A NOTE ON THE NAGELL-LJUNGGREN EQUATION $\frac{x^n-1}{x-1} = y^3$

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Abstract

In this note, we prove by elementary methods that the Nagell–Ljunggren equation $\dfrac{x^n-1}{x-1} = y^3$ has no integer solutions when x is congruent to $2, 3, 5,$ or 6 modulo 9.

– To my wife and our children

1. The Result

In 1920, Nagell [7, 8] obtained the first results on the Diophantine equation

$$\frac{x^n-1}{x-1} = y^q, \text{ in integers } |x| > 1, y > 1, n > 2, q \geq 2. \tag{1}$$

Two decades later, Ljunggren [6] resolved the remaining issues in Nagell's proof and established the following result, which is presented in [4].

Theorem 1 ([6]). *Except for the solutions*

$$\frac{3^5-1}{3-1} = 11^2, \ \frac{7^4-1}{7-1} = 20^2, \ \frac{18^3-1}{18-1} = 7^3, \ \textit{and} \ \frac{-19^3-1}{-19-1} = 7^3,$$

Equation (1) admits no other solutions (x, y, n, q) if one of the following conditions holds: (i) $q = 2$, (ii) $3 \mid n$, (iii) $4 \mid n$, and (iv) $q = 3$ with $n \not\equiv 5 \pmod 6$.

The generalized Nagell–Ljunggren equation has been extensively studied by many authors in the literature, notably Bugeaud, Mihăilescu, Mignotte, Shorey, and Bennett, among others; see, for example, [4, 1, 3, 5, 2]. However, a common feature is

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that the methods involved rely on linear forms in p-adic logarithms or other deep techniques.

Here, we use only very elementary observations, such as congruence properties and Euler's theorem, to derive a conclusion for a certain class of this equation. We observe that, in Theorem 1, condition (iv) for $q = 3$ is considered only for $n \not\equiv 5 \pmod 6$. This means that the case where $q = 3$ and $n \equiv 5 \pmod 6$ has not yet been considered. It appears that this case is difficult; from that time to the present, it has remained an open problem.
A special case of this form, namely $q = 3$ and $n = 5$, was treated very recently by Tran Duc Son [9], who proved that Equation (1) has no integer solutions (x, y) other than $(0, 1)$.

In this note, we focus on the case $q = 3$ and $n \equiv 5 \pmod 6$. We have the following theorem.

Theorem 2. *For $n \equiv 5 \pmod 6$, the Diophantine equation $\dfrac{x^n - 1}{x - 1} = y^3$ has no integer solutions (x, y, n) with $x = 9z + 2, 9z + 3, 9z + 5,$ or $9z + 6$, where z is an integer.*

Proof. Suppose that $n = 6m+5$, where $m \geq 0$ is an integer. In this case, we rewrite the equation as follows

$$\frac{x^{6m+5} - x}{x - 1} = y^3 - 1.$$

Since $x^{6m+5} - x = x(x^{6m+4} - 1) \equiv 0 \pmod 3$ and $x \not\equiv 1 \pmod 3$, it follows that $y^3 \equiv 1 \pmod 3$. Thus, $y \equiv 1 \pmod 3$. Hence, $y^3 \equiv 1 \pmod 9$.

On the other hand, we have $x^{6m+5} - x = (x^{6m} - 1)x^5 + x^5 - x \equiv x^5 - x \pmod 9$. Clearly, $x^5 - x$ is not divisible by 9 for any x of the form $9z + 2, 9z + 3, 9z + 5$, or $9z + 6$. This contradiction completes the proof. $\square$

By Theorem 1, when $q = 3$ and $n \not\equiv 5 \pmod 6$, the only solutions to Equation (1) are $x = 18$ and $x = -19$, which are not of the form $9z+2, 9z+3, 9z+5$, or $9z+6$. Combining Theorems 1 and 2, we immediately obtain the following corollary.

Corollary 1. *The Diophantine equation $\dfrac{x^n - 1}{x - 1} = y^3$ has no integer solutions (x, y, n) with $x = 9z + 2, 9z + 3, 9z + 5,$ or $9z + 6$, where z is an integer.*

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