EXACT COVERING SYSTEM DIGRAPHS OF DEGREE 2 AND NON-STANDARD BINARY REPRESENTATIONS OF INTEGERS

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Abstract

Given an exact covering system $S = \{x \equiv a_i \pmod{d_i} : 1 \leq i \leq r\}$, we define the corresponding exact covering system digraph (ECSD) G_S. The vertices of G_S are the integers, and the edges are $(n, d_i n + a_i)$ for each $n \in \mathbb{Z}$ and for each congruence in the covering system. We examine the structure of ECSDs corresponding to covering systems with two congruences, completely characterizing the cyclic vertices in each case. Using these results, we show that every integer has a representation as $\sum_{j=0}^{k} b_j(-2)^j$ for $b_j \in \{1, a\}$ and some $k \in \mathbb{N}$ if and only if $a = 1 \pm 3^m$ for some $m \in \mathbb{N}_0$. Moreover, we describe all non-standard binary representations that represent all integers.

1. Exact Covering System Digraphs

In this paper, we build upon the framework in [11] regarding exact covering system digraphs (ECSDs) and characterize all ECSDs of degree 2. We use these results to completely describe all non-standard binary representations that represent all integers.

For integers a_i, d_i, and $r \geq 1$, an *exact covering system* is a system of congruences $\{x \equiv a_i \pmod{d_i} : 1 \leq i \leq r\}$ such that every integer n satisfies $n \equiv a_i \pmod{d_i}$ for exactly one value of i. The most basic covering system, and the one we expand upon in this paper, is the system $\{x \equiv 0 \pmod 2,\ x \equiv 1 \pmod 2\}$. Covering systems and their properties have been extensively studied since their introduction by Erdős [7], and hundreds of papers have been written regarding them (see [4, 14, 15]).

As congruences in exact covering systems correspond to equivalence classes, choosing an equivalent representative a_i does not change the congruence, and choosing $-d_i$ as a modulus instead of d_i also does not change the congruence. We would like to distinguish between different representatives, so we necessarily require a *representative set* $S = \{(a_i, d_i) \in \mathbb{Z}^2 : 1 \leq i \leq r\}$. For example, we would use the

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representative sets $S_1 = \{(0,2),(1,2)\}$ and $S_2 = \{(2,-2),(5,2)\}$ to distinguish between the two equivalent covering systems $\{x \equiv 0 \pmod 2,\ x \equiv 1 \pmod 2\}$ and $\{x \equiv 2 \pmod{-2},\ x \equiv 5 \pmod 2\}$.

Given a representative set $S = \{(a_i, d_i) \in \mathbb{Z}^2 : 1 \le i \le r\}$ such that the system of congruences $\{x \equiv a_i \pmod{d_i} : 1 \le i \le r\}$ is an exact covering system, the corresponding *exact covering system digraph (ECSD)* is denoted $G_S := (V, E)$, where $V = \mathbb{Z}$, and $E = \{(n, d_i n + a_i) : n \in \mathbb{Z}, 1 \le i \le r\}$. To simplify notation, as shorthand we denote G_S as $G(d_1 n + a_1, \ldots, d_r n + a_r)$. This is notably an abuse of function notation, as the parameter $d_i n + a_i$ is not a number, but a representation of one of the particular integers to which each vertex $n \in \mathbb{Z}$ is sent. Thus, the technical details of choosing a covering system, representative set, and constructing the graph can be evocatively displayed as simply a set of r "rules" that describe the r out-neighbors of each vertex. We say that r is the *degree* of G_S.

An ECSD has indegree 1 at each vertex, and therefore every $n \in \mathbb{Z}$ has a unique *predecessor*, denoted $p(n)$. A vertex m is the *(k-)ancestor* of another vertex n (and we also say n is a *(k-)descendant* of m) if k successive predecessors of n give m, and we say $p^k(n) = m$. We call any 1-descendant a *successor*.

In our previous paper [11], we showed that in an ECSD of degree greater than or equal to 2, each of finitely many components has a single cycle. Thus, the structure of each component of an ECSD of degree $r \ge 2$ is a directed cycle, in which each vertex of the cycle, or *cyclic vertex*, is the root of a tree with $r - 1$ successors, each of which is the root of an infinite r-ary tree [11] (see Figures 2, 3, 4, 5, 6, 8, and 11 for examples). Here, one may observe that every ancestor of a cyclic vertex is itself a cyclic vertex.

Because of this structure, ECSDs of the same degree only differ up to isomorphism by the number of components and the number of cyclic vertices per component. Two such isomorphisms arise from automorphisms on the integers, although these isomorphisms alone do not account for every isomorphism of ECSDs.

Theorem 1 ([11])**.** *For each $k \in \mathbb{Z}$ and representative sets for exact covering systems*

$$S = \{(a_i, d_i)\ : 1 \le i \le r\}$$
$$S' = \{(a_i - (d_i - 1)k, d_i) : 1 \le i \le r\}$$

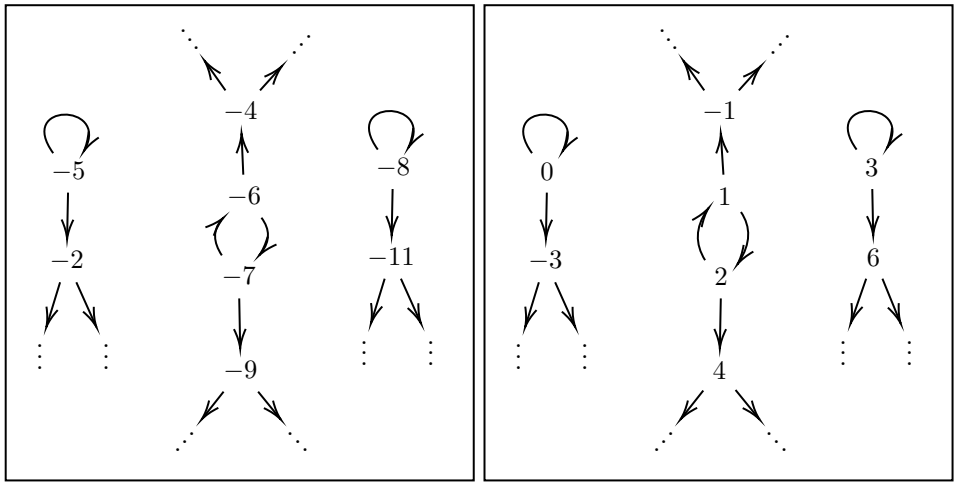
the map $\phi_{1,k} : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi_{1,k}(n) = n + k$ is a graph isomorphism between G_S and $G_{S'}$.

Theorem 2 ([11])**.** *For each $k \in \mathbb{Z}$ and representative sets for exact covering systems*

$$S = \{(a_i, d_i)\ : 1 \le i \le r\}$$
$$S' = \{(-a_i + (d_i - 1)k, d_i) : 1 \le i \le r\}$$

the map $\phi_{2,k} : \mathbb{Z} \to \mathbb{Z}$ *defined by* $\phi_{2,k}(n) = -(n+k)$ *is a graph isomorphism between* G_S *and* $G_{S'}$.

As an example, we can use Theorem 2 to determine that $G(2n+8, 2n+5)$ and $G(2n, 2n-3)$ are isomorphic via $\phi_{2,5}(n) = -(n+5)$. The two ECSDs are shown side by side in Figure 1, and we can indeed say that they have the same structure: the corresponding vertices shown in each ECSD are the images under $\phi_{2,5}$.



$G(2n+8,\ 2n+5)$ $G(2n,\ 2n-3)$

Figure 1: The two isomorphic ECSDs $G(2n+8, 2n+5)$ and $G(2n, 2n-3)$.

Since ECSDs are uniquely determined by their cycles, in Section 2 we focus our results on characterizing the cycles of all ECSDs of degree 2. We first use the above isomorphisms to simplify each of three main types of ECSDs to single-parameter families. Then, for every ECSD of degree 2, we determine the cyclic vertices, and in some cases, determine the cycles themselves.

In Section 3, we apply these results to non-standard binary representations of integers. If an ECSD with 0 a cyclic vertex has a single component, the ECSD corresponds to a non-standard binary representation that can represent every integer [11]. We find all ECSDs of degree 2 with a single component, and show that

$$\mathbb{Z} = \left\{ \sum_{j=0}^{k} b_j(-2)^j : b_j \in \{1, a\}, k \in \mathbb{N} \right\}$$

if and only if $a = 1 \pm 3^m$ for some $m \in \mathbb{N}_0$. Moreover, we describe all 2-digit non-standard binary representations that represent all integers.

2. Characterizing Cycles of ECSDs of Degree 2

In this section, we classify all ECSDs of degree 2, and describe the cycles of all degree 2 ECSDs. Every representative set for an exact covering system with two congruences must be of the form

$$S = \{(a_1, \pm 2), (a_2, \pm 2)\},$$

where a_1 and a_2 have opposite parity (and the two moduli need not have the same sign). Therefore, any ECSD G_S of degree 2 must be of the form

$$G_S = G(\pm 2n + a_1, \pm 2n + a_2).$$

Thus, we can see three natural categories of degree 2 ECSDs by choice of representative of the modulus: Type 1, where both pairs have modulus 2, Type 2, where one pair has modulus 2 and the other has modulus -2, and Type 3, where both pairs have modulus -2. The three types are illustrated in Table 1. The isomorphism statements follow from Theorems 3, 4, and 5.

Type	ECSD	Isomorphic to:
Type 1	$G(2n + a_1, 2n + a_2)$	$G(2n, 2n - a)$, $a \in \mathbb{N}$ odd
Type 2	$G(2n + a_1, -2n + a_2)$	$G(2n, -2n + a)$, $a \in \mathbb{N}$ odd
Type 3	$G(-2n + a_1, -2n + a_2)$	$G(-2n, -2n + a)$, $a \in \mathbb{N}$ odd $G(-2n + 1, -2n + a)$, $a \in \mathbb{N}$ even, $a \equiv 1 \pmod 3$ $G(-2n + 1, -2n + a)$, $a \in \mathbb{Z}$ even, $a \equiv 2 \pmod 3$

Table 1: The three main types of degree 2 ECSDs and their simplified forms.

In each of the three cases, we can use the two isomorphisms of ECSDs presented in the previous section to simplify our task when characterizing the cycles of such ECSDs. Indeed, instead of dealing with two parameters a_1 and a_2, in each case we can reduce to a single parameter using isomorphisms.

Theorem 3. *Every ECSD of Type 1 is isomorphic to an ECSD of the form $G(2n, 2n - a)$ with $a \in \mathbb{N}$ odd.*

Proof. By Theorem 1 with $k = a_1$, we have that

$$G(2n + a_1, 2n + a_2) \cong G(2n, 2n + a_2 - a_1).$$

Since a_1 and a_2 have opposite parity, $a_2 - a_1 = a$ is odd. By Theorem 2 with $k = 0$, we have that

$$G(2n, 2n + a) \cong G(2n, 2n - a),$$

allowing us to restrict to $G(2n, 2n - a)$ with $a \in \mathbb{N}$ odd. □

We choose to represent Type 1 ECSDs by $G(2n, 2n-a)$ instead of $G(2n, 2n+a)$ because the former has all of its cyclic vertices non-negative, whereas none of the latter's cyclic vertices are positive (as we will see later in this section). One example of an ECSD of Type 1 is $G(2n, 2n-9)$, shown in Figure 2.

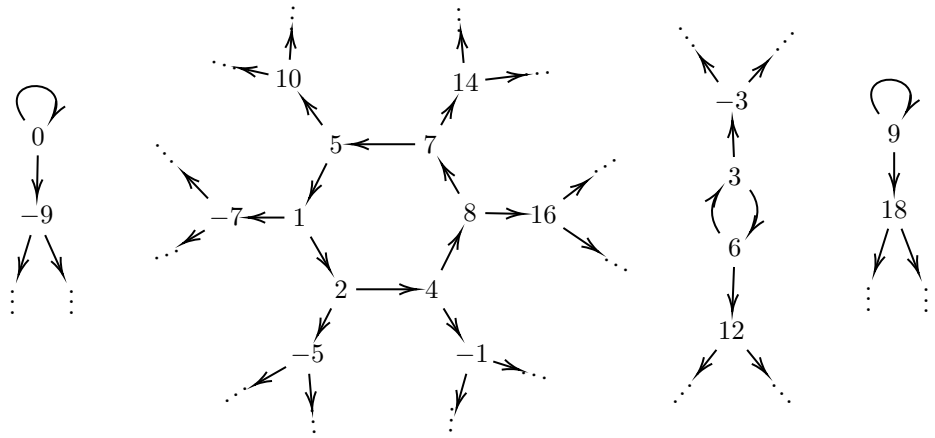


Figure 2: $G(2n, 2n-9)$, a Type 1 ECSD.

One might think that ECSDs of Type 2 would be more complicated than ECSDs of Type 1 due to the opposite signs of the bases, but in fact, they have just as simple a form. One example of an ECSD of Type 2 is $G(2n, -2n+9)$, shown in Figure 3.

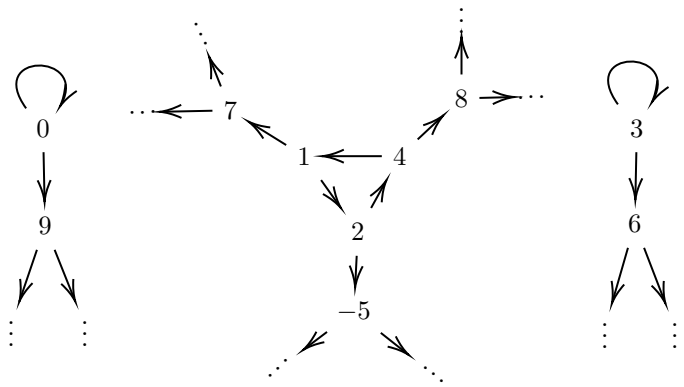


Figure 3: $G(2n, -2n+9)$, a Type 2 ECSD.

Theorem 4. *Every ECSD of Type 2 is isomorphic to an ECSD of the form* $G(2n, -2n + a)$ *with* $a \in \mathbb{N}$ *odd.*

Proof. Similarly to the proof of Theorem 3, we can eliminate the first parameter by Theorem 1 with $k = a_1$ and restrict to positive a by Theorem 2 with $k = 0$. □

Finally, ECSDs of Type 3 are the most complicated: they cannot be simplified to a single family of ECSDs with one parameter. Instead, they must be broken into three sub-types, each of which we will see are not isomorphic to any of the others: Type 3.1 ECSDs are the only Type 3 ECSDs with cycles of length 1 (loops), Type 3.2 ECSDs have an odd number of cyclic vertices, and Type 3.3 ECSDs have an even number of cyclic vertices (we prove these claims later in the section). Examples of each of the three sub-types can be seen in Figures 4, 5, and 6.

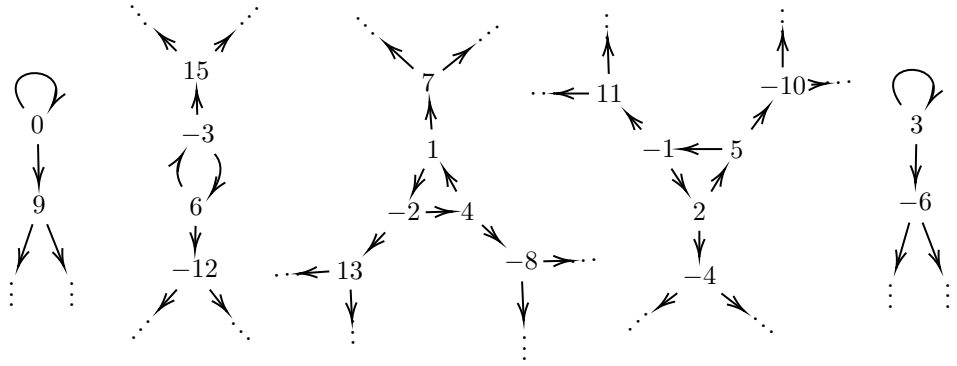


Figure 4: $G(-2n, -2n + 9)$, a Type 3.1 ECSD.

Theorem 5. *Every ECSD of Type 3 is isomorphic to an ECSD of one of the following forms:*

Type 3.1: $G(-2n, -2n + a)$ *for* $a \in \mathbb{N}$ *odd*
Type 3.2: $G(-2n + 1, -2n + a)$ *for* $a \in \mathbb{N}$ *even,* $a \equiv 1 \pmod 3$
Type 3.3: $G(-2n + 1, -2n + a)$ *for* $a \in \mathbb{Z}$ *even,* $a \equiv 2 \pmod 3$.

Proof. Consider an ECSD $G(-2n + a_1, -2n + a_2)$.
`Case 1:` Neither of the a_i are equal to 0 or 1. If one of the a_i is congruent to 0 modulo 3, without loss of generality say a_1 is, then, by the isomorphism in Theorem 2 with $k = -\frac{a_1}{3}$, we have that

$$G(-2n + a_1, -2n + a_2) \cong G(-2n, -2n + a_1 - a_2),$$

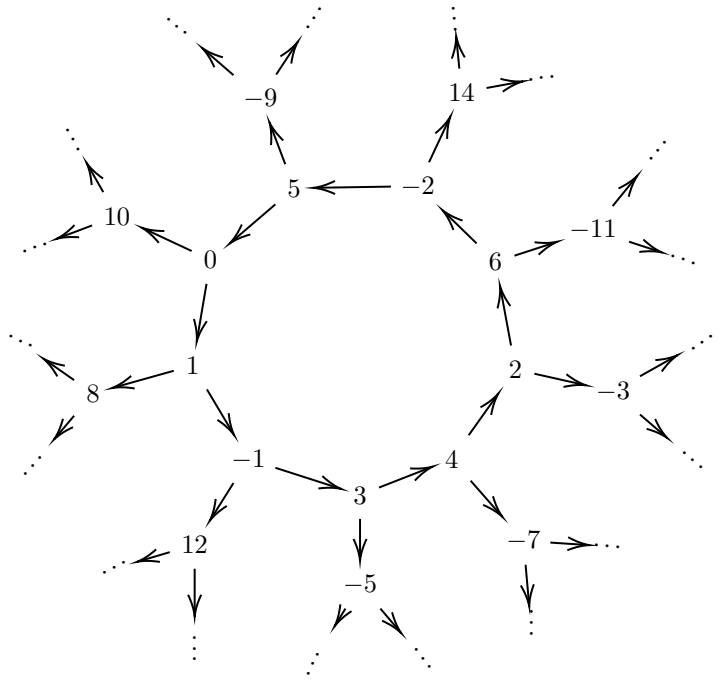


Figure 5: $G(-2n+1,-2n+10)$, a Type 3.2 ECSD.

and it is isomorphic to an ECSD where one of the a_i is equal to 0. If one of the a_i is congruent to 2 modulo 3, without loss of generality say a_1 is, then, by the isomorphism in Theorem 2 with $k=-\frac{a_1+1}{3}$, we have that

$$G(-2n+a_1,-2n+a_2)\cong G(-2n+1,-2n+a_1-a_2+1),$$

and it is isomorphic to an ECSD where $a_1=1$. Otherwise, we have that both a_i are congruent to 1 modulo 3. By the isomorphism in Theorem 1 with $k=-\frac{a_1-1}{3}$, we have that

$$G(-2n+a_1,-2n+a_2)\cong G(-2n+1,-2n+a_2-a_1+1),$$

and it is isomorphic to an ECSD where $a_1=1$. Thus, each ECSD of this type is isomorphic to an ECSD with one of the a_i equal to 0 or 1.

`Case 2:` One of the a_i is equal to 0. By Theorem 2 with $k=0$, we have that

$$G(-2n,-2n+a)\cong G(-2n,-2n-a),$$

so we only need to consider positive values of a, giving us Type 3.1.

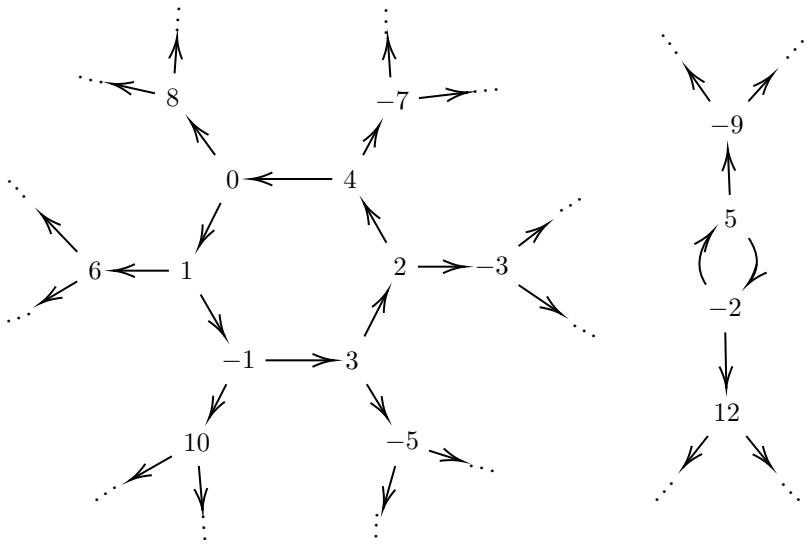


Figure 6: $G(-2n+1,-2n+8)$, a Type 3.3 ECSD.

`Case 3:` One of the a_i is equal to 1. Let a denote the other a_i, so a is even, and we have $G(-2n+1,-2n+a)$.

Suppose $a \equiv 0 \pmod 3$. By Theorem 2 with $k = -\frac{a}{3}$, we see that

$$G(-2n+1,-2n+a) \cong G(-2n+a-1,-2n)$$

and it is isomorphic to an ECSD with one of the two a_i equal to 0, which falls under Case 2. Otherwise, $a \equiv 1 \pmod 3$ or $a \equiv 2 \pmod 3$. If $a \equiv 2 \pmod 3$, we have Type 3.3.

If $a \equiv 1 \pmod 3$ and $a < 0$, then, by Theorem 1 with $k = -\frac{1}{3}(a-1)$, we have that

$$G(-2n+1,-2n+a) \cong G(-2n+1,-2n-a+2)$$

where $-a+2 > 0$, which allows us to restrict to $a \in \mathbb{N}$. This gives us Type 3.2. □

Using the following results, we will see that within each of the three main cases, the above simplifications are the best we can do: that is, within Type 1, Type 2, or Type 3, no two ECSDs are isomorphic. However, isomorphisms still exist between types, for example $G(2n,2n-a) \cong G(-2n,-2n+a)$ for certain values of a.

Since we have now determined five one-parameter types of degree 2 ECSDs, the remainder of this section will examine each type in turn.

2.1. Type 1: $G(2n, 2n-a)$, $a \in \mathbb{N}$ Odd

In this subsection we can completely describe the cycles of degree ECSDs of Type 1, which are isomorphic to $G(2n, 2n-a)$ with $a \in \mathbb{N}$ odd. These are the ECSDs of degree 2 that we know the most about, as they are nicely structured and we can completely determine their cycles. The 1 and 2-descendants of m are shown in Figure 7.

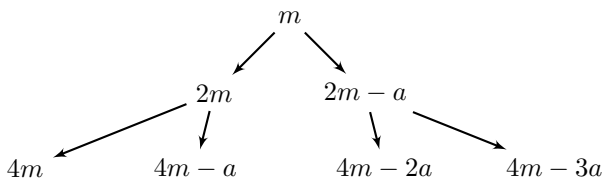


Figure 7: The 1- and 2-descendants of m in $G(2n, 2n-a)$.

First, we must introduce some notation. Let $\text{ord}(2, a)$ denote the multiplicative order of 2 in $\mathbb{Z}/a\mathbb{Z}$, and for $n \in \mathbb{Z}$, let $\overline{n}$ be the unique integer in $[0, a-1]$ such that $\overline{n} \equiv n \pmod{a}$.

We define the term cocycle similarly to the definition of coset in group theory.[1] If $C = (c_1, c_2, \ldots, c_k)$ is a cycle in an ECSD, its *cocycle* $\overline{bC}$ is $\left(\overline{bc_1}, \overline{bc_2}, \ldots, \overline{bc_k}\right)$, in this case where each element is reduced modulo a. We may now precisely describe the cycles of $G(2n, 2n-a)$ for $a > 0$.

Theorem 6. *Let $a \in \mathbb{N}$ be odd. The following statements are true about the ECSD $G(2n, 2n-a)$.*

(i) The cycles (0) and (a) are cycles of length 1, and these are the only cycles of length 1.

(ii) An integer m is a cyclic vertex if and only if $0 \le m \le a$.

(iii) The cycle $C = (\overline{2^k} : 0 \le k < \text{ord}(2, a))$ is a cycle of length $\text{ord}(2, a)$, and its cocycles $\{\overline{bC} : 1 \le b < a\}$ are also cycles, each of length $\text{ord}\left(2, \frac{a}{\gcd(a,b)}\right)$. Every cycle of length greater than 1 is of this form.

(iv) Whenever odd $a' \in N$ is such that $a = ma'$, we have that $C = (c_1, \ldots, c_t)$ is a cycle in $G(2n, 2n-a')$ if and only if $mC = (mc_1, \ldots, mc_t)$ is a cycle in $G(2n, 2n-a)$.

Proof. (i) Suppose (m) is a cycle of length 1. If $m = 2m$, then $m = 0$. If $m = 2m-a$, then $m = a$.

[1] Note that our use of the word cocycle is not related to its use in algebraic topology.

(ii) If $m > a$ or $m < 0$, each of its successors m' have $|m'| > |m|$, so m cannot be a descendant of itself and is therefore not a cyclic vertex.

Alternatively, if $0 \leq m \leq a$, it has a successor m_1 such that $0 \leq m_1 \leq a$. (Either $0 \leq 2m \leq a$ if $m \leq \frac{a}{2}$ or $0 \leq 2m - a \leq a$ if $m > \frac{a}{2}$.) Similarly m_1 has a successor between 0 and a, call this successor m_2. Thus, we can construct an infinite sequence $m, m_1, m_2, \ldots$ such that $0 \leq m_i \leq a$, so this sequence has repetitions and m is the ancestor of some cyclic vertex. Because an ancestor of a cyclic vertex must itself be a cyclic vertex, then m is also a cyclic vertex.

(iii) Each step along the cycle is either $(n, 2n)$ or $(n, 2n - a)$. We know that the elements in cycles of length more than 1 are limited to $\{1, 2, \ldots, a-1\}$, so the edge in the cycle is $(n, 2n - a)$ if and only if $2n \geq a$.

Because $2n - a \equiv 2n \pmod{a}$, we have that the elements in the cycle starting with 1 are $\{\overline{2^k} : 0 \leq k < \text{ord}(2, a)\}$, which are distinct by definition.

Cocycles $\{\overline{bC} : 1 \leq b < a\}$ are also cycles: suppose we start with b instead of 1. Consider the integers $\overline{b}, \overline{2b}, \ldots, \overline{b2^k}, \ldots$. The length of $\overline{bC}$ is the minimum k such that $b2^k \equiv b \pmod{a}$, or equivalently such that $a \mid (2^k - 1)b$. Let $g = \gcd(a, b)$, so $a = ga'$ and $b = gb'$, where $\gcd(a', b') = 1$. Then,

$$a \mid (2^k - 1)b \text{ if and only if } ga' \mid (2^k - 1)gb' \text{ if and only if } a' \mid 2^k - 1,$$

or equivalently $2^k \equiv 1 \pmod{a'}$.

Since we can let b be any element of $\{1, 2, \ldots, a-1\}$, every cycle of length greater than 1 must be a cocycle of the form $\overline{bC}$.

(iv) Let $C = (c_1, \ldots, c_t)$ be a cycle in $G(2n, 2n-a')$. Consider $mC = (mc_1, \ldots, mc_t)$. Let c_i be an element of C. Since $0 \leq c_i \leq a'$, we have that $0 \leq mc_i \leq ma' = a$. If $c_{i+1} = 2c_i$, then clearly $mc_{i+1} = 2mc_i$. Similarly, if $c_{i+1} = 2c_i - a'$, then $mc_{i+1} = 2mc_i - ma' = 2mc_i - a$ (replace c_{i+1} with c_1 if $i = t$). Thus, mC is a cycle of $G(2n, 2n - a)$ of the same length (and same edge types) as C.

Similarly, consider a cycle $K = (k_1, \ldots, k_t)$ in $G(2n, 2n - a)$ where $m \mid k_i$ for all $1 \leq i \leq t$. If k_i is an arbitrary element, $0 \leq \frac{k_i}{m} \leq \frac{a}{m} = a'$. Moreover, if $k_{i+1} = 2k_i$, then $\frac{k_{i+1}}{m} = 2k_i m$ and if $k_{i+1} = 2k_i - a$, then $\frac{k_{i+1}}{m} = 2\frac{k_i}{m} - a'$, so $K/m = \left(\frac{k_1}{m}, \ldots, \frac{k_t}{m}\right)$ is a cycle of $G(2n, 2n - a')$ of the same length (and same edge types) as K. □

The main distinction between cosets and cocycles is that different cocycles need not have the same length. Take the example of the ECSD $G(2n, 2n - 9)$, shown in Figure 2. We have the cycle $C = (1, 2, 4, 8, 7, 5)$ given to us by

$$C = (\overline{2^k} : 0 \leq k < \text{ord}(2, 9))$$

as in Theorem 6, Statement (iii), which has length $\text{ord}(2, 9) = 6$. The corresponding cocycle $\overline{3C}$ is

$$(\overline{1 \cdot 3}, \overline{2 \cdot 3}, \overline{4 \cdot 3}, \overline{8 \cdot 3}, \overline{7 \cdot 3}, \overline{5 \cdot 3}) = (3, 6, 3, 6, 3, 6) = (3, 6),$$

which has length ord $\left(2, \frac{9}{\gcd(9,3)}\right) = \text{ord}(2,3) = 2$.

The reason cocycles of non-equal size occur is because we are working in rings $\mathbb{Z}/a\mathbb{Z}$ where not every non-identity element is a generator. This would not happen in a field, as $\gcd(a,b)$ would always be 1, giving us the following observation.

Observation 1. If a is prime, then all the cycles of $G(2n, 2n-a)$ other than (0) and (a) must have the same size.

An example of this is $G(2n, 2n-17)$, shown in Figure 8. The cocycles are all length $\text{ord}(2,17) = 8$, and there are $16/8 = 2$ of them. Another example is $G(2n, 2n-31)$, which in addition to the two cycles of length 1 has 6 cycles of length 5.

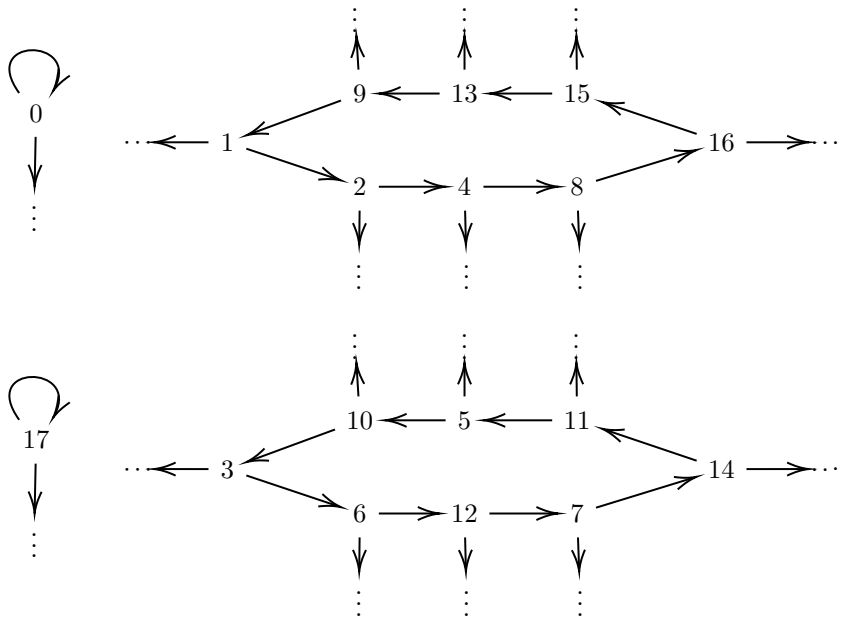


Figure 8: The ECSD $G(2n, 2n-17)$.

In most cases, finding the exact number of components of an ECSD is very difficult, but in the case of Type 1 ECSDs, we can determine the number of components easily because we know the size of the cycles. We can make the following observation about the number of components of an ECSD of Type 1.

Observation 2. The number of components of $G(2n, 2n-a)$ is

$$2 + \sum_{k=1}^{a-1} \frac{1}{\text{ord}\left(2, \frac{a}{\gcd(a,k)}\right)}.$$

Proof. There are two components with a cycle of length 1, and the other cyclic vertices are $k \in \mathbb{N}$ such that $1 \leq k \leq a-1$. For each of these cyclic vertices, the length of the cycle containing k is $\operatorname{ord}\left(2, \frac{a}{\gcd(a,k)}\right)$, so by summing $\frac{1}{\operatorname{ord}\left(2, \frac{a}{\gcd(a,k)}\right)}$ over all cyclic vertices we get the number of components that do not contain cycles of length 1. □

This sequence of the number of components of $G(2n, 2n-a)$ indexed by a is listed in the OEIS as Sequence A327551 [13]. In [2], it is made clear how this sequence arises in this context due to the *doubling modulo an odd integer* map given by $x \mapsto 2x \pmod{a}$ where a is an odd integer.

To help illustrate Theorem 6, Statement (iv), consider for example the ECSDs $G(2n, 2n-3)$, $G(2n, 2n-5)$, and $G(2n, 2n-15)$. The cycles of each are shown in Table 2. We can see that the cycle $(5, 10)$ in $G(2n, 2n-15)$ is the cycle $5C = (5 \cdot 1, 5 \cdot 2)$ for $C = (1, 2)$ in $G(2n, 2n-3)$. Similarly, the cycle $(3, 6, 12, 9)$ in $G(2n, 2n-15)$ is the cycle $3K$ for $K = (1, 2, 4, 3)$ in $G(2n, 2n-5)$. We can also see the cocycle structure: $(5, 10) = \overline{5C}$ for $C = (1, 2, 4, 8)$ in $G(2n, 2n-15)$, although the sizes of the cycles are different.

ECSD	Cycles
$G(2n, 2n-3)$	(0), $(1, 2)$, (3)
$G(2n, 2n-5)$	(0), $(1, 2, 4, 3)$, (5)
$G(2n, 2n-15)$	(0), $(1, 2, 4, 8)$, $(3, 6, 12, 9)$, $(5, 10)$, $(7, 14, 13, 11)$, (15)

Table 2: Cycles of the ECSDs $G(2n, 2n-3)$, $G(2n, 2n-5)$, and $G(2n, 2n-15)$.

2.2. Type 2: $G(2n, -2n+a)$

Although this case looks rather complicated due to the mismatched signs of bases, this case is similar to Type 1 ECSDs, although we cannot determine the lengths of cycles as easily. We begin by classifying the k-descendants of n in $G(2n, -2n+a)$.

Proposition 1. *The k-descendants of m in $G(2n, -2n+a)$ are $2^k m$, $\pm(2^k m - ba)$ for $1 \leq b < 2^{k-1} \in \mathbb{N}$, and $-(2^k - 2^{k-1}a)$.*

Proof. We induct on k. Figure 9 shows the 1 and 2-descendants of n. Assume that the k-descendants of m are $2^k m$, $\pm(2^k m - ba)$ for $1 \leq b < 2^{k-1} \in \mathbb{N}$, and $-(2^k - 2^{k-1}a)$. From a vertex $j \in \mathbb{Z}$, let $2j$ be called its *left successor* and $-2j + a$ be called its *right successor* (see Figure 9).

- We have that $2^{k+1}n$ is the left successor of $2^k n$, so it is a $k+1$-descendant of n.
- We have that$-(2^{k+1} - 2^k a)$ is the left successor of $-(2^k - 2^{k-1}a)$.

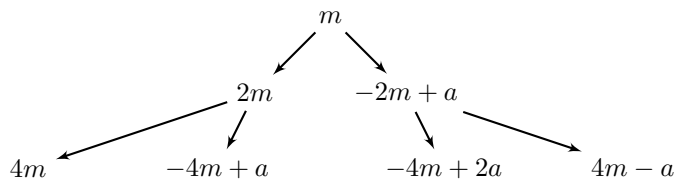


Figure 9: The 1 and 2-descendants of m in $G(2n, -2n+a)$.

- When $1 \leq c < 2^k \in \mathbb{N}$ is even and $c = 2b$, we have that $\pm(2^{k+1}m - ca)$ is the left successor of $\pm(2^k m - ba)$ with $1 \leq b < 2^{k-1} \in \mathbb{N}$.
- When $1 \leq c < 2^k \in \mathbb{N}$ is odd and $c = 2b-1$, we have that $2^{k+1}m - ca$ is the right successor of $-(2^k m - ba)$ with $1 \leq b \leq 2^{k-1}$ (note this includes $-2^k m + 2^{k-1}a$).
- When $1 \leq c < 2^k \in \mathbb{N}$ is odd and $c = 2b+1$, we have that $-(2^{k+1}m - ca)$ is the right successor of $2^k m - ba$ with $0 \leq b < 2^{k-1}$ (note this includes $2^k m$).

Thus, the $k+1$-descendants of m are $2^{k+1}m$, $\pm(2^{k+1}m - ca)$ for $1 \leq c < 2^k \in \mathbb{N}$, and $-(2^{k+1} - 2^k a)$. □

We now prove a similar theorem for Type 2 ECSDs as we proved for Type 1 ECSDs, including determining a consecutive list of cyclic vertices. Note that we do not have the same structure of powers of 2 in cycles, as the bases 2 and -2 are different.

Theorem 7. *Let $a \in \mathbb{N}$ be odd. The following statements are true about the ECSD $G(2n, -2n+a)$.*

(i) The cycle (0) is a cycle of length 1, and if $a \equiv 0 \pmod 3$, then $(a/3)$ is a second cycle of length 1. These are the only two cycles of length 1.

(ii) An integer m is a cyclic vertex if and only if $0 \leq m < \frac{1}{2}a$.

(iii) Whenever odd $a' \in \mathbb{N}$ is such that $a = ma'$, we have that $C = (c_1, \ldots, c_t)$ is a cycle in $G(2n, -2n+a')$ if and only if $mC = (mc_1, \ldots, mc_t)$ is a cycle in $G(2n, -2n+a)$.

Proof. (i) Suppose (m) is a cycle of length 1. If $m = 2m$, then $m = 0$. If $m = -2m + a$, then $m = \frac{a}{3}$ which is an integer if and only if $a \equiv 0 \pmod 3$.
(ii) Recall from Proposition 1 that the k-descendants of m in $G(2n, -2n+a)$ are $2^k m$, $\pm(2^k n - ba)$ for $1 \leq b < 2^{k-1} \in \mathbb{N}$, and $-(2^k m - 2^{k-1}a)$. Suppose $m < 0$. Its k-descendants $2^k m$ and $2^k m - ba$ are less than n, and the others, which are $-(2^k m - 2^{k-1}a)$ and $-(2^k m - ba)$, are positive. Alternatively, suppose $m > \frac{1}{2}a$

(note that this is equivalent to $a < 2m$). Its k-descendant $2^k m$ is greater than m. We can then see that $2^k m - ba > 2m$ as follows:

$$2^k m - ba > 2^k m - b(2m) > 2^k m - (2^{k-1} - 1)2m = 2m,$$

so the k-descendants $\pm(2^k m - ba)$ are all greater in modulus than m. We have that $-(2^k n - 2^{k-1}a) < -(2^k m - 2^{k-1}(2m)) = 0$, so this k-descendant is negative. Thus, if m is not in the range $0 \leq m < \frac{1}{2}a$, none of the k-descendants of m can be m, and m is not a cyclic vertex.

The proof of the other implication follows similarly to the proof of Theorem 6, Statement (ii): if $m \in \mathbb{Z}$ is in the range $0 \leq m < \frac{1}{2}a$, then it has an infinite sequence of descendants in the same range, which must have repeats and therefore m is an ancestor of some cyclic vertex, and is itself a cyclic vertex.
(iii) The proof of this statement follows analogously to the proof of Proposition 6, Statement (iv). □

Shown in Figure 3 is the ECSD $G(2n, -2n + 9)$. We can see that the cyclic vertices are indeed the integers from 0 to 4, and it has two 1-cycles, the second one at $\frac{9}{3} = 3$.

2.3. Type 3: $G(-2n + a_1, -2n + a_2)$

Unlike the previous two cases, for ECSDs of the form $G(-2n+a_1, -2n+a_2)$, we no longer have a single-parameter family. Recall that there are 3 separate sub-types of Type 3 ECSDs:

Type 3.1: $G(-2n, -2n + a)$ for $a \in \mathbb{N}$ odd
Type 3.2: $G(-2n + 1, -2n + a)$ for $a \in \mathbb{N}$ even, $a \equiv 1 \pmod 3$
Type 3.3: $G(-2n + 1, -2n + a)$ for $a \in \mathbb{Z}$ even, $a \equiv 2 \pmod 3$.

In fact, ECSDs in each of the three sub-types have different characteristics: each ECSD of Type 3 belongs to exactly one of Type 3.1, 3.2, or 3.3.

Before we consider each of the three types separately, we have a few general results that apply to all ECSDs of the form $G(-2n + a_1, -2n + a_2)$. The first such result characterizes the k-descendants of an integer m by their modulus. The 1 and 2-descendants of m are shown in Figure 10.

Theorem 8. *In an ECSD of the form $G(-2n + a_1, -2n + a_2)$, the k-descendants of m are all congruent to*

$$(-2)^k m + a_1 \left(\frac{1 - (-2)^k}{3} \right) \pmod{|a_1 - a_2|}.$$

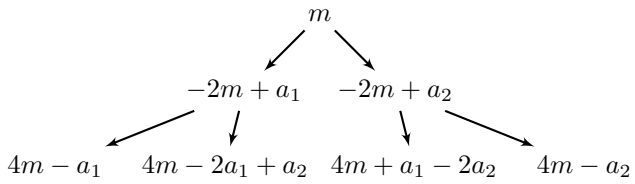


Figure 10: The 1 and 2-descendants of m in $G(-2n + a_1, -2n + a_2)$.

Proof. Both successors of m, $-2m + a_1$ and $-2m + a_2$, are congruent to $-2m + a_1 \pmod{|a_1 - a_2|}$. This is easily seen by adding $a_1 - a_2$ to $-2m + a_2$ to get $-2m + a_1$. If we iterate the function $f(m) = -2m + a_1$ k times, we see that the k^{th} iteration is

$$f^k(m) = (-2)^k m + a_1 \sum_{i=0}^{k-1} (-2)^k = (-2)^k m + a_1 \left(\frac{1 - (-2)^k}{3} \right). \qquad \square$$

Using this theorem, we can conclude the following about the cyclic vertices modulo $|a_1 - a_2|$.

Theorem 9. *Each of the $|a_1 - a_2|$ distinct congruence classes modulo $|a_1 - a_2|$ is represented by at least one cyclic vertex of $G(-2n + a_1, -2n + a_2)$.*

Proof. Let $0 \leq c \leq |a_1 - a_2|$. Because each component of an ECSD has exactly one cycle, c is the k-descendant of m for some m in a cycle and some k. Then by Proposition 8, all of the k-descendants of m must be congruent to c modulo $|a_1 - a_2|$, including its k-descendant on the cycle containing m. So an integer congruent to c modulo $|a_1 - a_2|$ appears in a cycle of $G(-2n + a_1, -2n + a_2)$. $\square$

Remark 1. Note that Proposition 9 holds for all Type 1 ECSDs as well: all are isomorphic to an ECSD of the form $G(2n, 2n - a)$ for $a \in \mathbb{N}$ odd. In this case, $|a_1 - a_2| = a$, and since the cyclic vertices are $0 \leq m \leq a$, each congruence class modulo a is represented. In fact, the congruence $x \equiv 0 \pmod{a}$ is represented twice, in both 0 and a. For example, $G(2n, 2n - 9)$, shown in Figure 2, has cycles (0), (9), $(3, 6)$, and $(1, 2, 4, 8, 7, 5)$. The integers 0 and 9 are two representatives of the congruence $x \equiv 0 \pmod{9}$, and each other congruence class is represented exactly once.

We will notice in the following exposition (particularly Type 3.2) that sometimes exactly one element from each congruence class appears, and otherwise one congruence class has two representatives which are the smallest and largest elements of cycles, as is the case with Type 1 ECSDs.

2.3.1. Type 3.1: $G(-2n, -2n + a)$, $a \in \mathbb{N}$ Odd

Now, we can split our discussion of Type 3 ECSDs into three distinct parts. First, we consider Type 3.1 ECSDs, which are isomorphic to some $G(-2n, -2n + a)$ with $a \in \mathbb{N}$ odd. This case is very similar to Type 1, where the cycles have length depending on the multiplicative order of -2, and cocycles are also cycles. ECSDs of Type 3.1 are the only Type 3 ECSDs to have cycles of length 1. See $G(-2n, -2n+9)$ in Figure 4 for an example of such an ECSD.

In the following proof, we use the following notation: for $n \in \mathbb{Z}$, define $\widehat{n}$ to be the unique integer in $(-\frac{1}{3}a, \frac{2}{3}a]$ such that $\widehat{n} \equiv n \pmod{a}$. Similar to Type 1, here, the cocycle $\widehat{bC}$ of cycle $C = (c_1, \ldots, c_k)$ is $\widehat{bC} = (\widehat{bc_1}, \ldots, \widehat{bc_k})$.

Theorem 10. *Let $a \in \mathbb{N}$ be odd. The following statements are true about the ECSD $G(-2n, -2n + a)$.*

(i) The cycle (0) is a cycle of length 1, and if $a \equiv 0 \pmod 3$, then $(a/3)$ is a second cycle of length 1. These are the only two cycles of length 1.

(ii) An integer m is a cyclic vertex if and only if $-\frac{1}{3}a \leq m \leq \frac{2}{3}a$.

(iii) If $a \equiv 1 \pmod 3$ or $a \equiv 2 \pmod 3$, there are a cyclic vertices and each congruence class modulo a has exactly one cyclic representative.

If $a \equiv 0 \pmod 3$, there are $a + 1$ cyclic vertices. Every congruence class modulo a has exactly one cyclic representative, except for the smallest and the largest cyclic vertices, which are congruent modulo a and are in a 2-cycle together.

(iv) The cycle $C = (\widehat{(-2)^k} : 0 \leq k < \operatorname{ord}(-2, a))$, is a cycle of length $\operatorname{ord}(-2, a)$, and its cocycles $\{\widehat{bC} : -\frac{1}{3}a < b < \frac{2}{3}a, b \notin \{0, a/3\}\}$ are also cycles, each of length $\operatorname{ord}\left(-2, \frac{a}{\gcd(a,b)}\right)$.

(v) Whenever odd $a' \in \mathbb{N}$ is such that $a = ma'$, we have that $C = (c_1, \ldots, c_t)$ is a cycle in $G(-2n, -2n + a')$ if and only if $mC = (mc_1, \ldots, mc_t)$ is a cycle in $G(-2n, -2n + a)$.

Proof. (i) Suppose (m) is a cycle of length 1. If $m = -2m$, then $m = 0$. If $m = -2m + a$, then $m = \frac{a}{3}$, which is an integer if and only if $a \equiv 0 \pmod 3$.
(ii) Suppose $m > \frac{2}{3}a$. Its successors $-2m$ and $-2m+a$ are both negative, as $2m > a$. Following one more step, it has four positive 2-descendants, the smallest of which is $-2(-2m + a) = 4n - 2a$. Since $a < \frac{3}{2}m$, we have that $4n - 2a > 4m - 3m = m$. Thus, its only positive descendants are greater than it, so m cannot be its own descendant and therefore it is not a cyclic vertex.

Suppose $m < -\frac{1}{3}a$. Its successors are clearly positive, so its four 2-descendants are negative, the largest of which is $-2(-2n) + a = 4m + a$. Since $a < -3m$, we

have that $4m+a < 4n-3n = m$. Thus, its only negative descendants are less than it, so m cannot be its own descendant and therefore it is not a cyclic vertex.

The proof of the other implication follows similarly to the proof of Theorem 6, Statement (ii): if $m \in \mathbb{Z}$ is in the range $-\frac{1}{3}a \leq m \leq \frac{2}{3}a$, then it has an infinite sequence of descendants in the same range, which must have repeats and therefore m is a cyclic vertex.

(iii) If $a \equiv 1 \pmod 3$ or $a \equiv 2 \pmod 3$, we observe that there are exactly a cyclic vertices, and each represents a different congruence class modulo a. If $a \equiv 0 \pmod 3$, both $-\frac{1}{3}a$ and $\frac{2}{3}a$ are integers, so we observe that there are $a+1$ cyclic vertices. Since $-\frac{1}{3}a + a = \frac{2}{3}a$, these two values are congruent modulo a and all other cyclic vertices represent a different congruence class modulo a. Moreover, since $-2(-\frac{1}{3}a) = \frac{2}{3}a$ and $-2(\frac{2}{3}a) + a = -\frac{1}{3}a$, we see that $(-\frac{1}{3}a, \frac{2}{3}a)$ is a 2-cycle.

(iv) The k-descendants of m in $G(-2n, -2n+a)$ are all congruent to $(-2)^k m$ by Theorem 8. Thus, if we let $m = 1$, we see that

$$C = (\widehat{(-2)^k} : 0 \leq k < \operatorname{ord}(-2, a))$$

is a cycle. It has length $\operatorname{ord}(-2, a)$. (Note that it never includes $-\frac{1}{3}a$ or $\frac{2}{3}a$, because if they are integers they are in a 2-cycle that does not include 1 by Part (iii) of this proof.)

If we consider starting with any element b such that $-\frac{1}{3}a < b < \frac{2}{3}a$, and b is not already a 1-cycle, we can see that the cosets

$$\left\{\widehat{bC} : -\frac{1}{3}a < b < \frac{2}{3}a, b \notin \{0, a/3\}\right\}$$

are also cycles. Similar to the proof of Theorem 6, Statement (iii), we consider the integers $\widehat{b}, \widehat{-2b}, \ldots, \widehat{(-2)^k b}, \ldots$. The length of $\widehat{bC}$ is the minimum k such that $(-2)^k b \equiv b \pmod a$, or equivalently, such that $a \mid ((-2)^k - 1)b$.

Let $g = \gcd(a, b)$, so $a = ga'$ and $b = gb'$, where $\gcd(a', b') = 1$. Then,

$$a \mid ((-2)^k - 1)b \text{ if and only if } ga' \mid ((-2)^k - 1)gb' \text{ if and only if } a' \mid (-2)^k - 1,$$

or equivalently $(-2)^k \equiv 1 \pmod{a'}$. So the length of bC is

$$\operatorname{ord}(-2, a') = \operatorname{ord}\left(-2, \frac{a}{\gcd(a, b)}\right).$$

(v) The proof of this statement follows analogously to the proof of Theorem 6, Statement (iv). □

2.3.2. Type 3.2: $G(-2n+1, -2n+a)$, $a \in \mathbb{N}$ Even, $a \equiv 1 \pmod 3$

Next, we consider ECSDs of Type 3.2, which are isomorphic to $G(-2n+1, -2n+a)$ for some $a \in \mathbb{N}$ even and $a \equiv 1 \pmod 3$. This is perhaps the most interesting type of

ECSD we cover in this paper, because several have only one component (see Section 3 for the results), and it has rich structure, though it is not as easily understood as ECSDs of other types. Specifically, these ECSDs are the only ECSDs of degree 2 which can possibly have a single component, as we will see in Section 3. For example, $G(-2n+1,-2n+10)$ shown in Figure 5 has a single component with a 9-cycle.

Theorem 11. *Let $a \in \mathbb{N}$ be even and $a \equiv 1 \pmod 3$. An integer m is a cyclic vertex of $G(-2n+1,-2n+a)$ if and only if $-\frac{1}{3}(a-1) < m < \frac{2}{3}a$.*

Proof. The proof of the "only if" statement follows similarly to the proof of Theorem 10, Statement (ii). If $m \geq \frac{2}{3}a$, its only positive descendants are greater than it, so it cannot be a cyclic vertex. If $m \leq -\frac{1}{3}(a-1)$, then its only negative descendants are less than it, so it cannot be a cyclic vertex.

Now assume that $-\frac{1}{3}(a-1) < m < \frac{2}{3}a$. There are exactly $a-1$ possible vertices in this range. By Theorem 9, each of these must represent a different congruence class modulo $a-1$ and are therefore all cyclic vertices. □

Because each congruence class modulo $a-1$ is represented exactly once, we can see a resemblance between ECSDs of Type 3.2 and ECSDs of Type 3.1 with $a \not\equiv 0 \pmod 3$. However, there are no isomorphisms between these two types, because ECSDs of Type 3.1 have 1-cycles whereas ECSDs of Type 3.2 do not: neither $-2m+1 = m$ or $-2m+a = m$ with $a \equiv 1 \pmod 3$ have integer solutions for m.

2.3.3. Type 3.3: $G(-2n+1,-2n+a)$, $a \in \mathbb{Z}$ Even, $a \equiv 2 \pmod 3$

This type follows similarly to Type 3.2 ECSDs, though we must consider positive and negative a separately here, because they are not isomorphic as in Type 3.2.

Theorem 12. *Let $a \in \mathbb{Z}$ be even, $a \equiv 2 \pmod 3$, and let $m \in \mathbb{Z}$ be a cyclic vertex of $G(-2n+1,-2n+a)$. Then, the following statements are true.*

(i) If $a \geq 2$, then $-\frac{1}{3}(a-1) < m < \frac{2}{3}a$.

(ii) If $a \leq -2$, then $-\frac{2}{3}(|a|+1) < m < \frac{1}{3}|a|+1$.

Proof. The proof of Statement (i) follows exactly as the proof of the reverse implication in Theorem 11. The proof of Statement (ii) is similar. If $m \geq \frac{1}{3}|a|+1$, its only positive descendants are greater than it, so it cannot be a cyclic vertex. If $m \leq -\frac{2}{3}(|a|+1)$, then its only negative descendants are less than it, so it cannot be a cyclic vertex. □

Theorem 13. *Let $a \in \mathbb{Z}$ be even, $a \equiv 2 \pmod 3$. Then, every $m \in \mathbb{Z}$ as in Theorem 12 is a cyclic vertex of $G(-2n+1,-2n+a)$. Every congruence class modulo $|a-1|$ has exactly one cyclic representative, except for the smallest and*

the largest cyclic vertices, which are congruent modulo $|a-1|$ and are in a 2-cycle together.

Proof. From Theorem 12, in either case $a \geq 2$ or $a \leq -2$, there are $|a-1|+1$ possible cyclic vertices. For $a \geq 2$, we have that

$$-\frac{1}{3}(a-2) \equiv -\frac{1}{3}(a-2) + (a-1) = \frac{2}{3}(a-2) + 1 \pmod{a-1}$$

and

$$-2\left(-\frac{1}{3}(a-2)\right) + 1 = \frac{2}{3}(a-2) + 1$$

as well as

$$-2\left(\frac{2}{3}(a-2)+1\right) + a = -\frac{1}{3}(a-2).$$

For $a \leq -2$, we have that

$$-\frac{2}{3}(|a|+2)+1 \equiv -\frac{2}{3}(|a|+2) + 1 + |a| + 1 = \frac{1}{3}(|a|+2) \pmod{|a|+1}$$

and

$$-2\left(\frac{1}{3}(|a|+2)\right) + 1 = -\frac{2}{3}(|a|+2) + 1$$

as well as

$$-2\left(-\frac{2}{3}(|a|+2)+1\right) - |a| = \frac{1}{3}(|a|+2).$$

Thus, in either case, the smallest and largest cyclic vertices are congruent modulo $|a-1|$ and are in a 2-cycle together. Since there are $|a-1|$ congruence classes and each must have a cyclic representative by Theorem 9, each other congruence class is represented exactly once. Thus, all $|a-1|+1$ possible cyclic vertices are indeed cyclic vertices. □

Because of the structure that the smallest and largest cyclic vertices are congruent modulo $|a-1|$ and in a 2-cycle together, we can see a resemblance between ECSDs of this type and ECSDs of Type 3.1 with $a \equiv 0 \pmod 3$. However, just as with ECSDs of Type 3.2, these ECSDs of Type 3.3 do not have any 1-cycles, so they cannot be isomorphic to an ECSD of Type 3.1.

3. Single-Component ECSDs of Degree 2 and Non-Standard Binary Representations of Integers

Every natural number has a *standard binary representation*, that is, every natural number can be written as a sum $\sum_{j=0}^{k} b_j 2^j$ for $b_j \in \{0,1\}$ and some $k \in \mathbb{N}$, which

is unique up to the appearance of leading zeroes. We consider *non-standard binary representations* of integers, both by using base -2 in place of 2 and by using other choices for the digits b_j. When making non-standard choices, we are able to not only represent every natural number, but in some cases, every integer using such a representation.[2]

Arithmetic in negative bases was first proposed by Grünwald in 1885 [8]. The concept gained interest during the advent of digital computing, especially with regards to the use of base -2, sometimes called *negabinary* (this term was introduced by DeRegt [6]). However, these publications always assumed a *standard digit set* (0 through $d-1$ in base d).

The most commonly known alternate digit set is a *symmetric* digit set, in which for an odd base d, the digits are taken to be $-\frac{d-1}{2}, \ldots, \frac{d-1}{2}$. This concept seems to be as old as civilization itself, as similar digit sets appear in number systems indigenous to West Africa [17], but was first introduced into the modern mathematical literature in 1726 [5] (see a summary of developments in [16]). In the case of $d = 3$, the system with symmetric digit set $\{-1, 0, 1\}$ is known as *balanced ternary*, which Knuth termed "perhaps the prettiest number system of all" [9].

Other non-standard digit sets were considered, even with negative bases (see [1] and [10]), but always included the digit 0 to account for uniqueness. In [11], we introduced the idea that even if 0 is not included in the digit set, a representation may still be unique up to leading instances of the shortest digital representation of 0.

In [11], we determined a relationship between single-component ECSDs and non-standard representations of integers.

Theorem 14 ([11])**.** *For some $d \geq 2 \in \mathbb{Z}$, let G_S be an ECSD with representative set $S = \{(a_i, D) : 1 \leq i \leq d\}$ and base $D = \pm d$. Then, every integer can be expressed as the sum*

$$\sum_{j=0}^{k} b_j D^j,\ b_j \in \{a_1, \ldots, a_d\} \tag{1}$$

for some $k \in \mathbb{N}$ if and only if G_S has a single component and its cycle contains 0. This representation is unique up to leading instances of the block of digits representing the path from 0 to itself in G_S.

In this section, we will show that the only single-component ECSDs of degree 2 are isomorphic to either $G(2n, -2n+1)$ or an ECSD of the form $G(-2n+1, -2n+a)$ with $a = 1 \pm 3^m$ for some $m \in \mathbb{N}_0$, and then use this to describe all non-standard binary representations that can represent every integer.

[2]In the literature, this property of a representation is called *completeness*, though the term is applied in different contexts to mean either representing every natural number or every integer. To avoid ambiguity we do not use this term.

Proposition 2. *If an ECSD of degree 2 has a single component, then it is isomorphic to either $G(2n, -2n+1)$ or an ECSD of the form $G(-2n+1, -2n+a)$ with $a = 0$, $a = 2$, or $a \equiv 1 \pmod 3$.*

Proof. We start by eliminating the different types of degree 2 ECSDs.

- Type 1 ECSDs, isomorphic to $G(2n, 2n-a)$ with $a \in \mathbb{N}$ odd, have two 1-cycles, and therefore at least two components (Theorem 6).
- Type 2 ECSDs, isomorphic to $G(2n, -2n+a)$ with $a \in \mathbb{N}$ odd, have a 1-cycle, and if $a > 1$, more than one cyclic vertex (Theorem 7). If $a = 1$, then $G(2n, -2n+1)$ has a single component.
- Type 3.1 ECSDs, isomorphic to $G(-2n, -2n+a)$ with $a \in \mathbb{N}$ odd, have a 1-cycle, and if $a > 1$, more than one cyclic vertex (Theorem 10). If $a = 1$, then $G(-2n, -2n+1)$ is of the form $G(-2n+1, -2n+a)$ with $a = 0$.
- Type 3.3 ECSDs, isomorphic to $G(-2n+2, -2n+a)$ with $a \in \mathbb{Z}$ even, $a \equiv 2 \pmod 3$, have a 2-cycle and $|a-1|+1$ cyclic vertices (Theorem 13). If $a = 2$, then it is of the form $G(-2n+1, -2n+a)$ with $a = 2$. If $|a| > 2$, then $|a-1|+1 > 2$ and there are more than 2 cyclic vertices, and so more than one component.

Thus, the only ECSDs other than $G(2n, -2n+1)$ that can have a single component are isomorphic to $G(-2n+1, -2n+a)$ with $a = 0$, or $a = 2$, or Type 3.2: $a \in \mathbb{N}$ even, $a \equiv 1 \pmod 3$. □

In the proof of the previous proposition we have already uncovered two interesting one-component ECSDs. The ECSD $G(2n, -2n+1)$ has a single component, and has quite an interesting structure (see the left graph in Figure 11), but since its two congruences have different bases (2 and -2) it does not correspond to any digital representation. The ECSD $G(-2n, -2n+1)$ corresponds to the standard negabinary representation, which is the only example of a binary representation of every integer where we ignore leading zeroes, since $G(-2n, -2n+1)$ has a single component and 0 is the only cyclic vertex (shown on the right side of Figure 11).

To find other degree 2 ECSDs that can represent every integer, we focus on Type 3.2 ECSDs. To prove our main result, Theorem 15, we need four lemmas regarding divisibility.

Lemma 1. *For any positive integer n, if $n \mid \frac{2^n+1}{3}$, then $3n \mid \frac{2^{3n}+1}{3}$.*

Proof. Suppose $n \mid \frac{2^n+1}{3}$. Then, $3n \mid 2^n + 1$, and for some integer k, we have that

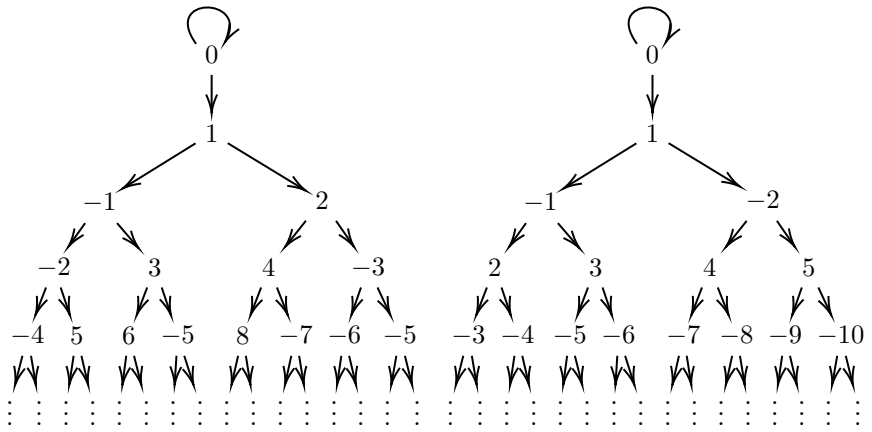


Figure 11: The ECSDs $G(2n, -2n+1)$ and $G(-2n, -2n+1)$.

$2^n = 3nk - 1$. So

$$\begin{aligned} 2^{3n} &= (3nk-1)^3 \\ &= (3nk)^3 - 3(3nk)^2 + 3(3nk) - 1 \\ &= 3^3n^3k^3 - 3^3n^2k^2 + 3^2nk - 1 \\ &= 9n(3n^2k^3 - 3nk^2 + 3k) - 1. \end{aligned}$$

Thus, we have that $9n \mid 2^{3n}+1$ and $3n \mid \frac{2^{3n}+1}{3}$. □

Lemma 2. *For any positive integers n and k, if $n \mid \frac{1-(-2)^k}{3}$, then $3n \mid \frac{1-(-2)^{3k}}{3}$.*

Proof. We can split the proof into cases for even and odd k. For odd k, we need to show that if $n \mid \frac{2^k+1}{3}$, then $3n \mid \frac{2^{3k}+1}{3}$. For even k, we need to show that if $n \mid \frac{1-2^k}{3}$, then $3n \mid \frac{1-2^{3k}}{3}$. Both statements follow by completely analogous proofs to the proof of Lemma 1. □

Lemma 3. *For any positive integers m and k, if $3^m \nmid \frac{2^k+1}{3}$, then $3^{m+1} \nmid \frac{2^{3k}+1}{3}$.*

Proof. If $3^m \nmid \frac{2^k+1}{3}$, then $3^m r + R = \frac{2^k+1}{3}$ where $1 \le R < 3^m$, in particular $3^m \nmid R$. Thus, we have that $3^{m+1}r + 3R - 1 = 2^k$, and $2^{3k} = (3^{m+1}r + 3R - 1)^3$.

Let $S = 3^{2m+2}r^3 + 3^{m+3}r^2 - 3^{m+2}r^2 + 27rR^2 - 18rR + 3r$. Then

$$\begin{aligned} 2^{3k} &= 3^{m+2}S + 27R^3 - 27R^2 + 9R - 1, \\ \frac{2^{3k}+1}{3} &= 3^{m+1}S + 9R^3 - 9R^2 + 3R, \\ \frac{2^{3k}+1}{3} &= 3^{m+1}S + 3R(3R^2 - 3R + 1). \end{aligned}$$

So for 3^{m+1} to divide $\frac{2^{3k}+1}{3}$, we would need $3^m \mid R(3R^2 - 3R + 1)$. Since

$$3R^2 - 3R + 1 \equiv 1 \pmod 3,$$

then 3^m must divide R. But this is not the case by the definition of R, and therefore $3^{m+1} \nmid \frac{2^{3k}+1}{3}$. □

Lemma 4 ([3])**.** *For any integer $n > 3$, if $n \mid \frac{2^n+1}{3}$, then n is divisible by 9.*

Proof. It is sufficient to show that if $n \mid 2^n + 1$, then $9 \mid n$. Observe that if $n \mid \frac{2^n+1}{3}$, then $n \mid 2^n + 1$.

Let $n \mid 2^n + 1$ and $n > 3$, and suppose for the sake of contradiction that $n = 3m$, where $\gcd(3, m) = 1$. Then, $2^{3m} \equiv -1 \pmod{3m}$.

Let p be the smallest prime such that $p \mid m$. By our assumptions, $p \geq 5$. We can see that $p \neq 2$ because if $n \mid 2^n + 1$, then n is odd, and $m > 1$ because $n > 3$, and $p \neq 3$ because $\gcd(3, m) = 1$. Then, $2^{3m} \equiv -1 \pmod{3p}$, and $2^{6m} \equiv 1 \pmod{3p}$. By Euler's theorem, we also know that $2^{\phi(3p)} \equiv 1 \pmod{3p}$, which is equivalent to $2^{2(p-1)} \equiv 1 \pmod{3p}$.

Thus, we have that $2^{\gcd(2(p-1),6m)} \equiv 1 \pmod{3p}$ and $2^{2\gcd(p-1,3m)} \equiv 1 \pmod{3p}$. Now, it must be the case that $\gcd(p-1, 3m)$ is 1 or 3, since $\gcd(p-1, m) = 1$ (p is the smallest prime dividing m). If $\gcd(p-1, 3m) = 1$, then $2^2 \equiv 1 \pmod{3p}$, and we would have $3 \equiv 0 \pmod{3p}$, which is a contradiction. Thus, we have that $\gcd(p-1, 3m) = 3$, and $2^6 \equiv 1 \pmod{3p}$, so $3p \mid 63$ and therefore, $p = 7$.

Now, we have that $n = 21t$, where $t = p/7$. Therefore, we have that $21t \mid 2^{21t} + 1$, so $2^{21t} \equiv -1 \pmod{21}$. We know that $2^{21} \equiv 2 \pmod 3$ and $2^{21} \equiv 1 \pmod 7$, so by the Chinese Remainder Theorem, we have that $2^{21} \equiv 8 \pmod{21}$. Due to the fact that $8^2 \equiv 1 \pmod{21}$, we can never have $8^{7t} = 2^{21t} \equiv -1 \pmod{21}$, and we have reached a contradiction. □

Now, we can prove our main result.

Theorem 15. *The ECSD $G(-2n+1, -2n+a)$ has one component if and only if $a = 1 \pm 3^m$ for some $m \in \mathbb{N}_0$.*

Proof. By Theorem 2, any ECSD $G(-2n+1, -2n+a)$ with one component must either have $a = 0$, $a = 2$, or $a \equiv 1 \pmod 3$. In the cases $a = 0$ and $a = 2$, we have

that $0 = 1 - 3^0$ and $2 = 1 + 3^0$, so these are indeed of the form $a = 1 \pm 3^m$ for some $m \in \mathbb{N}_0$. Thus, we may assume that $a \equiv 1 \pmod 3$. By Theorem 11, there are exactly $a - 1$ cyclic vertices in $G(-2n + 1, -2n + a)$ with $a \equiv 1 \pmod 3$ and $a \in \mathbb{N}$. For ease of notation, we now denote $c = a - 1$.

By Theorem 8, the k-descendants of 0 in $G(-2n+1, -2n+c+1)$ are congruent to

$$\frac{1 - (-2)^k}{3} \text{ modulo } c.$$

For an ECSD to have one component, it is necessary that the c-descendants be congruent to 0 modulo c, that is,

$$\frac{1 - (-2)^c}{3} \equiv 0 \pmod c.$$

Since a is even, this means $c = a - 1$ is odd, and so our criterion is that

$$\frac{2^c + 1}{3} \equiv 0 \pmod c.$$

Moreover, for an ECSD with one component, no k'-descendant with $1 \leq k' < c$ can be congruent to 0 modulo c, so we cannot have $\frac{2^{k'}+1}{3} \equiv 0 \pmod c$ for any such k'. An ECSD with more than one component does have such a k'. For example, let k' be the length of the cycle containing 0.

We claim that for any $m \in \mathbb{N}_0$, the ECSD $G(-2n + 1, -2n + 3^m + 1)$ has a single component. We prove this by induction on m. By inspection it is true for $m = 0, 1, 2$. Let $c = 3^m$. By our inductive hypothesis, the ECSD $G(-2n + 1, -2n + 3^{m-1} + 1)$ has a single component. Thus, as above, we have that $3^{m-1} \mid \frac{2^{3^{m-1}}+1}{3}$. Therefore, by Lemma 1, we have that $3^m \mid \frac{2^{3^m}+1}{3}$. That is, the 3^m-descendants of 0 are congruent to 0 modulo 3^m. Thus, the cycle containing 0 in $G(-2n + 1, -2n + 3^m + 1)$ must have length a factor of 3^m.

By our inductive hypothesis, we have that $3^{m-1} \mid \frac{2^{3^{m-1}}+1}{3}$, and moreover, for all k' such that $1 \leq k' < 3^{m-1}$, we have that $3^{m-1} \nmid \frac{2^{3^{k'}}+1}{3}$. More specifically, we know that $3^{m-1} \nmid \frac{2^{3^{m-2}}+1}{3}$. Thus, by Lemma 3, we have that $3^m \nmid \frac{2^{3^{m-1}}+1}{3}$, which means that the 3^{m-1}-descendants of 0 are not congruent to 0 modulo 3^m. Since the cycle containing 0 must have length a factor of 3^m, we gather that the cycle containing 0 in $G(-2n + 1, -2n + 3^m + 1)$ must have length 3^m and the ECSD must have a single component.

We further claim that if $c \neq 3^m$ for some $m \in \mathbb{N}_0$, then $G(-2n+1, -2n+c+1)$ has more than one component. We need only consider c such that $c \equiv 0 \pmod 3$ and $c \mid \frac{2^c+1}{3}$. Suppose c is not a power of 3. The smallest c with all of these properties is 171, so c is sufficiently large. Then, $c = 3^k c'$ for some c' not divisible by 3. By Lemma 4, we have that c is divisible by 9, so we conclude that $k \geq 2$.

Since any c such that $c \mid \frac{2^c+1}{3}$ must be divisible by 9, we have that $3c' \nmid \frac{2^{3c'}+1}{3}$, and therefore $G(-2n+1, -2n+3c'+1)$ has more than one component. This ECSD has $3c'$ cyclic vertices, but the cycle containing 0 does not contain all of the cyclic vertices. That is, the 0-cycle is length k' where $3c' \mid \frac{1-(-2)^{k'}}{3}$ and $1 \leq k' < 3c'$.

By repeated applications of Lemma 2, we have that

$$3^k c' \mid \frac{1-(-2)^{3^{k-1}k'}}{3}$$

with $1 \leq 3^{k-1}k' < 3^k c'$. Thus, the 0-cycle of $G(-2n+1, -2n+3^k c'+1)$ is at most $3^{k-1}k'$ in length, and therefore does not contain all of the $3^k c'$ cyclic vertices. Thus, the ECSD $G(-2n+1, -2n+3^k c'+1) = G(-2n+1, -2n+c+1)$ has more than one component.

Since $G(-2n+1, -2n+a) \cong G(-2n+1, -2n-a+2)$ by an application of Theorem 1, we may generalize this result about $a = 1+3^m$ for $m \in \mathbb{N}_0$ to $a = 1 \pm 3^m$. Thus, our theorem is proved. □

By applying Theorem 14 to Theorem 15, we conclude the following about non-standard binary representations.

Corollary 1. *For an integer a,*

$$\mathbb{Z} = \left\{ \sum_{j=0}^{k} b_j(-2)^j : b_j \in \{1, a\}, k \in \mathbb{N} \right\}$$

if and only if $a = 1 \pm 3^m$ for some $m \in \mathbb{N}_0$.

Given some $m \in \mathbb{Z}$, we can find the representation of m by tracing back the ancestors of m in $G(-2n+1, -2n+a)$ until we reach 0, because the choices of coefficients in the representation are given by the types of edges in the path from 0 to m [11]. For example, in the ECSD $G(-2n+1, -2n+10)$ shown in Figure 5, the integer -3 can be reached from 0 by the path

$$0 \xrightarrow{-2n+1} 1 \xrightarrow{-2n+1} -1 \xrightarrow{-2n+1} 3 \xrightarrow{-2n+10} 4 \xrightarrow{-2n+10} 2 \xrightarrow{-2n+1} -3,$$

which gives us the representation "1 1 1 10 10 1," or

$$\begin{aligned} 6 &= 1(-2)^5 + 1(-2)^4 + 1(-2)^3 + 10(-2)^2 + 10(-2)^1 + 1(-2)^0 \\ &= -32 + 16 - 8 + 40 - 20 + 1. \end{aligned}$$

In contrast, in the ECSD $G(-2n, -2n+1)$ shown in the right half of Figure 11, we can see that -3 can be reached from 0 by the path $0 \to 1 \to -1 \to 2 \to -3$, giving the representation "1 1 0 1." It should be noted that these representations can be rather long, as proved in the following theorem.

Theorem 16. *In the ECSD $G(-2n+1,-2n+a)$ where $a \notin \{0,2\}$, the paths from 0 to $a-1$ and from 0 to $-(a-1)$ each have $|a-1|$ edges.*

Proof. Without loss of generality we can assume a is positive, as the proof for negative a follows via graph isomorphism.

We first straightforwardly prove the claim that the path from 0 to $-(a-1)$ has $|a-1|$ edges. First, observe that the predecessor of 0 in $G(-2n+1,-2n+a)$ is $\frac{a}{2}$. The other successor of $\frac{a}{2}$ is $-2(\frac{a}{2})+1 = -(a-1)$. Thus, there are $a-2$ edges in the cycle from 0 to $\frac{a}{2}$, and one more edge off the cycle to arrive at $-(a-1)$, resulting in $a-1$ digits in the representation of $-(a-1)$.

We now claim that the path from 0 to $a-1$ has $|a-1|$ edges. The proof of this claim is more involved. Below, we will describe the k-ancestors of both 0 and $a-1$ to show that they must equal each other for the same value of k, that is, $p^K(0) = p^K(a-1)$ for some value of K (and this is the first common ancestor). Call this vertex P. Since P is an ancestor of 0, P is a cyclic vertex. Since the values of k are the same, there are K edges on the cycle from P to 0, and K edges off the cycle from P to $a-1$. Since there are a total of $a-1$ edges in the cycle from 0 to 0, the path from 0 to $a-1$ must also have $a-1$ edges (see Figure 12 for an illustration).

Now, we describe the ancestors of 0 and $a-1$. First, observe that $p(0) = \frac{a}{2}$ and $p(a-1) = \frac{a-2}{-2}$, and they have the opposite parity and sign (given $a > 2$). If $\frac{a}{2}$ is even and $\frac{a-2}{-2}$ is odd, their predecessors are both $\frac{a}{4}$ and we have a common ancestor $p^2(0) = p^2(a-1) = \frac{a}{4}$. Otherwise, $p(\frac{a}{2}) = \frac{a-2}{-4}$ and $p(\frac{a-2}{-2}) = \frac{3a-2}{4}$. Note that these 2-ancestors also have opposite sign and opposite parity. Let the positive ancestor be called $p^2_{(+)}$ and the negative ancestor be called $p^2_{(-)}$.

If we continue to assume that $p^k_{(+)}$ is odd and $p^k_{(-)}$ is even, we get that the k-ancestors are

$$p^k_{(+)} = \frac{c_{k+1}a - 2c_k}{2^k} \qquad \text{and} \qquad p^k_{(-)} = \frac{c_k(a-2)}{-2^k}$$

where c_k is the Jacobsthal sequence $1,1,3,5,11,21,43,\ldots$, given by the recurrence relation $c_k = c_{k-1} + 2c_{k-2}$ with initial conditions $c_1 = 1$ and $c_2 = 1$, with explicit formula $c_k = \frac{1}{3}(2^k - (-1)^k)$. These formulas can be proved inductively. Note that for all $k \geq 2$, $p^k_{(+)}$ and $p^k_{(-)}$ must have opposite sign and opposite parity, and therefore cannot be equal.

However, if at some point $p^k_{(+)}$ is even and $p^k_{(-)}$ is odd, their predecessors are equal:

$$p\left(p^k_{(+)}\right) = p\left(p^k_{(-)}\right) = \frac{c_k a - 2c_k + 2^k}{2^{k+1}}.$$

We know that this must happen for some $k < a-1$, because we know that $p^{a-1}(0) = 0$, and neither $p^k_{(+)}$ nor $p^k_{(-)}$ can equal 0 if $a > 2$. Moreover, we know that this must be the first common ancestor. If some $p^i(0) = p^j(a-1)$ for $i \neq j$, this common

ancestor would be a cyclic vertex, and from then on the successive predecessors would be the same. Thus, the k-ancestors of 0 and $a-1$ could never equal each other at the same value of k. □

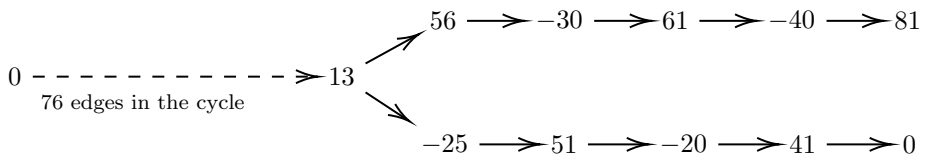


Figure 12: Paths from 0 to $a-1$ and from 0 to 0 in the ECSD $G(-2n+1, -2n+82)$.

Remark 2. The proof of Theorem 16 reveals alternating behavior in the path from 0 to $a-1$ in $G(-2n+1, -2n+a)$ (illustrated in Figure 12). As soon as the path from 0 to $a-1$ leaves the cycle, the first two numbers off of the cycle must have the same parity, but opposite sign. From there, the numbers in the path must alternate in parity and sign. Moreover, the two paths from the common ancestor P to $a-1$ and from P to 0 have opposite edge types, since the k-ancestors have opposite parity if they are not equal. Since the edge types determine the digital representation in the corresponding number system, the representations of 0 and $a-1$ also have this complementary structure.

Note that the previous proof does not rely on the graph having a single component, so it also provides a proof that the integers $a-1$ and $-(a-1)$ are always representable in the number system in base -2 with digits 1 and a. Further questions regarding the behavior of representations in these non-standard binary systems, bounds for the length of such representations, and non-representable integers in systems where $a \neq 1 \pm 3^m$ are interesting to this author and nontrivial, but are outside of the scope of this paper.

Up to ECSD isomorphism, Theorem 15 and Corollary 1 account for all non-standard binary representations that can represent all integers, though we must ensure that the isomorphic ECSD has 0 as a cyclic vertex. The specific conditions for these ECSDs are presented in the following theorem.

Theorem 17. *An ECSD of degree 2 has a single component with cyclic vertex 0 if and only if it is $G(2n, -2n \pm 1)$ or of the form $G(-2n+a_1, -2n+a_2)$ for which all three of the the following conditions are satisfied:*

(i) $|a_1 - a_2| = 3^m$ *for some* $m \in \mathbb{N}_0$,

(ii) $\{a_1, a_2\} = \{0, \pm 1\}$ *or* $a_1, a_2 \not\equiv 0 \pmod 3$,

(iii) $\max\{|a_1|, |a_2|\} \leq 2|a_1 - a_2| - 1$.

Proof. Combining the results of Proposition 2 and Theorem 15, we know that an ECSD of degree 2 has a single component if and only if it is isomorphic to $G(2n, -2n+1)$ or an ECSD of the form $G(-2n+1, -2n+a)$ with $a - 1 \pm 3^m$ for some $m \in \mathbb{N}_0$. We now work backwards in each case to expand the simplifications conducted in Theorems 4 and 5 using graph isomorphisms. Additionally, we must ensure that 0 is a cyclic vertex.

The ECSD $G(2n, -2n+1)$, depicted in the left half of Figure 11, has a single cyclic vertex at 0. The only ECSD isomorphic to this ESCD in which 0 remains a cyclic vertex is $G(2n, -2n-1)$, via the graph isomorphism $\phi_{2,0}$. Every isomorphism $\phi_{1,k}$ or $\phi_{2,k}$ with $k \neq 0$ ensures that the cyclic vertex is no longer 0.

Now we will show that $G(-2n + a_1, -2n + a_2)$ has 0 as a cyclic vertex and is isomorphic to $G(-2n+1, -2n+a)$ with $a-1 \pm 3^m$ if and only if all three conditions laid out in the statement of the theorem are satisfied.

In Theorem 5, we used both types of graph isomorphisms to simplify some of the Type 3 ECSDs into a single ECSD, where one of the $a_i's$ is equal to 1. When $a_1, a_2 \equiv 1 \pmod 3$, from Theorem 1 with $k = -\frac{a_1-1}{3}$, we have that

$$G(-2n + a_1, -2n + a_2) \cong G(-2n + 1, -2n + [a_2 - a_1] + 1)$$

by the map $\phi_{1,k}(n) = -(n+k)$. When some $a_i \equiv 2 \pmod 3$, from Theorem 2 with $k = -\frac{a_1+1}{3}$, we have that

$$G(-2n + a_1, -2n + a_2) \cong G(-2n + 1, -2n + [a_1 - a_2] + 1)$$

by the map $\phi_{2,k}(n) = n + k$.

In order to classify the ECSDs isomorphic to $G(-2n+1, -2n+a)$ with $a = 1 \pm 3^m$, in either isomorphism case we must have $|a_1 - a_2| = 3^m$, necessitating the first condition.

Next, we consider the a_i modulo 3. If either of the a_i is congruent to 0 modulo 3, from the proof of Theorem 5, it is isomorphic to an ECSD where one of the a_i is equal to 0. The only ECSD with one of the a_i equal to 0 that has a single component is $G(-2n, -2n+1)$ (see Theorem 10 and the ECSD depicted in the right half of Figure 11). Similar to our argument for $G(2n, -2n+1)$, this ECSD has a single cyclic vertex at 0, and the isomorphism $\phi_{2,0}$ gives us a single other ECSD with cyclic vertex 0, namely $G(-2n, -2n-1)$. Otherwise, neither a_1 or a_2 may be congruent to 0 modulo 3, giving us our second condition.

Lastly, when implementing the two isomorphisms above, we must ensure that 0 remains a cyclic vertex. For the map $\phi_{1,k}$ above, 0 is a cyclic vertex of $G(-2n + a_1, -2n + a_2)$ if and only if $k = -\frac{a_1-1}{3}$ is a cyclic vertex of $G(-2n+1, -2n+[a_2 - a_1]+1)$, since $\phi_{1,k}(0) = k$. So for a cyclic vertex j of $G(-2n+1, -2n+[a_2-a_1]+1)$, we must have $a_1 = -3j + 1$.

Similarly, for the map $\phi_{2,k}$, 0 is a cyclic vertex of $G(-2n + a_1, -2n + a_2)$ if and only if $-k = \frac{a_1+1}{3}$ is a cyclic vertex of $G(-2n + 1, -2n + [a_1 - a_2] + 1)$, since

$\phi_{2,k}(0) = -k$. So for a cyclic vertex j of $G(-2n+1, -2n+[a_1-a_2]+1)$, we must have $a_1 = 3j-1$.

Now, we consider the cyclic vertices of $G(-2n+1, -2n+a)$ when $a = 1 \pm 3^m$. For ease of computation, first consider the case $a > 0$. By Theorem 11 we know that the range of cyclic vertices j of $G(-2n+1, -2n+a)$ is $-\frac{1}{3}(a-1) < j < \frac{2}{3}a$ when $a > 0$. Thus, since $a = 1 + 3^m$, we have that $a-1$ is divisible by 3, so

$$-\frac{1}{3}(a-1) + 1 \leq j \leq \frac{2}{3}(a-1),$$

and this range of integers is entirely cyclic vertices.

For the map $\phi_{1,k}$, it follows that $-2(a-1)+1 \leq -3j+1 \leq (a-1)-2$, so the range of a_1 is

$$-2(a-1)+1 \leq a_1 \leq (a-1)-2.$$

Since $[a_2 - a_1] + 1 = a$, the range of a_2 is

$$-(a-1)+1 \leq a_2 \leq 2(a-1)-2.$$

This gives us all of the valid pairs a_1 and a_2 which are both congruent to 1 modulo 3.

Similarly, for the map $\phi_{2,k}$ where $a_1 = 3j-1$ and $[a_1 - a_2] + 1 = a$, we get the ranges

$$\begin{aligned} -(a-1)+2 \leq a_1 \leq 2(a-1)-1 \text{ and} \\ -2(a-1)+2 \leq a_2 \leq (a-1)-1. \end{aligned}$$

This gives us all of the valid pairs a_1 and a_2 which are both congruent to 2 modulo 3. Since $(a-1) = |a_1 - a_2|$ in both cases, we can summarize this by giving the third condition that $\max\{|a_1|, |a_2|\} \leq 2|a_1 - a_2| - 1$.

In the case where $a < 0$, we use the map in Theorem 1 with $k = -\frac{1}{3}(a-1)$ to get the isomorphism $G(-2n+1, -2n+a) \cong G(-2n+1, -2n-a+2)$. Using $\phi_{1,k}(j) = j + k$, we get that the range of cyclic vertices l of these graphs is

$$\frac{1}{3}(a-1) \leq l \leq -\frac{2}{3}(a-1) - 1.$$

Then, analogous arguments hold for determining the ranges of a_1 and a_2 for each isomorphism above. Notably, for a fixed value of a, the valid pairs a_1 and a_2 derived from $G(-2n+1, -2n+a)$ are the same as the valid pairs derived from $G(-2n+1, -2n-a+2)$. □

As an example, the ECSD $G(-2n+5, -2n+2)$ has a single component with 0 a cyclic vertex, because $5 = 2|a_1 - a_2| - 1$. In contrast, the ECSD $G(-2n+7, -2n+4)$ has a single component, but 0 is not a cyclic vertex. Both of these graphs are

isomorphic to $G(-2n+1,-2n+4)$. Consequently, the digit set $\{5,2\}$ represents all integers in base -2, but $\{7,4\}$ is not such a digit set.

Following directly from Theorems 17 and 14, we may now describe all 2-digit non-standard binary representations that represent all integers.

Corollary 2. *For integers a_1 and a_2,*

$$\mathbb{Z} = \left\{ \sum_{j=0}^{k} b_j(-2)^j : b_j \in \{a_1, a_2\}, k \in \mathbb{N} \right\}$$

if and only if all three of the following conditions are satisfied:

(i) $|a_1 - a_2| = 3^m$ *for some* $m \in \mathbb{N}_0$,

(ii) $\{a_1, a_2\} = \{0, \pm 1\}$ *or* $a_1, a_2 \not\equiv 0 \pmod 3$,

(iii) $\max\{|a_1|, |a_2|\} \leq 2|a_1 - a_2| - 1$.

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