$ in $P(x)$ (or $y = 5z$ in $Q(y)$) yields the polynomial $R(z) = (390z + 137)(390z + 139)(2340z + 827)(2340z + 829)$ in z. □

The least values of $z \geq 0$ for which the four factors of

$$R(z) := (390z + 137)(390z + 139)(2340z + 827)(2340z + 829)$$

are all primes are $z = 0, 55, 75$, and 81. For $z = 55$, $p = 21587$, $q = p+2$, $r = 129527$, and $s = r + 2$. Thus, p and r are quadratic nonresidues of 5 and 13, while q and s are quadratic residues. Moreover, $p+1(= q-1)$ and $r+1(= s-1)$ divide $R(z)-1$.

4. An Open Matter: Highly Carmichael Numbers

Recall that the *sign*, ϵ_σ, of a signature $\sigma = (\epsilon_1, \epsilon_2, \ldots, \epsilon_s)$ is the product $\prod_{i=1}^{s} \epsilon_i$. Denote by $C(\sigma)$ the set of numbers which are D-Carmichael of signature σ for some D, and by $\omega(n)$ the number of distinct prime factors of an integer n.

If a number $n = p_1 p_2 \cdots p_s$ is a D-Carmichael number, then it has a signature $\sigma = (\epsilon_1, \ldots, \epsilon_s)$. The number n will be a D-Carmichael number of signature σ for an associated set $S(n, \sigma)$ of discriminants D defined by the conditions $D \equiv 0, 1 \pmod 4$ and $(D \mid p_i) = \epsilon_i$, $i = 1, \ldots, s$. The set $S(n, \sigma)$ has a natural density within the set of integers which, by the Chinese remainder theorem, is $2^{-s-1} \prod_{i=1}^{s} (1 - \frac{1}{p_i})$. If $s = 3$, then the upper bound of all such densities is $1/16$ (at least presumably, if we assume there are infinitely many 3-factor D-Carmichael numbers with respect to a given D, and only finitely many with p_1 and p_2 fixed, as results of Section 2 permit thinking). But it is conceivable that the same n be a D-Carmichael number for some D outside of $S(n, \sigma)$, i.e., with respect to a different signature. Such numbers would be "highly" Carmichael, and it would be of interest to find which numbers n

have the largest associated density, $\delta_C(n)$, of discriminants D with respect to which they are D-Carmichael (or the upper bound of all such densities). If $s = 2$, then any D-Carmichael has to have signature $(-1, 1)$ by Theorem 4, so it cannot be D-Carmichael with other signatures. If $s = 3$ and $p_1 = 3$, then we see from Theorem 7 that this cannot happen either. Thus, the number 105, say, can only be a D-Carmichael number for $D \in S(105, (- + -))$, a set of natural density $1/35$ within the set of integers. But it can actually occur. Call such numbers *bi-Carmichael*. The number $n = 1{,}034{,}881 = 41 \cdot 43 \cdot 587 = pqr$ is 5-Carmichael with signature $+--$, and 13-Carmichael with signature $-+-$. The two signatures have the same sign, $+1$, and p and q are twin primes so that the requirements $p + 1 \mid n - 1$ and $q - 1 \mid n - 1$ are identical. Moreover, the number $n - 1 = 2^7 \cdot 3 \cdot 5 \cdot 7^2 \cdot 11$ is very smooth. Thus, $\delta_C(n)$ is at least $2^{-3}\big(1 - \frac{1}{41}\big)\big(1 - \frac{1}{43}\big)\big(1 - \frac{1}{587}\big) = \frac{5\times 42\times 586}{n}$, i.e., about 11.89%. We did not probe very far, but, among 3-factor numbers, we found a few dozens using two of the four discriminants $\{5, 13, -7, -11\}$. All bi-Carmichael numbers we found occur for two signatures σ and σ' of sign $+1$. In all, but one case, the two signatures are $+--$ and $-+-$. But the number $\ell = 71 \cdot 271 \cdot 521$ has signature $+++$ with respect to $D = 5$ (i.e., ℓ is a Carmichael number) and is a 13-Carmichael number of signature $--+$. Unsurprisingly, $\ell - 1 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ is very smooth. Are there infinitely many such cases? Again, with the generalized k-tuples conjecture coming to the rescue, we believe there are. Here is an example of a theorem, among similar ones, we can prove.

Theorem 11. *The H-hypothesis implies infinitely many numbers are simultaneously primary* 5*-Carmichael of signature* $+--$ *and primary* (-11)*-Carmichael of signature* $-+-$.

Proof. Consider the polynomial

$$Q(x) = (3x - 1)(3x + 1)(3x^2 - 1).$$

Note that $3x^2 - 1$ is irreducible over the rationals. Since $Q(0) = 1$, $Q(x)$ is not identically zero modulo any prime. Thus, the generalized prime k-tuples conjecture implies there are infinitely many integer values of x for which all three factors of $Q(x)$ are prime numbers. This is true if $x = 34$ for instance. In such cases, we write $p = 3x-1$, $q = p+2$, and $r = 3x^2-1 = \frac{(p+1)^2}{3} - 1$. Next we check that p^2-1, $q+1$, and $r+1$ all divide $pqr-1$. We have $p^2-1 = 3x(3x-2)$ and $pq = p^2-1+2p+1$, so that $pqr-1 \equiv (2p+1)r-1 = (6x-1)(3x^2-1)-1 = 3x(2x+1)(3x-2) \equiv 0 \pmod{p^2-1}$. Then, modulo $q + 1$, we have $1 - pqr \equiv 1 + pr \equiv 1 - 3r = 1 - (q + 1 - 2)^2 + 3 \equiv 0$. Also, $1 - pqr \equiv 1 + pq = 1 + p^2 + 2p = 3(r + 1) \equiv 0 \pmod{r + 1}$.

In order for pqr to be 5-Carmichael of signature $+--$, we may replace x by $10x + 4$ in $Q(x)$. We obtain

$$R(x) = (30x + 11)(30x + 13)\big(12(5x + 2)^2 - 1\big).$$

The generalized k-tuples conjecture applies to the polynomial $R(x)$. Assuming the three factors of $R(x)$ are prime numbers, we see that $(5 \mid r) = (48 - 1 \mid 5) = -1$ so the required signature is realized. Now, if we take $x \equiv 3, 4$ or $5 \pmod{11}$ in $R(x)$, then, respectively, $p \equiv 2, 7$ or $10 \pmod{11}$, and $q \equiv 4, 9$ or $1 \pmod{11}$. Hence, $(-11 \mid p) = -1$ and $(-11 \mid q) = +1$. Also, modulo 11 we find that

$$r = \frac{(p+1)^2 - 3}{3} = \frac{pq - 2}{3} \equiv 4(pq - 2) \equiv \begin{cases} 2, & \text{if } x \equiv 3, 4 \pmod{11}; \\ 10, & \text{if } x \equiv 5 \pmod{11}, \end{cases}$$

so that, in all cases, $(-11 \mid r) = -1$. Thus, the generalized prime k-tuples conjecture says that the three polynomials $R(3 + 11x)$, $R(4 + 11x)$, and $R(5 + 11x)$ generate infinitely often a product of three primes which have all the announced properties.

Suppose the three factors of $n := Q(x)$ are prime numbers p, q, and r. Write $\sigma = + - -$ and $\tau = - + -$. As $qr = 2\frac{p-2}{3} + \frac{p+4}{3} \cdot p^2$, we see that $s_p(n) = p$ for σ, and $s_p(n) = -p$ for τ. Also, $pr = 2\frac{q+2}{3} + \frac{q-4}{3} \cdot q^2$ so that $s_q(n) = q$ for σ, and $s_q(n) = -q$ for τ. Finally, $pq = 3r + 2$ so that $s_r(n) = 1$ for both σ and τ. This proves the numbers are primary with respect to both signatures. □

Theorem 11, if one believes the H-hypothesis, means the upper bound of all $\delta_C(n)$ is at least 1/8, or 12.5%. We do not know whether there are infinitely many numbers that are D-Carmichael with respect to three or more signatures. There are $\binom{8}{2} = 28$ possible pairings of signatures for 3-factor bi-Carmichael numbers. We found at least two pairings that we know occur. How many actually do? If they do, do they occur infinitely often? Theorems 12 and 13, next, show that if n is bi-Carmichael with respect to σ and τ, then σ and τ have to have the same sign. Hence, the number of possible pairings reduces from 28 to 12. Can any of the six pairings with two signatures of negative signs ever occur? This remains an open question.

These problems have their origin in a question of Hugh Williams [17] who asked about the existence of Carmichael numbers n that would also be D-Carmichael numbers for some D with $(D \mid n) = -1$. If n has s prime factors, then Williams showed the signature of n must be $(-1, -1, \ldots, -1)$ (s times -1's), s must be odd, at least 5, and $3 \nmid n$. This open question was reiterated as Problem 5 in the recent book [2, p. 271]. The next two theorems broaden the scope of the above-mentioned results of Williams following closely and pushing further the techniques he used.

Theorem 12. *Suppose $\sigma = (\epsilon_1, \ldots, \epsilon_s)$ and $\sigma' = (\epsilon'_1, \ldots, \epsilon'_s)$ are two signatures of opposite signs. If $n \in C(\sigma) \cap C(\sigma')$, then $\sigma' = -\sigma$, i.e., the two s-tuples have opposite entries, $\omega(n)$ is odd, and $3 \nmid n$.*

Proof. Suppose $p_1 < p_2 < \cdots < p_s$ are the prime factors of n. Let p be one of these prime factors. Then $p - \epsilon_p \mid n - \epsilon_\sigma$ and $p - \epsilon'_p \mid n + \epsilon_\sigma$. If $\epsilon_p = \epsilon'_p$, because $\gcd(n - \epsilon_\sigma, n + \epsilon_\sigma) = 2$, $p \geq 5$ is not possible. Thus, $p = p_1 = 3$, and if $q = p_i$,

$i \geq 2$, then $\epsilon_q = -\epsilon'_q$. One of $q \pm 1$ is divisible by 3. Therefore, one of the two numbers $n \pm 1$ is also divisible by 3. Hence, $3 \nmid n$, $p_1 > 3$, and $\epsilon'_i = -\epsilon_i$ for all i, $1 \leq i \leq s$. Hence, $\sigma' = -\sigma$. But $\epsilon_{\sigma'} = -\epsilon_\sigma$ and $\epsilon_{\sigma'} = (-1)^s \epsilon_\sigma$ implies $s = \omega(n)$ must be odd. □

Theorem 13. *There are no three-factor bi-Carmichael numbers with respect to two signatures of opposite signs.*

Proof. We proceed by contradiction. Suppose that $n = p_1 p_2 p_3$, or alternately pqr, $p < q < r$, is bi-Carmichael. By Theorem 12, the two signatures are $\pm(\epsilon_1, \epsilon_2, \epsilon_3)$. Put σ for the signature with sign $+1$ and assume without loss of generality $\sigma = (\epsilon_1, \epsilon_2, \epsilon_3)$. By hypothesis,

$$p_i - \epsilon_i \mid n - \epsilon_\sigma = n - 1 \quad \text{and} \quad p_i + \epsilon_i \mid n + \epsilon_\sigma = n + 1.$$

Hence, with $\{i, j, k\} = \{1, 2, 3\}$, since $p_i \equiv \pm\epsilon_i \pmod{p_i \mp \epsilon_i}$, we find that $p_i - \epsilon_i \mid p_j p_k - \epsilon_i$ and $p_i + \epsilon_i \mid p_j p_k - \epsilon_i$. Hence, $(p_i^2 - 1)/2 \mid p_j p_k - \epsilon_i$. Thus we obtain the strict inequalities between three integers

$$x := \frac{qr - \epsilon_1}{(p^2 - 1)/2} > y := \frac{pr - \epsilon_2}{(q^2 - 1)/2} > z := \frac{pq - \epsilon_3}{(r^2 - 1)/2}. \tag{11}$$

Since $r^2 - 1 > pq - \epsilon_3$, we find that $z = 1$, and

$$r^2 = 2pq + (1 - 2\epsilon_3) \leq 2pq + 3. \tag{12}$$

`Case 1:` $y \geq 3$. (Note that this case holds if $\epsilon_2 = 1$, for $y = 2$ would imply $pr - 1 = q^2 - 1$.) Then

$$qr + pr + pq - \sum \epsilon_i \geq 4 \cdot \frac{p^2 - 1}{2} + 3 \cdot \frac{q^2 - 1}{2} + \frac{r^2 - 1}{2},$$

which, using $p^2 + q^2 + r^2 > qr + pr + pq$ and multiplying through by 2, yields $2(p^2 + q^2 + r^2) > 4p^2 + 3q^2 + r^2 - 12$. Simplifying gives $r^2 > 2p^2 + q^2 - 12 \geq p^2 + q^2 + 13 > 2pq + 3$, using $p \geq 5$ which we know from Theorem 12. But the last inequality contradicts (12).

`Case 2:` $y = 2$ and $x = 3$ or 4. Then $\epsilon_2 = -1$ so that, as $\epsilon_\sigma = 1$, we have $\epsilon_1 \epsilon_3 = -1$ and $q^2 = pr + 2$. Put $X = qr$.

Assume first $\epsilon_3 = -1$. Then $r^2 = 2pq + 3$ and $2qr - 2 = x(p^2 - 1)$. We will show that $x \geq 5$. Multiplying q^2 by r^2 yields $X^2 = 2p^2 X + 4pq + 3pr + 6$. But $p^2 = (2X + (x - 2))/x$ so that

$$\frac{4 - x}{x} X^2 + \frac{2x - 4}{x} X + 4pq + 3pr + 6 = 0.$$

Since $x = 3$ or $x = 4$, the first term is nonnegative and the others are positive, yielding a contradiction.

If $\epsilon_3 = 1$, then instead we have $r^2 = 2pq - 1$ and $2qr + 2 = x(p^2 - 1)$. Thus, $p^2 = (2X + (2 + x))/x$. Instead we obtain by the same process

$$\frac{4-x}{x}X^2 + \frac{2x+4}{x}X + 4pq - pr - 2 = 0.$$

But $x = 3$ or $x = 4$ implies the left-hand side of the above equality is at least equal to $3X + 4pq - pr - 2$, which is positive since $X > pr$, leading to a contradiction.

`Case 3:` $y = 2$ and $x \geq 5$. Recall that we may assume $\epsilon_2 = -1$, or else $y \geq 3$ and we fall under the Case 1 contradiction. Thus, $q^2 = pr + 2$. This implies 2 is a quadratic residue of p and of r. Thus, $p \geq 7$, $q \geq 11$, and $r \geq 17$. Put $c_1 = 1 + \frac{3}{2pq}$ and $c_2 = 1 + \frac{2}{pr}$. Then, by (12) and $q^2 = pr + 2$, we see that $r^2 \leq 2pqc_1$, which implies $r^4 \leq 4p^2q^2c_1^2 \leq 4p^2(prc_2)c_1^2$. Dividing through by r gives $r^3 \leq 4p^3c_1^2c_2$. Taking cube roots yields $r \leq \sqrt[3]{4}\, p\, c_1^{2/3}\, c_2^{1/3}$. Now

$$q = \sqrt{c_2 pr} \leq \sqrt{p} \cdot (\sqrt[3]{2}\sqrt{p}\, c_1^{1/3} c_2^{1/6}) \cdot c_2^{1/2} = \sqrt[3]{2}\, p\, c_1^{1/3} c_2^{2/3},$$

so that

$$qr \leq 2p^2 c_1 c_2.$$

From (11), we obtain, following what we did in Case 1,

$$2qr + 2pr + 2pq \geq 5p^2 + 2q^2 + r^2 - 5 - 2 - 1 + 2\sum \epsilon_i. \tag{13}$$

We see that $4c_1c_2p^2 + (2q^2 - 4) + (r^2 + 1)$ is an upper bound for the left-hand side of (13). Simplifying by the common terms in q^2 and in r^2 gives $4c_1c_2p^2 - 4 + 1 \geq -10 + 5p^2$. (Since $\epsilon_\sigma = 1$ and $\epsilon_2 = -1$, we know $\sum \epsilon_i = -1$.) Thus, $7 \geq (5 - 4c_1c_2)p^2$. Using $c_1 \leq 1 + \frac{1}{2\cdot 7\cdot 11}$ and $c_2 \leq 1 + \frac{1}{7\cdot 17}$ gives $4c_1c_2 < 4.06$. Thus, $(5 - 4c_1c_2)p^2 > 0.94 \times 49 > 7$ and we have reached a contradiction. □

Remark 5. (The rationale governing the calculations for Case 3 in the above proof came from the observation that we roughly have $q^2 = pr$ and $r^2 = 2pq$. Thus, $qr = 2p^2$. Using (11), as we did in Case 1, but with $x \geq 5$ and, neglecting the small constants, leads to $2p^2 + \frac{r^2}{2} + q^2 \geq qr + pr + pq \geq \frac{5}{2}p^2 + q^2 + \frac{r^2}{2}$. Hence, $2p^2 \geq \frac{5}{2}p^2$, a contradiction.)

Acknowledgements. The referee said the initial draft of the paper was well-written, yet suggested many changes in the exposition and made many useful remarks. We cheerfully thank the referee for his involvement. This paper was made possible by the lists of D-Carmichael numbers tabulated by C. Helmreich and J.

Webster. It is by long time gazing at these numbers that we came up with several of our theorems. Thanks are also directed to my dear colleague Denis Simon, in Caen, who is always generous in sharing his expertise with my favorite programming software PARI gp.

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