AN ADDITIVE UNIQUENESS PAIR OF SETS FOR MULTIPLICATIVE FUNCTIONS: SQUARES AND TRIANGULAR NUMBERS

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Abstract

For $u \in \mathbb{N}$, let $t_u = u(u+1)/2$ denote the u-th triangular number. We prove that if a multiplicative function f satisfies $f\big(s^2 + t_u\big) = f\big(s^2\big) + f(t_u)$ for all positive integers s and u, then f is the identity function.

1. Introduction

An arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is called *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all relatively prime positive integers m and n. Spiro [7] proved that if a multiplicative function f satisfies

$$f(p + q) = f(p) + f(q)$$

for all primes p and q, and if $f(p_0) \neq 0$ for some prime p_0, then f must be the identity function.

Let $\mathfrak{S}$ be a set of arithmetic functions. A subset $E \subseteq \mathbb{N}$ is called *an additive uniqueness set for* $\mathfrak{S}$ if there exists exactly one function $f \in \mathfrak{S}$ such that

$$f(a + b) = f(a) + f(b) \tag{1}$$

for all $a, b \in E$.

Let $\mathfrak{F}$ denote the set of all multiplicative functions. For each integer $m \geq 3$, we define $P(m)$ to be the set of all m-gonal numbers and $GP(m)$ to be the set of all

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positive generalized m-gonal numbers; that is,

$$P(m) = \left\{ \frac{(m-2)u^2 - (m-4)u}{2} : u \in \mathbb{N} \right\},$$

and

$$GP(m) = \mathbb{N} \cap \left\{ \frac{(m-2)u^2 - (m-4)u}{2} : u \in \mathbb{Z} \right\}.$$

It is easily verified that $P(m) = GP(m)$ if and only if $m \in \{3, 4\}$.

Chung and Phong [2] showed that $P(3)$, the set of triangular numbers, is an additive uniqueness set for $\mathfrak{F}$. In contrast, Chung [1] proved that $P(4)$, the set of all squares of positive integers, is not an additive uniqueness set for $\mathfrak{F}$. The second author and his collaborators [4] established that both $P(5)$ and $P(6)$ are additive uniqueness sets for $\mathfrak{F}$. In [3], it was shown that $GP(8)$ fails to be an additive uniqueness set for $\mathfrak{F}$, whereas the second author [6] proved that $GP(9)$ does possess this property.

We investigate additive uniqueness when a and b in Equation (1) are taken from two distinct subsets of $\mathbb{N}$. Let A and B be subsets of $\mathbb{N}$. We say that the pair (A, B) is an *additive uniqueness pair of sets* for $\mathfrak{F}$ if there exists exactly one function $f \in \mathfrak{F}$ satisfying

$$f(a+b) = f(a) + f(b)$$

for every $a \in A$ and every $b \in B$. De Koninck, Kátai, and Phong [5] proved that the pair $(\mathcal{P}, P(4))$ is an additive uniqueness pair of sets for $\mathfrak{F}$, where $\mathcal{P}$ is the set of primes. The main result of this article is that the pair $(P(4), P(3))$ is an additive uniqueness pair of sets for $\mathfrak{F}$. For simplicity, we use the notation

$$t_u = \frac{u(u+1)}{2}, \quad u \in \mathbb{Z}$$

throughout the paper.

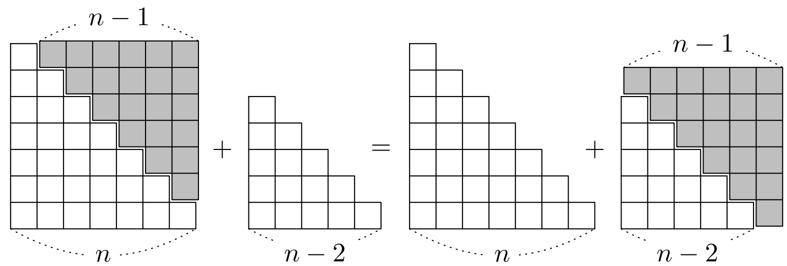
Theorem 1. *Let f be a multiplicative function such that $f\big(u^2 + t_v\big) = f\big(u^2\big) + f(t_v)$ for all positive integers u and v. Then f is the identity function.*

2. Proof of the Main Theorem

Lemma 1. *For any integer $n \geq 2$, we have*

$$n^2 + t_{n-2} = (n-1)^2 + t_n.$$

Proof. One sees that



$\square$

Lemma 2. *Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function such that $f\big(u^2 + t_v\big) = f\big(u^2\big) + f(t_v)$ for all positive integers u and v. Let $N \geq 3$ be an integer, and assume that $f(n) = n$ for all $n = 1, 2, \ldots, N-1$. Then we have $f(m^2) = m^2$ for all $m = 1, 2, \ldots, N-2$.*

Proof. Note that

$$f(t_v) = \begin{cases} f\left(\dfrac{v}{2}\right) f(v+1) & \text{if } \ v \equiv 0 \pmod 2, \\ f(v)\, f\left(\dfrac{v+1}{2}\right) & \text{if } \ v \equiv 1 \pmod 2 \end{cases}$$

for $v = 1, 2, \ldots, N-2$. Hence, by assumption, we have $f(t_v) = t_v$ for $v = 1, 2, \ldots, N-2$.

To see that $f\big(m^2\big) = m^2$ for $m = 1, 2, \ldots, N-2$, one may use induction on m. Note that $f(1) = 1$. Now, assume that $f\big((m-1)^2\big) = (m-1)^2$ for some $m \geq 2$. On the one hand, we have

$$f\big(m^2 + t_{m-2}\big) = f\big(m^2\big) + f(t_{m-2}) = f\big(m^2\big) + t_{m-2}.$$

On the other hand, we have

$$f\big((m-1)^2 + t_m\big) = f\big((m-1)^2\big) + f(t_m) = (m-1)^2 + t_m.$$

From these and Lemma 1, it follows that

$$f\big(m^2\big) + t_{m-2} = (m-1)^2 + t_m.$$

Thus, we have $f\big(m^2\big) = m^2$. This completes the proof. $\square$

Lemma 3. *Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function such that $f(u^2 + t_v) = f(u^2) + f(t_v)$ for all positive integers u and v. Let $N \geq 4$ be an integer, and assume that $f(n) = n$ for all $n = 1, 2, \ldots, N-1$. Then $f(2^{2k-1}) = 2^{2k-1}$ for any positive integer k satisfying $2^k \leq N-2$.*

Proof. Let k be a positive integer such that $2^k \le N-2$. By Lemma 2, we have $f\bigl(2^{2k}\bigr) = 2^{2k}$. On the one hand, we have

$$\begin{aligned} f\left(2^{2k} + \frac{2^{2k}(2^{2k}+1)}{2}\right) &= f\bigl(2^{2k}\bigr) + f\left(\frac{2^{2k}(2^{2k}+1)}{2}\right) \\ &= f(2^{2k}) + f(2^{2k-1})f(2^{2k}+1) \\ &= f\bigl(2^{2k}\bigr) + f\bigl(2^{2k-1}\bigr)\bigl(f\bigl(2^{2k}\bigr) + f(1)\bigr) \\ &= 2^{2k} + f\bigl(2^{2k-1}\bigr)\bigl(2^{2k}+1\bigr). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} f\left(2^{2k} + \frac{2^{2k}(2^{2k}+1)}{2}\right) &= f\bigl(2^{2k-1}(2^{2k}+3)\bigr) \\ &= f\bigl(2^{2k-1}\bigr)f\bigl(2^{2k}+3\bigr) \\ &= f\bigl(2^{2k-1}\bigr)\bigl(f\bigl(2^{2k}\bigr) + f(3)\bigr) \\ &= f\bigl(2^{2k-1}\bigr)\bigl(2^{2k}+3\bigr). \end{aligned}$$

By combining these, we get $f\bigl(2^{2k-1}\bigr) = 2^{2k-1}$. This completes the proof. $\square$

We are now ready to prove the main theorem.

Proof of Theorem 1. To prove that f is the identity function, we show that $f(N) = N$ for every positive integer N. We directly show this for $N = 1, 2, \ldots, 10$, and then use induction on N for all $N \ge 11$. One sees that

$$\begin{aligned} &f(2) = f(1+1) = f(1) + f(1) = 2, \\ &f(4) = f(1+3) = f(1) + f(3) = 1 + f(3). \end{aligned}$$

So we have

$$\begin{aligned} &f(5) = f(4+1) = f(4) + f(1) = f(4) + 1 = f(3) + 2, \\ &f(6) = f(2)f(3) = 2f(3), \\ &f(7) = f(4+3) = f(4) + f(3) = 2f(3) + 1, \end{aligned}$$

and

$$f(10) = f(4+6) = f(4) + f(6) = 3f(3) + 1.$$

Since

$$2f(3) + 4 = 2f(5) = f(2)f(5) = f(10) = 3f(3) + 1,$$

we have $f(3) = 3$. This yields that $f(N) = N$ for $N \in \{1, 2, \ldots, 10\} \setminus \{8, 9\}$. For $N = 9$, one has

$$f(9) = f(9) + f(3) - 3 = f(9+3) - 3 = f(4)f(3) - 3 = 9.$$

Since

$$3f(8) = f(3)f(8) = f(24) = f(9+15) = f(9) + f(15) = 9 + f(3)f(5) = 24,$$

we have $f(8) = 8$. Therefore, we have $f(N) = N$ for every $N = 1, 2, \ldots, 10$.

Let $N \geq 11$ be an integer and assume that $f(n) = n$ for all $n = 1, 2, \ldots, N-1$. We shall show that $f(N) = N$. Since f is multiplicative, we may assume that $N = p^s$ for a prime p and $s \in \mathbb{N}$.

First, assume that $p = 2$ and s is even. Then $N = 2^{2r}$ for some integer $r > 1$. Noting that $2^r \leq 2^{2r} - 2$, we have $f\big(2^{2r}\big) = 2^{2r}$ by Lemma 2.

Second, assume that $p = 2$ and s is odd. Then $N = 2^{2r-1}$ for some integer $r > 2$. Since $2^r \leq 2^{2r-1} - 2$, we have $f(2^{2r-1}) = 2^{2r-1}$ by Lemma 3.

Last, assume that p is odd so that $N = p^s$ is an odd integer greater than or equal to 11. Define M to be the unique even integer in the set $\left\{\dfrac{3N \pm 1}{2}\right\}$. We first show that $f(M) = M$ and use this to prove that $f(N) = N$. Let d be the positive integer satisfying $2^d \,\|\, M$. If $M \neq 2^d$, then the two positive integers $\dfrac{M}{2^d}$ and 2^d are both less than N and $\gcd\!\left(\dfrac{M}{2^d}, 2^d\right) = 1$ so that

$$f(M) = f\left(\frac{M}{2^d} \cdot 2^d\right) = f\left(\frac{M}{2^d}\right) f\big(2^d\big) = \frac{M}{2^d} \cdot 2^d = M.$$

Hence, we assume that $M = 2^d$. Define k to be the positive integer such that $d \in \{2k-1, 2k\}$. Then one may easily check that $2^k \leq N - 2$. We use Lemma 3 if $d = 2k-1$ and Lemma 2 if $d = 2k$ to see that $f(M) = M$.

Noting that $M - N = \dfrac{N \pm 1}{2}$ is coprime to N and is less than or equal to $N - 2$, one has

$$\begin{aligned}
f\big((M-N)^2 + N(M-N)\big) &= f((M-N)^2) + f(N(M-N)) \\
&= (M-N)^2 + f(N)f(M-N) \\
&= (M-N)^2 + f(N) \cdot (M-N).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
f\big((M-N)^2 + N(M-N)\big) &= f((M-N)M) \\
&= f(M-N)f(M) \\
&= (M-N)M,
\end{aligned}$$

where we use the fact that $\gcd(M-N, M) = 1$ for the second equality. Combining these, we get $f(N) = N$. This completes the proof. $\square$

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