UNIFIED CATALAN-TYPE POLYNOMIALS WITH MATRIX FORMULATIONS AND APPLICATIONS

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Abstract

This paper presents a comprehensive study of generalized Catalan-type polynomials constructed via generating functions involving pseudo-Bessel functions, Hermite polynomials, and Humbert-type structures. Explicit formulas, recurrence relations, and operational identities are derived for multi-variable extensions. Matrix representations, including lower-triangular, Hankel, and Toeplitz forms, are developed, and their determinantal properties are analyzed. Applications in differential equations are discussed, highlighting the analytical utility and structural richness of the proposed generalizations.

1Introduction

The *Hermite polynomials* $H_n(x)$ are defined through the generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

These polynomials occupy a central position in the theory of special functions and have served as a foundational tool for extending classical families such as Bessel

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functions (see [1, 3, 5]), Laguerre polynomials (see [1, 4]), Chebyshev polynomials, and Humbert polynomials (see [8]). Their algebraic richness and operational versatility make them particularly suitable for generalizations and applications in mathematical physics, quantum optics, and combinatorics.

For example, in the context of quantum optics, Nieto et al. [12, 13] derived generalized generating functions for ordinary Hermite polynomials that play an important role in the analysis of higher-order coherent and squeezed states. The *Hermite numbers* H_n are defined by evaluating the Hermite polynomials at the origin:

$$H_n = H_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \dfrac{(-1)^{n/2} n!}{(n/2)!}, & \text{if } n \text{ is even.} \end{cases}$$

Although Hermite polynomials possess a rich algebraic structure, relatively few explicit summation identities and closed-form representations are available. To address this limitation, Nieto et al. [12, 13] introduced the generalized generating function

$$F(j,k,x,t) = \sum_{m=0}^{\infty} \frac{t^{jm+k} H_{jm+k}(x)}{(jm+k)!}, \tag{1}$$

where $j \geq 1$ and $k \geq 0$ are integers. The corresponding closed-form expression is

$$F(j,k,x,t) = e^{-t^2} \frac{1}{j} \sum_{\ell=1}^{j} \exp\Big(2xt\, e^{2\pi i \ell/j}\Big) \left(e^{2\pi i \ell/j}\right)^k .$$

Important special cases are

$$\begin{aligned} F(1,0,x,t) &= e^{-t^2+2xt}, \\ F(2,0,x,t) &= e^{-t^2} \cosh(2xt), \\ F(2,1,x,t) &= e^{-t^2} \sinh(2xt), \end{aligned}$$

which correspond to the decomposition of ordinary, even, and odd coherent states into harmonic-oscillator number states. Nieto et al. [12] also considered the generating function

$$G(j,k,x,y,t) = \sum_{m=0}^{\infty} \frac{t^{jm+k} H_{jm+k}(x) H_{jm+k}(y)}{(jm+k)!},$$

whose closed-form representation is

$$G(j,k,x,y,t) = \frac{1}{j} \sum_{l=1}^{j} \frac{1}{\big[\exp(2\pi i l/j)\big]^k \sqrt{1-4z_{lj}^2}} \exp\left(\frac{4x^2 z_{lj}^2 - 4y^2 z_{lj}^2 + 4xyz_{lj}}{1-4z_{lj}^2} \right),$$

where

$$z_{lj} = z \exp\left(\frac{2\pi i l}{j}\right).$$

A further extension introduced in [13] is the function

$$D(j,k,p,t,x) = \sum_{r=0}^{\infty} \frac{t^{jr+k} H_{jr+k}(x)}{(pr)!},$$

for integers $j, p \geq 1$ and $k \geq 0$. Particular cases include

$$D(2,0,1,t,x) = (1+4t^2)^{-1} \exp\left(\frac{4t^2x^2}{1+4t^2}\right),$$
$$D(2,1,1,t,x) = \frac{2tx}{\sqrt{1+4t^2}} \exp\left(\frac{4t^2x^2}{1+4t^2}\right).$$

Combining these expressions yields

$$\sum_{r=0}^{\infty} \frac{t^r H_r(x)}{\lfloor r/2 \rfloor!} = \frac{1+2tx+4t^2}{(1+4t^2)^{3/2}} \exp\left(\frac{4t^2x^2}{1+4t^2}\right).$$

The *two-variable Hermite polynomials*, also known as Kampé de Fériet polynomials (see [1, 17]), are defined by

$$H_n^m(x,y) = n! \sum_{r=0}^{\lfloor n/m \rfloor} \frac{y^r x^{n-mr}}{r!\,(n-mr)!}, \qquad H_n^2(x,y) = H_n(x,y), \tag{2}$$

and possess the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n^2(x,y) \frac{t^n}{n!}. \tag{3}$$

Here, $\lfloor n/m \rfloor$ denotes the greatest integer less than or equal to n/m. A further generalization introduced in [6] is

$$H(x,y,t|k) = e^{xt+yt^2} H_k(x+2yt, y) = \sum_{n=0}^{\infty} H_{n+k}(x,y) \frac{t^n}{n!}.$$

Equation (3) corresponds to the special case $m = 2$ of the Kampé de Fériet polynomials, whereas the latter generating function provides a natural extension involving the parameter k.

The study of generating functions has led to numerous generalizations of classical polynomial families. In 1989, Sinha [16] introduced a class of polynomials through the generating relation

$$\left[1 - 2xt + t^2(2x-1)\right]^{-\nu} = \sum_{n=0}^{\infty} S_n^{\nu}(x) t^n,$$

where
$$S_n^{\nu}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (\nu)_{n-k} (2x)^{n-2k} (2x-1)^k}{k!(n-2k)!}.$$
These polynomials generalize the Shrestha polynomials $S_n(x)$ (see [15]) and possess rich algebraic and operational properties. Subsequently, Milovanović and Djordjević [7] proposed a broader generalization involving an arbitrary positive integer m and a parameter $\lambda > -1/2$, namely
$$(1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,m}^{\lambda}(x) t^n,$$
whose coefficients are given by
$$p_{n,m}^{\lambda}(x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k (\lambda)_{n-(m-1)k} (2x)^{n-mk}}{k!(n-mk)!}. \tag{4}$$
This family includes several well-known polynomial sequences as special cases. In particular, when $m = 1$, the polynomials $p_{n,1}^{\lambda}(x)$ reduce to the Horadam polynomials (see [9]); when $m = 2$, they coincide with the classical Gegenbauer polynomials; and when $m = 3$, they yield the Horadam–Pethe polynomials (see [10]). These relationships highlight the unifying character of the generating function and its ability to recover a variety of classical polynomial families through suitable parameter choices. A further extension was introduced by Pathan and Khan [14], who considered the more general generating function
$$[c - axt + bt^m (2x-1)^d]^{-\nu} = \sum_{n=0}^{\infty} p_{n,m,a,b,c,d}^{\nu}(x) t^n = \sum_{n=0}^{\infty} \Theta_n(x) t^n,$$
where
$$\Theta_n(x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k c^{-\nu-n+(m-1)k} (\nu)_{n+(1-m)k} (ax)^{n-mk} [b(2x-1)^d]^k}{k!(n-mk)!}.$$
The additional parameters a, b, c, and d provide substantial flexibility in shaping the generating kernel. More precisely, a and b govern the linear and nonlinear contributions in t, respectively, c acts as a scaling or shift parameter, and d controls the influence of the factor $(2x-1)^d$. Consequently, this generalized framework encompasses numerous previously studied polynomial families as special cases and provides a convenient setting for symbolic manipulation, operational techniques, and further analytical investigations. The developments described above demonstrate the effectiveness of generating-function methods in constructing broad classes of polynomial sequences with rich algebraic, combinatorial, and analytical structures. Motivated

by these results, we introduce and investigate a new family of generalized Catalan-type polynomials. Our approach combines pseudo-Bessel functions, Hermite-type structures, and generalized Humbert kernels within a unified generating-function framework. The resulting polynomial families admit explicit representations, recurrence relations, operational identities, and multivariable extensions, while simultaneously maintaining connections with several classical sequences. The main objective of this paper is to develop a systematic theory of these generalized Catalan-type polynomials and to explore their structural properties and applications. In particular, we derive explicit formulas and generating-function identities, establish relationships with Hermite, Humbert, and Jacobsthal-type polynomials, develop matrix representations including lower-triangular, Hankel, and Toeplitz forms, and investigate determinantal properties arising from these constructions. We also demonstrate how the proposed polynomials can be employed in the study of differential equations and related operational problems.

The paper is organized as follows. In Section 2, we introduce a generalized class of Catalan-type polynomials based on pseudo-Bessel functions and derive their fundamental properties, including explicit formulas and recurrence relations. Section 3 presents a broader family involving parameterized generating kernels and establishes connections with Humbert-type polynomials. In Section 4, we develop matrix representations of these polynomial families, including lower-triangular, Hankel, and Toeplitz matrices, and derive corresponding determinantal identities. Finally, Section 5 is devoted to applications, with particular emphasis on differential equations and the construction of polynomial solutions.

2Generalized Class of Catalan-Type Polynomials

In this section, we introduce a generalized family of Catalan-type polynomials based on the pseudo-Bessel function and derive several explicit formulas, recurrence relations, and generating-function identities.

Definition 1. Let $j \geq 1$, $k \geq 0$, and $t \in \mathbb{R}$. The *pseudo-Bessel function* is defined by

$$S(j,k,t) = \sum_{m=0}^{\infty} \frac{t^{jm+k}}{(jm+k)!}. \tag{5}$$

Special cases include

$$S(1,0,t) = e^t, \qquad S(2,0,t) = \frac{e^t + e^{-t}}{2},$$

and

$$S(3,0,t) = \frac{d}{dt} S(3,1,t) = \frac{d}{dt}\left[\frac{1}{3}e^t - \frac{2}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{3}\right)\right].$$

Further representations are given by

$$S(1,j,t) = \frac{t^j}{\Gamma(j+1)}\, {}_1F_1(1; j+1; t),$$

$$S(2,j,t) = \frac{t^j}{\Gamma(j+1)}\, {}_1F_2\left(1; \frac{j+1}{2}, \frac{j+2}{2}; \frac{t^2}{4}\right),$$

$$S(1,j,e^t-1) = \frac{(e^t-1)^j}{\Gamma(j+1)} \sum_{n=0}^{\infty} B_{n,j} \frac{t^n}{n!},$$

where $B_{n,j}$ denotes the jth Bell polynomial.

Definition 2. Let $S(j,k,t)$ be defined by Equation (5). The *generalized Catalan-type polynomials* $S_{n,m}^{j.k;\lambda}(x,y)$ are defined through the generating function

$$\frac{S(j,k,yt)}{(1-mt+xt^m)^\lambda} = \sum_{n=0}^{\infty} S_{n,m}^{j.k;\lambda}(x,y) t^n. \tag{6}$$

Theorem 1. *The explicit expression for* $S_{n,m}^{j.k;\lambda}(x,y)$ *is given by*

$$S_{n,m}^{j.k;\lambda}(x,y) = \sum_{r=0}^{\left\lfloor \frac{n-k}{j} \right\rfloor} \frac{y^{jr+k}\, p_{n-jr-k,m}^{\lambda}(x)}{(jr+k)!},$$

where $p_{n,m}^{\lambda}(x)$ *is defined by Equation* (4).

Proof. Using Equation (6) together with the generating function of $p_{n,m}^{\lambda}(x)$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} S_{n,m}^{j.k;\lambda}(x,y) t^n &= \left(\sum_{r=0}^{\infty} \frac{(yt)^{jr+k}}{(jr+k)!}\right)\left(\sum_{n=0}^{\infty} p_{n,m}^{\lambda}(x) t^n\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{\infty} \frac{y^{jr+k}}{(jr+k)!} p_{n-jr-k,m}^{\lambda}(x)\right) t^n. \end{aligned}$$

Since $jr+k \le n$, we must have

$$r \le \left\lfloor \frac{n-k}{j} \right\rfloor.$$

Hence,

$$\sum_{n=0}^{\infty} S_{n,m}^{j.k;\lambda}(x,y) t^n = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{\lfloor (n-k)/j \rfloor} \frac{y^{jr+k}}{(jr+k)!} p_{n-jr-k,m}^{\lambda}(x)\right) t^n.$$

Comparing coefficients of t^n completes the proof. $\square$

Remark 1. Setting $\lambda = 1$ in Equation (6) gives

$$S^{j.k;1}_{n,m}(x,y) = \sum_{r=0}^{\lfloor (n-k)/j \rfloor} \frac{y^{jr+k}\mathfrak{J}_{n-jr-k,m}(x)}{(jr+k)!},$$

where the generalized Jacobsthal polynomials are defined by

$$\mathfrak{J}_{n,m}(x) = \sum_{s=0}^{\lfloor (n-1)/m \rfloor} \binom{n-1-(m-1)s}{s} (2x)^s.$$

These reduce to the classical Jacobsthal numbers when $x = 1$ and $m = 2$. Furthermore,

$$\mathfrak{J}_{n,m}(x) = P_n\Big(m, \frac{x}{m}, -2, -1, 1\Big),$$

showing that they are special cases of Humbert polynomials. The particular cases $\mathfrak{J}_{n,2}$ and $j_{n,2}$ correspond to the classical Jacobsthal and Jacobsthal–Lucas numbers, respectively.

Definition 3. The *generalized class of three-variable polynomials* $H^{k;\lambda}_{n,m}(x,y,z)$ and $h^{k;\lambda}_{n,m}(x,y,z)$ is defined by the generating functions

$$\frac{e^{xt+yt^2} H_k(x+2yt,y)}{(1-mt+zt^m)^\lambda} = \sum_{n=0}^{\infty} H^{k;\lambda}_{n,m}(x,y,z)t^n, \tag{7}$$

and

$$\frac{e^{xt+yt^2} H_k(x+2yt,y)}{(1-mzt+t^m)^\lambda} = \sum_{n=0}^{\infty} h^{k;\lambda}_{n,m}(x,y,z)t^n,$$

where $H_k(x,y)$ is defined by Equations (2) and (3).

Theorem 2. *The polynomials $H^{k;\lambda}_{n,m}(x,y,z)$ and $h^{k;\lambda}_{n,m}(x,y,z)$ admit the explicit representations*

$$H^{k;\lambda}_{n,m}(x,y,z) = \sum_{r=0}^{n} \frac{H_{n-r+k}(x,y)\, P^\lambda_{r,m}(z)}{(n-r)!}, \tag{8}$$

and

$$h^{k;\lambda}_{n,m}(x,y,z) = \sum_{r=0}^{n} \frac{H_{n-r+k}(x,y)\, h^\lambda_{r,m}(z)}{(n-r)!}. \tag{9}$$

Proof. We prove Equation (8). Using Equation (7) together with Equation (3), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} H^{k;\lambda}_{n,m}(x,y,z)t^n &= \left(\sum_{n=0}^{\infty} H_{n+k}(x,y)\frac{t^n}{n!}\right)\left(\sum_{r=0}^{\infty} P^\lambda_{r,m}(z)t^r\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} H_{n-r+k}(x,y)\frac{P^\lambda_{r,m}(z)}{(n-r)!}\right) t^n.\end{aligned}$$

Comparing coefficients of t^n on both sides yields Equation (8). The proof of Equation (9) follows similarly. $\square$

Definition 4. The *generalized polynomial families* $F_{n,m}^{j.k;\lambda}(x,y)$ and $f_{n,m}^{j.k;\lambda}(x,y)$ are defined by

$$\frac{F(j,k,y,t)}{(1-mt+xt^m)^\lambda} = \sum_{n=0}^{\infty} F_{n,m}^{j.k;\lambda}(x,y)t^n, \tag{10}$$

and

$$\frac{F(j,k,y,t)}{(1-mxt+t^m)^\lambda} = \sum_{n=0}^{\infty} f_{n,m}^{j.k;\lambda}(x,y)t^n, \tag{11}$$

where $F(j,k,y,t)$ is defined by Equation (1).

Theorem 3. *For integers $j \geq 1$, $k \geq 0$, $r \geq 0$, and $m \geq 1$, let $F_{r,m}^{j.k;\lambda}(x,y)$ and $f_{r,m}^{j.k;\lambda}(x,y)$ be the generalized polynomial families defined by the generating functions* (10) *and* (11)*, respectively. Then these polynomials admit the explicit representations*

$$F_{r,m}^{j.k;\lambda}(x,y) = \sum_{n=0}^{\lfloor (r-k)/j \rfloor} \frac{H_{jn+k}(y)\, P_{r-jn-k,m}^{\lambda}(x)}{(jn+k)!},$$

and

$$f_{r,m}^{j.k;\lambda}(x,y) = \sum_{n=0}^{\lfloor (r-k)/j \rfloor} \frac{H_{jn+k}(y)\, h_{r-jn-k,m}^{\lambda}(x)}{(jn+k)!}.$$

Proof. The result follows by substituting the generating function $F(j,k,y,t)$ into Equations (10) and (11), expanding the corresponding series, and comparing coefficients of equal powers of t. $\square$

Definition 5. Let $A(x) \in \mathbb{Z}[x]$. The *generalized Catalan-type polynomials* $P_{n,m}^{\lambda,A}(x,y)$ and $p_{n,m}^{\lambda,A}(x,y)$ are defined by

$$\frac{1+A(x)t}{(1-mt+yt^m)^\lambda} = \sum_{n=0}^{\infty} P_{n,m}^{\lambda,A}(x,y)t^n,$$

and

$$\frac{1+A(x)t}{(1-myt+t^m)^\lambda} = \sum_{n=0}^{\infty} p_{n,m}^{\lambda,A}(x,y)t^n. \tag{12}$$

Setting $y=x$ yields the one-variable specialization

$$\frac{1+A(x)t}{(1-mxt+t^m)^\lambda} = \sum_{n=0}^{\infty} \Pi_{n,m}^{\lambda,A}(x)t^n, \tag{13}$$

where $\Pi_{n,m}^{\lambda,A}(x)$ denotes the corresponding reduced polynomial sequence.

From Equation (12), we obtain the recurrence

$$p_{n,m}^{\lambda,A}(x,y) = p_{n,m}^{\lambda}(x,y) + A(x)\, p_{n-1,m}^{\lambda,A}(x,y).$$

In the special case $y = x$, this becomes

$$\Pi_{n,m}^{\lambda,A}(x) = \Pi_{n,m}^{\lambda}(x) + A(x)\, \Pi_{n-1,m}^{\lambda,A}(x),$$

where $\Pi_{n,m}^{\lambda,A}(x)$ is defined by Equation (13). Some notable examples are

$$\sum_{n=0}^{\infty} G_{n,A}^{\lambda}(x)t^n = \frac{1 + A(x)t}{(1 - 2t + xt^2)^{\lambda}}, \qquad (y = x,\ m = 2),$$

and

$$\sum_{n=0}^{\infty} M_n(x)t^n = \frac{1 + (1 - x)t}{1 - xt + t^2}, \qquad \left(A(x) = 1 - x,\ y = \frac{x}{2}\right).$$

The initial values are

$$P_{0,m}^{\lambda,A}(x) = 1, \qquad P_{1,m}^{\lambda,A}(x) = A(x) + \lambda m, \qquad M_0(x) = M_1(x) = 1.$$

These generalized Catalan-type polynomials contain several classical families as special cases, including Gegenbauer and Humbert-type polynomials.

Definition 6. Let $A(x) \in \mathbb{Z}[x]$. The *unified Catalan-type polynomial family* is defined by

$$f_{m,\lambda,A}^{j,k}(x,y,t) = \frac{S(j,k,yt) + A(x)t}{(1 - mt + xt^m)^{\lambda}} = \sum_{n=0}^{\infty} P_{n,m}^{j.k;\lambda,A}(x,y)t^n. \tag{14}$$

For $j = 1$ and $k = 0$, Equation (14) reduces to

$$\frac{e^{yt} + A(x)t}{(1 - mt + xt^m)^{\lambda}} = \sum_{n=0}^{\infty} P_{n,m}^{1.0;\lambda,A}(x,y)t^n.$$

Furthermore, setting $y = 0$ yields

$$\frac{1 + A(x)t}{(1 - mt + xt^m)^{\lambda}} = \sum_{n=0}^{\infty} P_{n,m}^{\lambda,A}(x)t^n.$$

The following symmetry relation holds:

$$P_{n,m}^{1.0;\lambda,A}(x,y) + P_{n,m}^{1.0;\lambda,-A}(x,y) = 2P_{n,m}^{1.0;\lambda,0}(x,y).$$

Theorem 4. *The identity*

$$S_{n,m}^{j.k;\lambda}(x,y) = P_{n,m}^{1.0;\lambda,A}(x,y) + P_{n,m}^{\lambda,0}(x) - P_{n,m}^{\lambda,A}(x)$$

holds, where $S_{n,m}^{j.k;\lambda}(x,y)$ is defined by Equation (6).

Proof. Using Equation (14), we obtain

$$\frac{S(j,k,yt)}{(1-mt+xt^m)^\lambda} = \frac{S(j,k,yt)+A(x)t}{(1-mt+xt^m)^\lambda} + \frac{1+A(x)t}{(1-mt+xt^m)^\lambda} - \frac{1+A(x)t}{(1-mt+xt^m)^\lambda} - \frac{1}{(1-mt+xt^m)^\lambda}.$$

Expressing each term by its generating series gives

$$\sum_{n=0}^{\infty} S_{n,m}^{j.k;\lambda}(x,y)t^n = \sum_{n=0}^{\infty} P_{n,m}^{1.0;\lambda,A}(x,y)t^n + \sum_{n=0}^{\infty} P_{n,m}^{\lambda,0}(x)t^n - \sum_{n=0}^{\infty} P_{n,m}^{\lambda,A}(x)t^n.$$

Comparing coefficients of equal powers of t completes the proof. □

3More Generalized Class of Catalan-Type Polynomials

In this section, we introduce a broader family of Catalan-type polynomials associated with the generating kernel

$$R(x,t) = \left[c - axt + bt^m(2x-1)^d\right]^\lambda.$$

We derive recurrence relations, establish structural identities, and demonstrate connections with Humbert-type polynomial families.

Definition 7. Let

$$R(x,t) = \left[c - axt + bt^m(2x-1)^d\right]^\lambda.$$

The *generalized Catalan-type polynomial family* $P_{n,m}^{j.k;a,b,c,d,\lambda,A}(x,y)$ is defined by

$$g_{m,\lambda,A}^{j,k;a,b,c,d}(x,y,t) = \frac{S(j,k,yt)+A(x)t}{R(x,t)} = \sum_{n=0}^{\infty} P_{n,m}^{j.k;a,b,c,d,\lambda,A}(x,y)t^n, \tag{15}$$

where $A(x) \in \mathbb{Z}[x]$.

Some noteworthy specializations are

$$\frac{S(j,k,yt)}{R(x,t)} = \sum_{n=0}^{\infty} P_{n,m}^{j.k;a,b,c,d,\lambda,0}(x,y)t^n,$$

$$\frac{e^{yt}+A(x)t}{R(x,t)} = \sum_{n=0}^{\infty} P_{n,m}^{1.0;a,b,c,d,\lambda,A}(x,y)t^n,$$

and

$$\frac{1+A(x)t}{R(x,t)} = \sum_{n=0}^{\infty} \Psi_{n,m}^{\lambda,A}(x)t^n. \tag{16}$$

Theorem 5. *The polynomial sequence $\Psi^{\lambda,A}_{n,m}(x)$ satisfies the recurrence relation*

$$x^{-n/m}\Psi^{\lambda,A}_{n,m}(x) = \Psi^{\lambda,A}_{n,m}(x^{-1/m}) + x^{-1/m}A(x)\Psi^{\lambda,A}_{n-1,m}(x^{-1/m}).$$

In particular, when $A(x) = 0$,

$$x^{-n/m}\Psi^{\lambda}_{n,m}(x) = \Psi^{\lambda}_{n,m}(x^{-1/m}),$$

and consequently

$$A(x) = \frac{x^{-n/m}\Psi^{\lambda,A}_{n,m}(x) - \Psi^{\lambda,A}_{n,m}(x^{-1/m})}{x^{-1/m}\Psi^{\lambda,A}_{n-1,m}(x^{-1/m})}.$$

Proof. From Equation (16), we obtain

$$\sum_{n=0}^{\infty}\Psi^{\lambda,A}_{n,m}(x)t^n = \frac{1}{R(x,t)} + \frac{A(x)t}{R(x,t)}.$$

Since

$$R(x,t) = R(x^{-1/m}, x^{1/m}t),$$

it follows that

$$\sum_{n=0}^{\infty}\Psi^{\lambda,A}_{n,m}(x)t^n = \sum_{n=0}^{\infty}\Psi^{\lambda,A}_{n,m}(x^{-1/m})x^{n/m}t^n + x^{-1/m}A(x)\sum_{n=0}^{\infty}\Psi^{\lambda,A}_{n-1,m}(x^{-1/m})x^{n/m}t^n.$$

Comparing coefficients of equal powers of t yields the stated recurrence relation. The remaining identities follow immediately. $\square$

Several important instances of the generalized generating function are

$$g^{j,k;2,1,1,0}_{m,\lambda,A}(x,y,t) = \frac{S(j,k,yt) + A(x)t}{(1-2xt+t^m)^{\lambda}} = \sum_{n=0}^{\infty}P^{j.k;2,1,1,0,\lambda,A}_{n,m}(x,y)t^n,$$

$$g^{j,k;2,1,1,1}_{2,\lambda,A}(x,y,t) = \frac{S(j,k,yt) + A(x)t}{[1-2xt+t^2(2x-1)]^{\lambda}} = \sum_{n=0}^{\infty}P^{j.k;2,1,1,1,\lambda,A}_{n,2}(x,y)t^n,$$

$$g^{1,0;2,1,1,1}_{2,\lambda,A}(x,0,t) = \frac{1 + A(x)t}{[1-2xt+t^2(2x-1)]^{\lambda}} = \sum_{n=0}^{\infty}P^{1.0;2,1,1,1,\lambda,A}_{n,2}(x)t^n.$$

Theorem 6. *The identity*

$$P^{j.k;a,b,c,d,\lambda,A}_{n,m}(x,y) = P^{j.k;a,b,c,d,\lambda,0}_{n,m}(x,y) + P^{1.0;a,b,c,d,\lambda,A}_{n,m}(x) - P^{1.0;a,b,c,d,\lambda,0}_{n,m}(x) \tag{17}$$

holds.

Proof. Using Equation (15), we decompose the generating function as

$$\frac{S(j,k,yt)+A(x)t}{R(x,t)}=\frac{S(j,k,yt)}{R(x,t)}+\frac{1+A(x)t}{R(x,t)}-\frac{1}{R(x,t)}.$$

Expanding each term into its corresponding power series and comparing coefficients of like powers of t yields Equation (17). $\square$

Definition 8. Let

$$R(x,t)=[c-axt+bt^m(2x-1)^d]^\lambda.$$

The *auxiliary polynomial family* $Q_{n,m}^{\lambda,A,B}(x,y)$ is defined by

$$h_{m,\lambda,A,B}(x,y,t)=\frac{A(x)t+B(y)t}{R(x,t)}=\sum_{n=0}^{\infty}Q_{n,m}^{\lambda,A,B}(x,y)t^n,$$

where $A(x)$ and $B(y)$ are polynomials.

Important special cases are given by

$$\frac{A(x)t+B(y)t}{(1-mt+xt^m)^\lambda}=\sum_{n=0}^{\infty}U_{n,m}^{\lambda,A,B}(x,y)t^n,$$

$$\frac{A(x)t+B(y)t}{(1-myt+t^m)^\lambda}=\sum_{n=0}^{\infty}V_{n,m}^{\lambda,A,B}(x,y)t^n,$$

and

$$\frac{A(x)t+B(y)t}{[1-2xt+t^2(2x-1)]^\lambda}=\sum_{n=0}^{\infty}W_{n,m}^{\lambda,A,B}(x,y)t^n.$$

Consequently,

$$U_{n,m}^{\lambda,A,B}(x,y)=P_{n,m}^{\lambda,A}(x)+P_{n,m}^{\lambda,B}(x,y)-2P_{n,m}^{\lambda}(x,y),$$

$$V_{n,m}^{\lambda,A,B}(x,y)=p_{n,m}^{\lambda,A}(x)+p_{n,m}^{\lambda,B}(x,y)-2p_{n,m}^{\lambda}(x,y),$$

$$W_{n,m}^{\lambda,A,B}(x,y)=P_{n,2}^{1.0;2,1,1,1,\lambda,A}(x)+P_{n,2}^{1.0;2,1,1,1,\lambda,B}(x,y)-2P_{n,2}^{\lambda}(x).$$

4Matrix Representations and Determinantal Identities of Generalized Catalan-Type Polynomials

Matrix representations of polynomial sequences provide a powerful algebraic framework for studying recurrence relations, convolution identities, and structural properties. In this section, we develop matrix formulations for the generalized Catalan-type polynomials $P_{n,m}^{\lambda,A}(x)$ and establish determinantal identities that reveal both their algebraic structure and computational significance.

Definition 9. Let
$$\{P_{n,m}^{\lambda,A}(x)\}_{n=0}^{N}$$
be a sequence of generalized Catalan-type polynomials. The associated *lower-triangular matrix*
$$\mathcal{P}_N^{(m,\lambda,A)}(x) \in \mathbb{R}^{(N+1)\times(N+1)}$$
is defined by
$$\left[\mathcal{P}_N^{(m,\lambda,A)}(x)\right]_{i,j} = \begin{cases} P_{i-j,m}^{\lambda,A}(x), & \text{if } i \geq j, \\ 0, & \text{otherwise,} \end{cases} \qquad 0 \leq i,j \leq N. \tag{18}$$

This matrix encodes the polynomial sequence in a structured form and facilitates operations such as convolution, recurrence extraction, and transformation analysis.

Theorem 7. *Let $\mathcal{P}_N^{(m,\lambda,A)}(x)$ be defined by Equation* (18)*. Then*
$$\det\Big(\mathcal{P}_N^{(m,\lambda,A)}(x)\Big) = \big(P_{0,m}^{\lambda,A}(x)\big)^{N+1}, \tag{19}$$
provided that $P_{0,m}^{\lambda,A}(x) \neq 0$.

Proof. The matrix $\mathcal{P}_N^{(m,\lambda,A)}(x)$ is lower triangular. Hence, its determinant is equal to the product of its diagonal entries. By Equation (18),
$$\left[\mathcal{P}_N^{(m,\lambda,A)}(x)\right]_{i,i} = P_{0,m}^{\lambda,A}(x), \qquad 0 \leq i \leq N.$$
Therefore,
$$\det\Big(\mathcal{P}_N^{(m,\lambda,A)}(x)\Big) = \prod_{i=0}^{N} P_{0,m}^{\lambda,A}(x) = \big(P_{0,m}^{\lambda,A}(x)\big)^{N+1},$$
which completes the proof. □

Remark 2. Although Equation (19) follows directly from the triangular structure of the matrix, it has useful computational implications. In particular, the matrix $\mathcal{P}_N^{(m,\lambda,A)}(x)$ permits efficient evaluation of polynomial sequences by forward substitution. Moreover, its lower-triangular structure facilitates recursive computation since each entry depends only on previously computed values.

Definition 10. The polynomial vector associated with the sequence is
$$\mathbf{P}^{(m,\lambda,A)}(x) = \Big[P_{0,m}^{\lambda,A}(x) \quad P_{1,m}^{\lambda,A}(x) \quad \cdots \quad P_{N,m}^{\lambda,A}(x)\Big]^{\top}.$$

This vector representation serves as the foundation for constructing structured matrices, including Hankel and Toeplitz matrices.

Definition 11. The *Hankel matrix*

$$H_N^{(m,\lambda,A)}(x) \in \mathbb{R}^{(N+1)\times(N+1)}$$

is defined by

$$\left[H_N^{(m,\lambda,A)}(x)\right]_{i,j} = P_{i+j,m}^{\lambda,A}(x), \qquad 0 \le i,j \le N. \tag{20}$$

Theorem 8. *Let N be a positive integer and let $H_N^{(m,\lambda,A)}(x)$ be the Hankel matrix defined in Definition 11. Assume that the polynomial sequence $\{P_{n,m}^{\lambda,A}(x)\}_{n\ge 0}$ satisfies a linear recurrence relation of order r, where $1 \le r \le N$. Then*

$$\det\Big(H_N^{(m,\lambda,A)}(x)\Big) = 0. \tag{21}$$

Proof. Assume that the sequence satisfies the recurrence relation

$$\sum_{k=0}^{r} c_k P_{n-k,m}^{\lambda,A}(x) = 0, \qquad n \ge r, \tag{22}$$

where the constants c_k are not all zero. Each row of the Hankel matrix in Equation (20) consists of consecutive elements of the polynomial sequence. By Equation (22), every term $P_{n,m}^{\lambda,A}(x)$ with $n \ge r$ can be expressed as a linear combination of the preceding r terms. Consequently, every row with index $i \ge r$ is a linear combination of the first r rows. Hence, the rows of $H_N^{(m,\lambda,A)}(x)$ are linearly dependent, implying that the matrix is singular. Therefore,

$$\det\Big(H_N^{(m,\lambda,A)}(x)\Big) = 0.$$

□

Remark 3. The vanishing of the Hankel determinant in Equation (21) provides a practical criterion for detecting linear recurrence relations. In particular, if

$$\det\Big(H_N^{(m,\lambda,A)}(x)\Big) = 0,$$

then the corresponding sequence satisfies a recurrence relation of order at most N. This criterion is useful in symbolic computation for identifying minimal recurrence relations and verifying algebraic dependencies among polynomial sequences.

Definition 12. The *Toeplitz matrix*

$$T_N^{(m,\lambda,A)}(x)$$

is defined by

$$\left[T_N^{(m,\lambda,A)}(x)\right]_{i,j} = \begin{cases} P_{i-j,m}^{\lambda,A}(x), & i \ge j, \\ 0, & i < j. \end{cases}$$

Toeplitz matrices play an important role in convolution operations and admit efficient numerical algorithms because of their highly structured form.

Example 1. Let $m = 2$, $\lambda = 1$, and $A(x) = 1 - x$. Using

$$P_{0,2}^{1,A}(x) = 1, \qquad P_{1,2}^{1,A}(x) = 3 - x, \qquad P_{2,2}^{1,A}(x) = 5 - 5x + x^2,$$

and

$$P_{3,2}^{1,A}(x) = 7 - 14x + 7x^2 - x^3,$$

we obtain

$$\mathcal{P}_2^{(2,1,A)}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 3 - x & 1 & 0 \\ 5 - 5x + x^2 & 3 - x & 1 \end{bmatrix}.$$

The corresponding Hankel matrix is

$$H_2^{(2,1,A)}(x) = \begin{bmatrix} 1 & 3 - x & 5 - 5x + x^2 \\ 3 - x & 5 - 5x + x^2 & 7 - 14x + 7x^2 - x^3 \\ 5 - 5x + x^2 & 7 - 14x + 7x^2 - x^3 & P_{4,2}^{1,A}(x) \end{bmatrix}.$$

Evaluating at $x = 1$, we obtain

$$\mathcal{P}_2^{(2,1,A)}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix},$$

which illustrates the numerical structure of the lower-triangular matrix.

Example 2. For $N = 3$, the matrix $\mathcal{P}_3^{(2,1,A)}(x)$ is given by

$$\mathcal{P}_3^{(2,1,A)}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 - x & 1 & 0 & 0 \\ P_{2,2}^{1,A}(x) & 3 - x & 1 & 0 \\ P_{3,2}^{1,A}(x) & P_{2,2}^{1,A}(x) & 3 - x & 1 \end{bmatrix}.$$

This example highlights the recursive structure of the matrix and its dependence on the underlying polynomial sequence.

5Applications in Differential Equations

Generalized Catalan-type polynomials and their extensions possess rich algebraic structures that make them suitable for solving a variety of differential equations. In this section, we illustrate how these polynomials can be used to construct explicit solutions of linear differential equations, derive recurrence relations for polynomial coefficients, and provide basis functions for operator methods.

Example 3. We present a complete construction of a polynomial solution using the generating-function approach. Consider the differential equation

$$\frac{dy}{dt} = (1 - 2t + xt^2)\, y(t), \qquad y(0) = 1, \tag{23}$$

corresponding to the choice $m = 2$, $\lambda = 1$, and $A(x) = 1$. We seek a solution in the form

$$y(t) = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{24}$$

where $P_n(x) \equiv P_{n,2}^{1,1}(x)$. Differentiating Equation (24) term by term gives

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}. \tag{25}$$

Substituting Equations (24) and (25) into Equation (23), we obtain

$$\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = (1 - 2t + xt^2) \sum_{n=0}^{\infty} P_n(x)t^n.$$

Expanding the right-hand side yields

$$\sum_{n=0}^{\infty} P_n(x)t^n - 2\sum_{n=0}^{\infty} P_n(x)t^{n+1} + x\sum_{n=0}^{\infty} P_n(x)t^{n+2}.$$

After reindexing,

$$\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = \sum_{n=0}^{\infty} P_n(x)t^n - 2\sum_{n=1}^{\infty} P_{n-1}(x)t^n + x\sum_{n=2}^{\infty} P_{n-2}(x)t^n.$$

Comparing coefficients of equal powers of t, we obtain the recurrence relation

$$(n + 1)P_{n+1}(x) = P_n(x) - 2P_{n-1}(x) + xP_{n-2}(x), \qquad n \geq 1, \tag{26}$$

subject to the initial conditions

$$P_0(x) = 1, \qquad P_1(x) = 1.$$

Using Equation (26), we compute

$$P_2(x) = \frac{1}{2}\big(P_1(x) - 2P_0(x)\big) = -\frac{1}{2},$$

$$P_3(x) = \frac{1}{3}\big(P_2(x) - 2P_1(x) + xP_0(x)\big) = \frac{x - \frac{5}{2}}{3}.$$

Consequently,

$$y(t) = 1 + t - \frac{1}{2}t^2 + \frac{x - \frac{5}{2}}{3}t^3 + \cdots .$$

This example illustrates the general procedure: substitution of a generating series into the differential equation, coefficient comparison, derivation of a recurrence relation, and explicit computation of polynomial coefficients.

We now recall the classical differential equation satisfied by the Hermite polynomials,

$$y''(x) - 2xy'(x) + 2ny(x) = 0,$$

which arises naturally in the theory of the quantum harmonic oscillator. Motivated by this equation, we introduce the *generalized differential operator*

$$\mathcal{L}_m y = (1 - mt + xt^m)^\lambda \frac{d^2y}{dx^2} - \lambda mt \frac{dy}{dx}.$$

The operator $\mathcal{L}_m$ incorporates the generating kernel $(1 - mt + xt^m)^\lambda$, thereby reflecting the algebraic structure of the generalized Catalan-type polynomial family. We seek polynomial solutions of the form

$$y(x) = P^\lambda_{n,m}(x)$$

satisfying the eigenvalue equation

$$\mathcal{L}_m \left[P^\lambda_{n,m}(x)\right] = \mu_n P^\lambda_{n,m}(x),$$

where μ_n is the corresponding eigenvalue. This formulation yields a parameterized family of differential equations whose solutions can be constructed through generating functions. Let

$$\mathcal{D}_x = \frac{d}{dx}$$

and define the operator

$$\mathcal{T}_m = (-\mathcal{D}_x + mx)^n.$$

When applied to monomials, this operator generates polynomial families associated with Hermite-type structures. For example,

$$\mathcal{T}_2[x^n] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2x)^{\,n-k} H_k(x),$$

where $H_k(x)$ denotes the Hermite polynomial of degree k. We next consider the generalized differential equation

$$(1 - mt + xt^m)^\lambda \frac{dy}{dt} + A(x)y(t) = 0, \tag{27}$$

with initial condition $y(0) = 1$. Assuming the series representation

$$y(t) = \sum_{n=0}^{\infty} P_{n,m}^{\lambda,A}(x) t^n,$$

and substituting it into Equation (27), one obtains recurrence relations for the coefficients $P_{n,m}^{\lambda,A}(x)$ by comparing coefficients of equal powers of t. Finally, consider the modified heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta x \frac{\partial u}{\partial x},$$

subject to the initial condition

$$u(x, 0) = P_{n,m}^{\lambda}(x).$$

Using separation of variables, the solution may be expanded in the form

$$u(x,t) = \sum_{n=0}^{\infty} c_n(t) P_{n,m}^{\lambda}(x),$$

where the coefficient functions $c_n(t)$ satisfy a corresponding system of differential equations. This representation demonstrates the usefulness of generalized Catalan-type polynomials as basis functions in diffusion-type problems and related operator-theoretic investigations.

6Conclusion

This paper has introduced and analyzed a generalized class of Catalan-type polynomials constructed through generating functions involving pseudo-Bessel functions, Hermite polynomials, and Humbert-type structures. Explicit expressions, recurrence relations, and multi-variable extensions were derived, along with matrix representations such as lower-triangular, Hankel, and Toeplitz forms. These formulations revealed determinantal identities and structural properties that connect the generalized polynomials to classical families, including Jacobsthal, Gegenbauer, and Horadam polynomials.

Applications in differential equations demonstrated the analytical utility of the proposed polynomials. The framework developed here provides a unified approach to generalizing classical polynomial sequences and opens avenues for further research in symbolic computation, orthogonal polynomials, and special function theory.

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