ON CERTAIN PRODUCTS OF THREE STIRLING NUMBERS

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Abstract

We present conjectural inequalities between triple products of elementary or complete homogeneous symmetric polynomial functions whose arguments are consecutive terms of a log-concave sequence of positive real numbers. These conjectures stem from numerical observations in the particular case of the Stirling numbers. We provide algebraic proofs, but for a few particular instances. It is eventually suggested that these inequalities might receive general bijective proofs involving the dominance ordering of certain restricted integer partitions.

1. Introduction

The *log-concavity* of a sequence of real numbers a_k, indexed with $k \in \mathbb{N}$, is characterized by an inequality between products of two terms of the sequence, namely $a_k^2 \geq a_{k-1}a_{k+1}$, for all $k \geq 1$. Moreover, when $a_i a_j \geq a_{i-1}a_{j+1}$ for all $j \geq i-1 \geq 0$, the sequence is said to be *strongly log-concave.* Log-concave sequences of positive real numbers are strongly log-concave. Log-concavity inequalities have been studied for centuries; Newton's inequality is among the earliest examples, and the literature around this topic is vast and still growing (see [2] for instance). Let us denote the binomial coefficient by $\binom{n}{k}$. A two-parameter sequence $a_{n,k}$ ($k \leq n$) of non-negative real numbers is *row log-concave* when it is log-concave in k for fixed n, and *ultra log-concave* when $a_{n,k}\binom{n}{k}^{-1}$ is row log-concave.

Many classical two-parameter integer sequences, like the binomial coefficients or the Stirling numbers, are ultra log-concave. Their ultra log-concavity is guaranteed when the polynomial generating function of the sequence is real-rooted. This is the case of elementary and complete homogeneous symmetric polynomial functions of a fixed number of non-negative real variables, considered as sequences indexed by the degree of the polynomial. Moreover, Sagan [7] has shown that other sequences formed with elementary and complete homogeneous symmetric polynomial functions, but at fixed degree or at fixed sum of the degree and the number of variables,

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are strongly log-concave, provided that their arguments are themselves consecutive terms of a strongly log-concave sequence.

In this paper, we present new inequalities between products of three terms of certain two-parameter sequences, which strengthen, in some cases, their ultra log-concavity. As our interest in this work was triggered by Stirling numbers, we start with them. They are denoted by ${n \brack k}$ and ${n \brace k}$, respectively, for the first and second kind, as in [4]. We then propose more general inequalities involving triple products of symmetric polynomial functions, whose arguments are consecutive terms of a log-concave sequence of positive real numbers. We provide algebraic proofs, but for a few particular instances. We also show how, in the general case, these inequalities might receive bijective proofs involving the dominance order of certain restricted integer partitions. Finally, some connections with other inequalities are discussed.

2. Bounding a Binomial Coefficient with Stirling Numbers

Theorem 2.1. *Let n, p, and j be positive integers such that $n \geq p$. We have*

$$\frac{{p+j \brace j}{n+j \brace p+j}}{{n+j \brace j}} \geq \binom{n}{p} \geq \frac{{p+j \brack j}{n+j \brack p+j}}{{n+j \brack j}}.$$

Proof. Assuming that $1 \leq j \leq p \leq n$, this is equivalent to showing that

$${n \brace j}\binom{n-j}{p-j} \leq {p \brace j}{n \brace p} \tag{2.1}$$

and

$${n \brack j}\binom{n-j}{p-j} \geq {p \brack j}{n \brack p}. \tag{2.2}$$

These inequalities are not new. A bijective proof of Inequality (2.1) appears online in an answer to a post on the MathOverflow forum [10]. We provide inductive proofs that utilize only the basic recurrences of the Stirling numbers. Inequality (2.1) clearly holds for $n = p = j = 1$. Suppose that it also holds for all j and p such that $j \leq p \leq n$. Then, for $j \leq p \leq n$, we have

$$\begin{aligned}
{p \brace j}{n+1 \brace p} &= p{p \brace j}{n \brace p} + {p \brace j}{n \brace p-1} \\
&= p{p \brace j}{n \brace p} + j{p-1 \brace j}{n \brace p-1} + {p-1 \brace j-1}{n \brace p-1} \\
&\geq p{n \brace j}\binom{n-j}{p-j} + j{n \brace j}\binom{n-j}{p-1-j} + {n \brace j-1}\binom{n-j+1}{p-j}
\end{aligned}$$

$$
\begin{aligned}
&\geq j\left\{{n \atop j}\right\}\binom{n-j}{p-j}+j\left\{{n \atop j}\right\}\binom{n-j}{p-1-j}+\left\{{n \atop j-1}\right\}\binom{n-j+1}{p-j}\\
&\geq j\left\{{n \atop j}\right\}\binom{n+1-j}{p-j}+\left\{{n \atop j-1}\right\}\binom{n-j+1}{p-j}\\
&=\left\{{n+1 \atop j}\right\}\binom{n+1-j}{p-j}.
\end{aligned}
$$

Moreover, if $p = n + 1$, the above inequality trivially holds for all $j \leq p$.

Similarly, Inequality (2.2) clearly holds for $n = p = j = 1$. Suppose that it also holds for all j and p such that $j \leq p \leq n$. Then, for $j \leq p \leq n$, we have

$$
\begin{aligned}
\left[{p \atop j}\right]\left[{n+1 \atop p}\right] &= n\left[{p \atop j}\right]\left[{n \atop p}\right]+\left[{p \atop j}\right]\left[{n \atop p-1}\right]\\
&= n\left[{p \atop j}\right]\left[{n \atop p}\right]+(p-1)\left[{p-1 \atop j}\right]\left[{n \atop p-1}\right]+\left[{p-1 \atop j-1}\right]\left[{n \atop p-1}\right]\\
&\leq n\left[{n \atop j}\right]\binom{n-j}{p-j}+(p-1)\left[{n \atop j}\right]\binom{n-j}{p-1-j}+\left[{n \atop j-1}\right]\binom{n-j+1}{p-j}\\
&\leq n\left[{n \atop j}\right]\binom{n-j}{p-j}+n\left[{n \atop j}\right]\binom{n-j}{p-1-j}+\left[{n \atop j-1}\right]\binom{n-j+1}{p-j}\\
&\leq n\left[{n \atop j}\right]\binom{n-j+1}{p-j}+\left[{n \atop j-1}\right]\binom{n-j+1}{p-j}\\
&=\left[{n+1 \atop j}\right]\binom{n+1-j}{p-j}.
\end{aligned}
$$

Moreover, if $p = n + 1$, the above inequality trivially holds for all $j \leq p$. $\square$

One may wonder how tight the bounds for the binomial coefficient presented in Theorem 2.1 are. In Figure 1, we plot $\ell(j) := \frac{\left\{{p+j \atop j}\right\}\left\{{n+j \atop p+j}\right\}}{\left\{{n+j \atop j}\right\}}$ and $r(j) := \frac{\left[{p+j \atop j}\right]\left[{n+j \atop p+j}\right]}{\left[{n+j \atop j}\right]}$ as functions of j in the case $n = 5$ and $p = 3$. We observe that r (respectively ℓ) is an increasing (respectively decreasing) function of j and that these bounds become tighter as j increases. In fact, there is no limit to their tightness, since $\ell(i)$ and $r(i)$ are asymptotically equal ($\sim$) to the binomial coefficient $\binom{n}{p}$. Indeed, it is known ([4], p. 271) that both $\left\{{n+j \atop j}\right\}$ and $\left[{n+j \atop j}\right]$ are polynomial functions of j of degree $2n$, given by

$$\left\{{n+j \atop j}\right\} = \sum_{k=0}^{n-1}\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle\binom{j+2n-1-k}{2n} \tag{2.3}$$

and

$$\left[{n+j \atop j}\right] = \sum_{k=0}^{n-1}\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle\binom{j+n+k}{2n}.$$

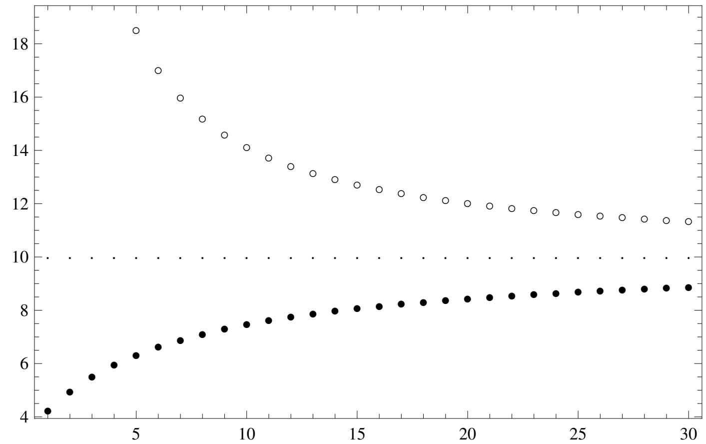

Figure 1: r ($\bullet$) and ℓ ($\circ$) as functions of j, for $p = 3$, $n = 5$.

Here, $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ denotes the *second-order Eulerian numbers* ([4], p. 270), whose row-sum satistfies $\sum_{k=0}^{n-1} \left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle = \frac{(2n)!}{2^n n!}$. It follows that $\left\{ {n+j \atop j} \right\} \sim \left[{n+j \atop j} \right] \sim \frac{j^{2n}}{2^n n!}$ and then

$$\ell(j) \sim r(j) \sim \frac{\frac{j^{2p}}{2^p p!} \frac{j^{2n-2p}}{2^{n-p}(n-p)!}}{\frac{j^{2n}}{2^n n!}} = \binom{n}{p}.$$

The numerically observed monotonicity of $\ell(j)$ and $r(j)$ leads us to the following conjecture.

Conjecture 2.1. Let n, p, and j be three integers such that $1 \le j \le p \le n$. We have

$$\left\{ {p-1 \atop j-1} \right\} \left\{ {n-1 \atop p-1} \right\} \left\{ {n \atop j} \right\} \ge \left\{ {p \atop j} \right\} \left\{ {n \atop p} \right\} \left\{ {n-1 \atop j-1} \right\}$$

and

$$\left[{p-1 \atop j-1} \right] \left[{n-1 \atop p-1} \right] \left[{n \atop j} \right] \le \left[{p \atop j} \right] \left[{n \atop p} \right] \left[{n-1 \atop j-1} \right].$$

Remark 2.1.1. If the Stirling numbers are replaced by binomial coefficients or by their q-analogs, these conjectured inequalities become elementary equalities.

These inequalities are rather strong statements. For example, in the first one, after incrementing n, p, and j by 1 and setting $j = p - 1$, we obtain (for $p \ge 2$)

$$\left\{ {n \atop p} \right\}^2 \ge \left\{ {n \atop p+1} \right\} \left\{ {n \atop p-1} \right\} \Big(\frac{p+1}{p}\Big)^2 \frac{p}{p-1} + \frac{2}{p(p-1)} \left\{ {n \atop p-1} \right\} \left\{ {n \atop p} \right\}. \tag{2.4}$$

Inequality (2.4) is stronger than the row log-concavity of the Stirling numbers, stronger than $\left\{ {n \atop p} \right\}^2 \geq \frac{p}{p-1} \left\{ {n \atop p+1} \right\} \left\{ {n \atop p-1} \right\}$ proved in [6], stronger than their ultra log-concavity $\left\{ {n \atop p} \right\}^2 \geq \frac{p+1}{p} \frac{n-p+1}{n-p} \left\{ {n \atop p+1} \right\} \left\{ {n \atop p-1} \right\}$, and even stronger than the refined ultra log-concavity $\left\{ {n \atop p} \right\}^2 \geq \frac{p+1}{p-1} \frac{n-p+1}{n-p} \left\{ {n \atop p+1} \right\} \left\{ {n \atop p-1} \right\}$ established in [9], as we now prove. After some manipulation, this requires showing that

$$\left\{ {n \atop p} \right\} > \left\{ {n \atop p+1} \right\} \frac{(2p-n)(p+1)}{2(n-p)}. \tag{2.5}$$

We need the following Lemma.

Lemma 2.1. *Let n and k be integers such that $0 < k \leq n$. Then*

$$n \left\{ {n \atop n-k} \right\} > (n-2k) \left\{ {n+1 \atop n+1-k} \right\}.$$

Proof. If $0 < i \leq k$, then clearly $(n+i)(n-2k) < n(n+i-2k)$. Incidentally, note that the log-concavity of a sequence of consecutive integers is used here. It follows that $n\binom{n+i-1}{2k} > (n-2k)\binom{n+i}{2k}$. Now, from Equation (2.3), it holds that $\left\{ {n \atop n-k} \right\} = \sum_{i=1}^{k} \left\langle\!\left\langle {k \atop k-i} \right\rangle\!\right\rangle \binom{n+i-1}{2k}$ and then $n\left\{ {n \atop n-k} \right\} > (n-2k)\left\{ {n+1 \atop n+1-k} \right\}$. □

Finally, setting $k = n-p$ in the lemma yields $n\left\{ {n \atop p} \right\} > (2p-n)\left\{ {n+1 \atop p+1} \right\}$. That is, $n\left\{ {n \atop p} \right\} > (2p-n)(p+1)\left\{ {n \atop p+1} \right\} + (2p-n)\left\{ {n \atop p} \right\}$ which readily implies Inequality (2.5).

3. Symmetric Polynomials

We recall e_k and h_k, respectively the *elementary* and *complete homogeneous symmetric polynomials of degree* k, in the n indeterminates $\mathbf{x_n} := (x_1, x_2, \ldots, x_n)$, which are defined as

$$e_k(\mathbf{x_n}) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_n}$$

and

$$h_k(\mathbf{x_n}) := \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_n},$$

together with the convention $e_0(\mathbf{x_n}) = h_0(\mathbf{x_n}) := 1$. From these definitions, the following recurrence relations can be derived

$$e_k(\mathbf{x_n}) = x_n e_{k-1}(\mathbf{x_{n-1}}) + e_k(\mathbf{x_{n-1}}) \tag{3.1}$$

and

$$h_k(\mathbf{x_n}) = x_n h_{k-1}(\mathbf{x_n}) + h_k(\mathbf{x_{n-1}}).$$

Their ordinary generating functions are

$$\sum_{k\geq 0} e_k(\mathbf{x_n})t^k = \prod_{i=1}^{n}(1+x_i t) \tag{3.2}$$

and

$$\sum_{k\geq 0} h_k(\mathbf{x_n})t^k = \prod_{i=1}^{n}\frac{1}{1-x_i t}. \tag{3.3}$$

Let $\mathbf{q_n} \coloneqq (q^0, q^1, \ldots, q^{n-1})$. We then see that there are $e_k(\mathbf{1_n}) = \binom{n}{k}$ monomials in the expansion of $e_k(\mathbf{x_n})$, and $h_k(\mathbf{1_n}) = \binom{n+k-1}{k}$ monomials in that of $h_k(\mathbf{x_n})$. Comparing Equations (3.2) and (3.3) with the known ordinary generating functions of the Stirling numbers, $\sum_{k\geq 0} {n \brack k} t^k = \prod_{j=0}^{n-1}(t+j)$ and $\sum_{n\geq 0} {n \brace k} t^n = \prod_{j=1}^{k}\frac{t}{1-jt}$, we obtain

$$\begin{bmatrix} n+1 \\ n+1-k \end{bmatrix} = e_k(1,2,\ldots,n)$$

and

$$\begin{Bmatrix} n+k \\ n \end{Bmatrix} = h_k(1,2,\ldots,n),$$

so that, setting $\mathbf{v_n} \coloneqq (1,2,\ldots,n)$, the inequalities of Conjecture 2.1 read

$$\frac{e_{n-p}(\mathbf{v_n})}{e_{n-p}(\mathbf{v_{n-1}})}\frac{e_{p-j}(\mathbf{v_p})}{e_{p-j}(\mathbf{v_{p-1}})} \geq \frac{e_{n-j}(\mathbf{v_n})}{e_{n-j}(\mathbf{v_{n-1}})}$$

and

$$\frac{h_{n-p}(\mathbf{v_{p+1}})}{h_{n-p}(\mathbf{v_p})}\frac{h_{p-j}(\mathbf{v_{j+1}})}{h_{p-j}(\mathbf{v_j})} \leq \frac{h_{n-j}(\mathbf{v_{j+1}})}{h_{n-j}(\mathbf{v_j})}.$$

Note that, if we replace $\mathbf{v}$ by $\mathbf{q}$ in the above inequalities, they become equalities, since $e_k(\mathbf{q_n}) = q^{\binom{k}{2}}\binom{n}{k}_q$ and $h_k(\mathbf{q_n}) = \binom{n+k-1}{k}_q$, where $\binom{n}{k}_q$ denotes the q-binomial coefficient. The sequence $v_i = i$ is clearly log-concave, whereas the sequence q^i is *log-straight*, that is, simultaneously log-concave and log-convex. A sequence of real numbers x_i is log-straight if and only if $x_i = q^i$, for some real q, up to a real multiplicative constant.

We then propose the following generalisation of Conjecture 2.1.

Conjecture 3.1. Let x_k $(k \geq 1)$ be a log-concave sequence of positive real numbers, and let n, p, and j be integers such that $1 \leq j \leq p \leq n$. Then

$$\frac{e_{n-p}(\mathbf{x_n})}{e_{n-p}(\mathbf{x_{n-1}})}\frac{e_{p-j}(\mathbf{x_p})}{e_{p-j}(\mathbf{x_{p-1}})} \geq \frac{e_{n-j}(\mathbf{x_n})}{e_{n-j}(\mathbf{x_{n-1}})}$$

and

$$\frac{h_{n-p}(\mathbf{x_{p+1}})}{h_{n-p}(\mathbf{x_p})}\frac{h_{p-j}(\mathbf{x_{j+1}})}{h_{p-j}(\mathbf{x_j})} \leq \frac{h_{n-j}(\mathbf{x_{j+1}})}{h_{n-j}(\mathbf{x_j})}.$$

If the sequence is log-convex, then the directions of the inequalities are reversed.

We provide algebraic proofs of the above conjecture in several simple special cases. We exclude the trivial cases in which two of n, p, and j are equal. After replacing n and p by $n+p+j$ and $p+j$, respectively, these cases are summarized in the following theorem.

Theorem 3.1. *Let n, p, and j be positive integers, two of which are equal to 1, and let x_k $(k \geq 1)$ be a log-concave sequence of positive real numbers. Then*

$$\frac{e_n(\mathbf{x_{n+p+j}})}{e_n(\mathbf{x_{n+p+j-1}})}\frac{e_p(\mathbf{x_{p+j}})}{e_p(\mathbf{x_{p+j-1}})} \geq \frac{e_{n+p}(\mathbf{x_{n+p+j}})}{e_{n+p}(\mathbf{x_{n+p+j-1}})}$$

and

$$\frac{h_n(\mathbf{x_{p+j+1}})}{h_n(\mathbf{x_{p+j}})}\frac{h_p(\mathbf{x_{j+1}})}{h_p(\mathbf{x_j})} \leq \frac{h_{n+p}(\mathbf{x_{j+1}})}{h_{n+p}(\mathbf{x_j})}.$$

If the sequence is log-convex, then the directions of the inequalities are reversed.

Proof. We deal with the complete homogeneous symmetric polynomials first. Let $\Delta(n,p,j) := h_n(\mathbf{x_{p+j}})h_p(\mathbf{x_j})h_{n+p}(\mathbf{x_{j+1}}) - h_n(\mathbf{x_{p+j+1}})h_p(\mathbf{x_{j+1}})h_{n+p}(\mathbf{x_j})$. We are going to show that

$$\Delta(1,1,j) = h_1(\mathbf{x_{j+1}})\sum_{a=1}^{j} h_1(\mathbf{x_a})\big(x_{j+1}x_{a+1} - x_{j+2}x_a\big), \tag{3.4}$$

$$\Delta(1,p,1) = x_1^p\sum_{b=0}^{p} x_2^b\big(x_2^{p-b+1}x_{b+1} - x_1^{p-b+1}x_{p+2}\big), \tag{3.5}$$

and

$$\Delta(n,1,1) = h_1(\mathbf{x_1})h_1(\mathbf{x_2})\sum_{c=1}^{n} h_{n-c}(\mathbf{x_1})h_{n-c}(\mathbf{x_2})\big(x_2^{2c} - x_1^c x_3^c\big). \tag{3.6}$$

From the strong log-concavity of x_k, $\Delta(1,1,j)$ and $\Delta(n,1,1)$ are clearly non-negative. The non-negativity of $\Delta(1,p,1)$ is shown by consecutively using the strong log-concavity inequality $p-b+1$ times for the summands corresponding to $b \geq 1$ and p times for the summand corresponding to $b=0$.

The proof of Equation (3.4) is obtained by the following derivation. We have

$$\begin{aligned}
\Delta(1,1,j) &= h_1(\mathbf{x_{j+1}})\big(h_1(\mathbf{x_j})h_2(\mathbf{x_{j+1}}) - h_1(\mathbf{x_{j+2}})h_2(\mathbf{x_j})\big)\\
&= h_1(\mathbf{x_{j+1}})\Big(\sum_{a=1}^{j} x_a \sum_{a=1}^{j+1} x_a h_1(\mathbf{x_a}) - \sum_{a=1}^{j+2} x_a \sum_{a=1}^{j} x_a h_1(\mathbf{x_a})\Big)\\
&= h_1(\mathbf{x_{j+1}})\Big(\sum_{a=1}^{j} x_a \sum_{a=1}^{j} x_a h_1(\mathbf{x_a}) + \sum_{a=1}^{j} x_a x_{j+1} h_1(\mathbf{x_{j+1}})\\
&\qquad - x_{j+1}\sum_{a=1}^{j} x_a h_1(\mathbf{x_a}) - x_{j+2}\sum_{a=1}^{j} x_a h_1(\mathbf{x_a}) - \sum_{a=1}^{j} x_a \sum_{a=1}^{j} x_a h_1(\mathbf{x_a})\Big)
\end{aligned}$$

$$
\begin{aligned}
&= h_1(\mathbf{x_{j+1}})\Big(x_{j+1}\sum_{a=1}^{j} x_a \sum_{i=a+1}^{j+1} x_i - x_{j+2}\sum_{a=1}^{j} x_a \sum_{i=1}^{a} x_i)\Big)\\
&= h_1(\mathbf{x_{j+1}})\Big(x_{j+1}\sum_{i=1}^{j} x_i \sum_{a=i+1}^{j+1} x_a - x_{j+2}\sum_{a=1}^{j} x_a \sum_{i=1}^{a} x_i)\Big)\\
&= h_1(\mathbf{x_{j+1}})\Big(x_{j+1}\sum_{i=1}^{j} x_i \sum_{a=i}^{j} x_{a+1} - x_{j+2}\sum_{a=1}^{j} x_a \sum_{i=1}^{a} x_i)\Big)\\
&= h_1(\mathbf{x_{j+1}})\Big(x_{j+1}\sum_{a=1}^{j} x_{a+1} \sum_{i=1}^{a} x_i - x_{j+2}\sum_{a=1}^{j} x_a \sum_{i=1}^{a} x_i)\Big)\\
&= h_1(\mathbf{x_{j+1}})\sum_{a=1}^{j} h_1(\mathbf{x_a})\big(x_{j+1}x_{a+1} - x_{j+2}x_a\big).
\end{aligned}
$$

The proof of Equation (3.5) is obtained by the following derivation. We have

$$
\begin{aligned}
\Delta(1,p,1) &= x_1^p(h_1(\mathbf{x_{p+1}})h_{p+1}(\mathbf{x_2}) - x_1h_1(\mathbf{x_{p+2}})h_p(\mathbf{x_2}))\\
&= x_1^p(h_1(\mathbf{x_{p+1}})(h_{p+1}(\mathbf{x_2}) - x_1h_p(\mathbf{x_2})) - x_1x_{p+2}h_p(\mathbf{x_2}))\\
&= x_1^p(h_1(\mathbf{x_{p+1}})x_2^{p+1} - x_1x_{p+2}h_p(\mathbf{x_2}))\\
&= x_1^p\Big(\sum_{b=1}^{p+1} x_b x_2^{p+1} - x_1x_{p+2}\sum_{b=0}^{p} x_1^{p-b}x_2^b\Big)\\
&= x_1^p\sum_{b=0}^{p} x_2^b\big(x_2^{p-b+1}x_{b+1} - x_1^{p-b+1}x_{p+2}\big).
\end{aligned}
$$

Finally, for the proof of Equation (3.6), let

$$a(n) \coloneqq h_1(\mathbf{x_1})h_{n+1}(\mathbf{x_2})h_n(\mathbf{x_2})$$

and

$$b(n) \coloneqq h_1(\mathbf{x_2})h_{n+1}(\mathbf{x_1})h_n(\mathbf{x_3}),$$

with $n \geq 1$, so that $\Delta(n,1,1) = a(n) - b(n)$. We have

$$
\begin{aligned}
a(n) &= h_1(\mathbf{x_1})\big(x_2h_n(\mathbf{x_2}) + x_1^{n+1}\big)\big(x_2h_{n-1}(\mathbf{x_2}) + x_1^n\big)\\
&= x_2^2a(n-1) + h_1(\mathbf{x_1})h_1(\mathbf{x_2})h_n(\mathbf{x_1})h_n(\mathbf{x_2})
\end{aligned}
$$

and

$$
\begin{aligned}
b(n) &= h_1(\mathbf{x_2})\big(x_1h_n(\mathbf{x_1})x_3h_{n-1}(\mathbf{x_3}) + x_1^{n+1}h_n(\mathbf{x_2})\big)\\
&= x_1x_3b(n-1) + h_1(\mathbf{x_1})h_1(\mathbf{x_2})h_n(\mathbf{x_1})h_n(\mathbf{x_2}).
\end{aligned}
$$

Let $M \coloneqq \left(\begin{smallmatrix} x_2^2 & 0 \\ 0 & x_1x_3 \end{smallmatrix}\right)$, $U_n \coloneqq \left(\begin{smallmatrix} a(n) \\ b(n) \end{smallmatrix}\right)$, and $U_0 \coloneqq \left(\begin{smallmatrix} h_1(\mathbf{x_1})h_1(\mathbf{x_2}) \\ h_1(\mathbf{x_1})h_1(\mathbf{x_2}) \end{smallmatrix}\right)$, then

$$U_n = M \cdot U_{n-1} + h_1(\mathbf{x_1})h_1(\mathbf{x_2})h_n(\mathbf{x_1})h_n(\mathbf{x_2})M^0$$

and, by telescoping, we obtain

$$U_n = h_1(\mathbf{x_1})h_1(\mathbf{x_2})\sum_{c=0}^{n} h_{n-c}(\mathbf{x_1})h_{n-c}(\mathbf{x_2})M^c,$$

which is the desired result.

Now, we deal with the elementary symmetric polynomials. In the case $n = p = 1$, we have to show that $e_1(\mathbf{x_{j+2}})e_2(\mathbf{x_{j+1}}) \geq e_1(\mathbf{x_j})e_2(\mathbf{x_{j+2}})$. Using the recurrence relation Equation (3.1), together with the identity $e_1(\mathbf{x_j})e_1(\mathbf{x_{j+1}}) - e_2(\mathbf{x_{j+1}}) = h_2(\mathbf{x_j})$, we see that it suffices to prove that $x_{j+1}e_2(\mathbf{x_{j+1}}) \geq x_{j+2}h_2(\mathbf{x_j})$, which readily follows from the strong log-concavity of x_k. Indeed, we have

$$\begin{aligned} x_{j+1}e_2(\mathbf{x_{j+1}}) &= x_{j+1}\sum_{1\leq\alpha<\beta\leq j+1} x_\alpha x_\beta \\ &= x_{j+1}\sum_{1\leq\alpha\leq\beta\leq j} x_\alpha x_{\beta+1} \\ &\geq x_{j+2}\sum_{1\leq\alpha\leq\beta\leq j} x_\alpha x_\beta = x_{j+2}h_2(\mathbf{x_j}). \end{aligned}$$

For the case $j = p = 1$, we must prove that

$$e_n(\mathbf{x_{n+2}})e_1(\mathbf{x_2})e_{n+1}(\mathbf{x_{n+1}}) \geq e_n(\mathbf{x_{n+1}})e_1(\mathbf{x_1})e_{n+1}(\mathbf{x_{n+2}}). \tag{3.7}$$

Let $y_k = x_k^{-1}$ for all $k \geq 1$ (which is licit since all x_k are positive). Clearly, y_k is a log-convex sequence of positive integers, and it satisfies

$$e_k(\mathbf{x_n}) = \frac{e_{n-k}(\mathbf{y_n})}{\prod_{i=1}^{n} y_i}. \tag{3.8}$$

Substituting Equation (3.8) into Inequality (3.7) and simplifying (which is valid since all y_i are positive), we are reduced to proving

$$e_1(\mathbf{y_2})e_2(\mathbf{y_{n+2}}) \geq y_2 e_1(\mathbf{y_{n+1}})e_1(\mathbf{y_{n+2}}).$$

That is,

$$\begin{aligned} y_1 e_2(\mathbf{y_{n+2}}) &\geq y_2\big(e_1(\mathbf{y_{n+1}})e_1(\mathbf{y_{n+2}}) - e_2(\mathbf{y_{n+2}})\big) \\ &\geq y_2 h_2(\mathbf{y_{n+1}}). \end{aligned}$$

The above inequality holds, as it was already shown (in the opposite direction, but for a log-concave sequence) in the previous case $n = p = 1$.

Finally, for the case $n = j = 1$, we have to prove that

$$e_1(\mathbf{x_{p+2}})e_p(\mathbf{x_{p+1}})e_{p+1}(\mathbf{x_{p+1}}) \geq e_1(\mathbf{x_{p+1}})e_p(\mathbf{x_p})e_{p+1}(\mathbf{x_{p+2}}).$$

Since $e_{p+1}(\mathbf{x_{p+1}}) = x_{p+1}e_p(\mathbf{x_p})$, this is equivalent to

$$e_1(\mathbf{x_{p+2}})e_p(\mathbf{x_{p+1}})x_{p+1} \geq e_1(\mathbf{x_{p+1}})e_{p+1}(\mathbf{x_{p+2}}). \tag{3.9}$$

We have

$$\begin{aligned}
&e_1(\mathbf{x_{p+2}})e_p(\mathbf{x_{p+1}})x_{p+1} - e_1(\mathbf{x_{p+1}})e_{p+1}(\mathbf{x_{p+2}})\\
&= x_{p+2}e_p(\mathbf{x_{p+1}})x_{p+1} + e_1(\mathbf{x_{p+1}})e_p(\mathbf{x_{p+1}})x_{p+1}\\
&\qquad - x_{p+2}e_1(\mathbf{x_{p+1}})e_p(\mathbf{x_{p+1}}) - e_1(\mathbf{x_{p+1}})e_{p+1}(\mathbf{x_{p+1}})\\
&= x_{p+2}\big(x_{p+1}e_p(\mathbf{x_{p+1}}) - e_1(\mathbf{x_{p+1}})e_p(\mathbf{x_{p+1}})\big)\\
&\qquad + e_1(\mathbf{x_{p+1}})e_p(\mathbf{x_{p+1}})x_{p+1} - e_1(\mathbf{x_{p+1}})e_{p+1}(\mathbf{x_{p+1}})\\
&= -x_{p+2}e_p(\mathbf{x_{p+1}})e_1(\mathbf{x_p}) + e_1(\mathbf{x_{p+1}})\big(e_p(\mathbf{x_{p+1}})x_{p+1} - e_{p+1}(\mathbf{x_{p+1}})\big)\\
&= -x_{p+2}e_p(\mathbf{x_{p+1}})e_1(\mathbf{x_p}) + e_1(\mathbf{x_{p+1}})x_{p+1}\big(e_p(\mathbf{x_{p+1}}) - e_p(\mathbf{x_p})\big)\\
&= -x_{p+2}e_p(\mathbf{x_{p+1}})e_1(\mathbf{x_p}) + e_1(\mathbf{x_{p+1}})x_{p+1}^2e_{p-1}(\mathbf{x_p}).
\end{aligned}$$

Then, Inequality (3.9) is equivalent to

$$x_{p+1}^2e_1(\mathbf{x_{p+1}})e_{p-1}(\mathbf{x_p}) \geq x_{p+2}e_1(\mathbf{x_p})e_p(\mathbf{x_{p+1}}). \tag{3.10}$$

Let L (respectively R) be the left-hand (respectively right-hand) side of Inequality (3.10). We have

$$R = \sum_{j=1}^{p}\sum_{i=1}^{p+1} x_{p+2}x_j \prod_{k\neq i}^{p+1} x_k,$$

where the product runs over all $k = 1, \cdots, p+1$, except i. We introduce $R(w)$ by reindexing this double sum with the new index $w = p+1+j-i$, which ranges from 1 to $2p$. Then

$$R = \sum_{w=1}^{2p}\sum_{i=imin}^{imax} x_{p+2}x_{w-p-1+i} \prod_{k\neq i}^{p+1} x_k = \sum_{w=1}^{2p} R(w),$$

with

$$R(w) \coloneqq \sum_{i=imin}^{imax} x_{p+2}x_{w-p-1+i} \prod_{k\neq i}^{p+1} x_k,$$

with bounds $imin = \max(p-w+2, 1)$ and $imax = \min(p+1, 2p-w+1)$. Actually, $imin = p-w+2$ and $imax = p+1$ if $w \leq p$, whereas $imin = 1$ and $imax = 2p-w+1$ if $w \geq p+1$. Similarly, introducing $L(w)$, we have

$$L = \sum_{i=1}^{p+1}\sum_{j=1}^{p} x_{p+1}^2 x_i \prod_{k\neq j}^{p} x_k = \sum_{w=1}^{2p}\sum_{i=imin}^{imax} x_{p+1}^2 x_i \prod_{k\neq p+i-w}^{p} x_k = \sum_{w=1}^{2p} L(w),$$

with

$$L(w) := \sum_{i=imin}^{imax} x_{p+1}^2 x_i \prod_{k\neq p+i-w}^{p} x_k,$$

with bounds $imin = 1$ and $imax = w$ if $w \leq p$, whereas $imin = w - p + 1$ and $imax = p + 1$ if $w \geq p + 1$. To complete the proof, it suffices to prove that $L(w) \geq R(w)$ for all w. The case $w \leq p$ is considered first. We obtain

$$\begin{aligned}
L(w) - R(w) &= \sum_{i=1}^{w} x_{p+1}^2 x_i \prod_{k\neq p+i-w}^{p} x_k - \sum_{i=p-w+2}^{p+1} x_{p+2} x_{w-p-1+i} \prod_{k\neq i}^{p+1} x_k \\
&= \sum_{i=1}^{w} x_{p+1}^2 x_i \prod_{k\neq p+i-w}^{p} x_k - \sum_{i=1}^{w} x_{p+2} x_{w-i+1} \prod_{k\neq p+2-i}^{p+1} x_k \\
&= \sum_{i=1}^{w} x_{p+1}^2 x_i \prod_{k\neq p+i-w}^{p} x_k - \sum_{i=1}^{w} x_{p+2} x_i \prod_{k\neq p+1+i-w}^{p+1} x_k \\
&= \sum_{i=1}^{w} x_i \Big(x_{p+1}^2 \prod_{k\neq p+i-w}^{p} x_k - \sum_{i=1}^{w} x_{p+2} \prod_{k\neq p+1+i-w}^{p+1} x_k \Big).
\end{aligned}$$

In this sum, each summand is non-negative since, multiplied by $x_{p+i-w} x_{p+1+i-w}$, it is $(x_{p+1} x_{p+1+i-w} - x_{p+2} x_{p+i-w}) \prod_{k=1}^{p+1} x_k$, which is non-negative by the strong log-concavity of x_k. For the case $w \geq p + 1$, we do the same derivation, but with the appropriate bounds $imin$ and $imax$, and we have

$$\begin{aligned}
L(w) - R(w) &= \sum_{i=w-p+1}^{p+1} x_{p+1}^2 x_i \prod_{k\neq p+i-w}^{p} x_k - \sum_{i=1}^{2p-w+1} x_{p+2} x_{w-p-1+i} \prod_{k\neq i}^{p+1} x_k \\
&= \sum_{i=1}^{2p-w+1} x_{p+1}^2 x_{p+2-i} \prod_{k\neq 2p+2-i-w}^{p} x_k - \sum_{i=1}^{2p-w+1} x_{p+2} x_{w-p-1+i} \prod_{k\neq i}^{p+1} x_k \\
&= \sum_{i=1}^{2p-w+1} x_{p+1}^2 x_{w-p+i} \prod_{k\neq i}^{p} x_k - \sum_{i=1}^{2p-w+1} x_{p+2} x_{w-p-1+i} \prod_{k\neq i}^{p+1} x_k \\
&= \sum_{i=1}^{2p-w+1} \prod_{k\neq i}^{p+1} x_k \Big(x_{p+1} x_{w-p+i} - x_{p+2} x_{w-p+i-1} \Big).
\end{aligned}$$

This sum is non-negative, again because each summand is non-negative by the log-concavity of x_k. □

The above proof of Inequality (3.9) is lengthy, but somehow insightful, as we shall observe at the end of this section. Upon reviewing, the anonymous referee pointed out the following, which is a much more concise inductive argument. Let $A_p := e_1(\mathbf{x_{p+1}})e_{p-1}(\mathbf{x_p})x_p$ and $B_p := e_1(\mathbf{x_p})e_p(\mathbf{x_{p+1}})$. Inequality (3.9) reads $A_{p+1} \geq B_{p+1}$ and Inequality (3.10) reads $x_{p+1}^2 A_p \geq x_p x_{p+2} B_p$. Since $A_1 = B_1 = x_1^2 + x_1x_2$ and $x_{p+1}^2 \geq x_p x_{p+2}$, we readily obtain an inductive proof.

In each case of Theorem 3.1, the proof uses different tricks from elementary algebra. But, as soon as at least two of the three indices n, p, and j are larger than 1, such techniques no longer seem to work. The algebraic derivations become very intricate. Nevertheless, we were able to confirm Conjecture 3.1 in the case where two indices are equal to 2 and the third equals 1. As an illustration of this increasing complexity, we write hereafter what we obtained, laboriously, for $\Delta(2,1,2)$.

$$\begin{aligned}\Delta(2,1,2) = (x_3^4 - x_2^2x_4^2)(2x_1^2 + 2x_1x_2 + x_2^2 + x_1x_3 + x_2x_3) + (x_3^2 - x_2x_4)P \\ + (x_2^2x_3^2 - x_1^2x_4^2)x_1(x_1 + x_2 + x_3) + (x_2x_3 - x_1x_4)x_1Q,\end{aligned}$$

where P and Q are polynomials with non-negative coefficients, given by

$$\begin{aligned}P &= x_1^4 + 2x_1^3x_2 + 3x_1^2x_2^2 + 3x_1x_2^3 + x_2^4 + 2x_1^3x_3 + 5x_1^2x_2x_3 + 4x_1x_2^2x_3 + 2x_2^3x_3 \\ &\quad + 2x_1x_2x_3^2 + x_2^2x_3^2 \\ \text{and } Q &= x_1^3 + 2x_1^2x_2 + 2x_1x_2^2 + x_2^3 + 2x_1^2x_3 + 2x_1x_2x_3 + x_1x_3^2 + x_2x_3^2 + x_1x_2x_4 \\ &\quad + x_2x_3x_4.\end{aligned}$$

Let us denote $\mathbb{E}_k^n$ (respectively $\mathbb{H}_k^n$) the set of the monomials of $e_k(\mathbf{x_n})$ (respectively $h_k(\mathbf{x_n})$). Since their cardinals are $|\mathbb{E}_k^n| = \binom{n}{k}$ and $|\mathbb{H}_k^n| = \binom{n+k-1}{k}$, it is easy to verify that

$$|\mathbb{E}_{n-p}^n \times \mathbb{E}_{p-j}^p \times \mathbb{E}_{n-j}^{n-1}| = |\mathbb{E}_{n-p}^{n-1} \times \mathbb{E}_{p-j}^{p-1} \times \mathbb{E}_{n-j}^n|$$

and

$$|\mathbb{H}_{n-p}^p \times \mathbb{H}_{p-j}^j \times \mathbb{H}_{n-j}^{j+1}| = |\mathbb{H}_{n-p}^{p+1} \times \mathbb{H}_{p-j}^{j+1} \times \mathbb{H}_{n-j}^j|.$$

In the cases of Theorem 3.1, it can be observed that each monomial on the left-hand side of an inequality is paired one-to-one with a monomial on the right-hand side, in such a way that the left monomial is larger than the corresponding right monomial (or vice versa). Therefore, one would need to construct such bijections between the (multi-)set of monomials of $e_n(\mathbf{x_{n+p+j}})e_p(\mathbf{x_{p+j}})e_{n+p}(\mathbf{x_{n+p+j-1}})$ and the (multi-)set of monomials of $e_n(\mathbf{x_{n+p+j-1}})e_p(\mathbf{x_{p+j-1}})e_{n+p}(\mathbf{x_{n+p+j}})$, and similarly, for complete homogeneous polynomials, between the the (multi-)set of monomials of $h_n(\mathbf{x_{p+j+1}})h_p(\mathbf{x_{j+1}})h_{n+p}(\mathbf{x_j})$ and that of $h_n(\mathbf{x_{p+j}})h_p(\mathbf{x_j})h_{n+p}(\mathbf{x_{j+1}})$. Moreover, in the first proof of the last case of Theorem 3.1, we also observe that there are proper subsets of monomials on one side of the inequality that are in bijective correspondence with subsets of monomials on the other side. To clarify these observations, it is convenient to translate them into the language of integer partitions. This is the purpose of the next section.

4. Integer Partitions

Let us recall some basic notions from the theory of integer partitions [1]. An integer partition λ (respectively, an integer partition into distinct parts) is a finite non-increasing (respectively, strictly decreasing) sequence of non-negative integers called the parts. The *length* of λ, denoted $\ell(\lambda)$, is the number of positive parts, while the *size* of λ is the sum of all its parts and is denoted $|\lambda|$. Let $w = |\lambda|$. Then λ is said to be a partition of w, written $\lambda \vdash w$. The finite set of all partitions of a given integer (equivalently, all partitions of a fixed size) can be partially ordered by the *dominance order* [3]. In this order, a partition λ *precedes* the partition μ (written $\lambda \preceq \mu$) if, for every $k \geq 1$, the sum of the k largest parts of λ is less than or equal to the sum of the k largest parts of μ. From their definition, each monomial in the complete homogeneous (respectively, elementary) symmetric polynomial can be indexed by an integer partition (respectively, an integer partition in distinct parts), and we denote such a monomial by X_λ. These partitions are, however, *restricted*. They must have the same length, equal to the degree of the polynomial, and their parts cannot exceed the number of indeterminates, although their sizes may vary.

We use the same notation $\mathbb{H}_k^n$ (respectively, $\mathbb{E}_k^n$) for the set of partitions in k parts (respectively, k distinct parts) not exceeding n and for the set of the monomials of $h_k(\mathbf{x_n})$ (respectively, $e_k(\mathbf{x_n})$), and then we may write

$$h_k(\mathbf{x_n}) = \sum_{\lambda \in \mathbb{H}_k^n} X_\lambda$$

and

$$e_k(\mathbf{x_n}) = \sum_{\lambda \in \mathbb{E}_k^n} X_\lambda.$$

The size of a partition in $\mathbb{H}_k^n$ ranges from k to kn, whereas the size of a partition in $\mathbb{E}_k^n$ ranges from $\frac{k(k+1)}{2}$ to $k(n+1) - \frac{k(k+1)}{2}$. In *superscript* notation, a partition $\lambda \in \mathbb{H}_k^n$ is written as $\lambda = 1^{a_1} \cdots n^{a_n}$, where $(a_1, \ldots, a_n)$ is a *weak composition* of k; that is, $a_i \geq 0$ for all $1 \leq i \leq n$ and $\sum_{i=1}^n a_i = k$. Similarly, a partition $\lambda \in \mathbb{E}_k^n$ is written $\lambda = 1^{a_1} \cdots n^{a_n}$, where $(a_1, \ldots, a_n)$ also verifies $\sum_{i=1}^n a_i = k$, but $1 \geq a_i \geq 0$ for all $1 \leq i \leq n$. Clearly, given two partitions λ and μ, the product of two monomials $X_\lambda \cdot X_\mu$ is the monomial $X_{\lambda\mu}$ where $\lambda\mu$ denotes the partition built by collecting together the parts of λ and μ. Clearly, $\ell(\lambda\mu) = \ell(\lambda) + \ell(\mu)$ and $|\lambda\mu| = |\lambda| + |\mu|$. Moreover, by Karamata's theorem [5], given two partitions λ and μ, if the sequence x_k is strongly log-concave, then the corresponding monomials satisfy the inequality $X_\lambda \geq X_\mu$ if and only if $\lambda \preceq \mu$. Note that this implies that partitions indexing two comparable monomials must have the same size.

We can now state our conjectures in the language of integer partitions.

Conjecture 4.1. *Let n, p, and j be positive integers. Let*

$$\begin{aligned}
\mathcal{H}^w_{n,p,j} &:=\{(\lambda_1,\lambda_2,\lambda_3)\in \mathbb{H}^{p+j}_n\times\mathbb{H}^{j}_p\times\mathbb{H}^{j+1}_{n+p};\ \ |\lambda_1|+|\lambda_2|+|\lambda_3|=w\},\\
\tilde{\mathcal{H}}^w_{n,p,j} &:=\{(\lambda_1,\lambda_2,\lambda_3)\in \mathbb{H}^{p+j+1}_n\times\mathbb{H}^{j+1}_p\times\mathbb{H}^{j}_{n+p};\ \ |\lambda_1|+|\lambda_2|+|\lambda_3|=w\},\\
\mathcal{E}^w_{n,p,j} &:=\{(\lambda_1,\lambda_2,\lambda_3)\in \mathbb{E}^{n+p+j}_n\times\mathbb{E}^{p+j}_p\times\mathbb{E}^{n+p+j-1}_{n+p};\ \ |\lambda_1|+|\lambda_2|+|\lambda_3|=w\},\ \textit{and}\\
\tilde{\mathcal{E}}^w_{n,p,j} &:=\{(\lambda_1,\lambda_2,\lambda_3)\in \mathbb{E}^{n+p+j-1}_n\times\mathbb{E}^{p+j-1}_p\times\mathbb{E}^{n+p+j}_{n+p};\ \ |\lambda_1|+|\lambda_2|+|\lambda_3|=w\}.
\end{aligned}$$

For each admissible w, there exists a bijective map from $\mathcal{H}^w_{n,p,j}$ to $\tilde{\mathcal{H}}^w_{n,p,j}$, sending $(\lambda_1,\lambda_2,\lambda_3)$ on (μ_1,μ_2,μ_3) such that $\lambda_1\lambda_2\lambda_3$ precedes $\mu_1\mu_2\mu_3$ in the dominance order. Similarly, there exists a bijective map from $\mathcal{E}^w_{n,p,j}$ to $\tilde{\mathcal{E}}^w_{n,p,j}$, sending $(\lambda_1,\lambda_2,\lambda_3)$ on (μ_1,μ_2,μ_3) such that $\lambda_1\lambda_2\lambda_3$ precedes $\mu_1\mu_2\mu_3$ in the dominance order.

Remark 4.1.1. If Conjecture 4.1 holds, then Conjecture 3.1 and Conjecture 2.1 hold as well.

Remark 4.1.2. In the complete homogeneous case, w ranges from $2(n+p)$ to $np+(2j+1)(n+p)$, while in the elementary case, w ranges from $n^2+p^2+np+n+p$ to $(n+p+2j)(n+p)$.

Conjecture 4.1 is obviously true in the complete homogeneous case for the minimal $w=2(n+p)$ because in this case $\mathcal{H}^{2(n+p)}_{n,p,j}=\tilde{\mathcal{H}}^{2(n+p)}_{n,p,j}=\{(1^n,1^p,1^{n+p})\}$. It is also true for the maximal $w=(2j+1)(n+p)+np$ because in this case $\mathcal{H}^w_{n,p,j}=\{((p+j)^n,j^p,(j+1)^{n+p})\}$ and $\tilde{\mathcal{H}}^w_{n,p,j}=\{((p+j+1)^n,(j+1)^p,j^{n+p})\}$. In both cases, the unique bijection between these one-element sets clearly satisfies the dominance condition. It is equally straightforward to verify Conjecture 4.1 in the elementary case for the minimal and maximal admissible w, where the corresponding sets again consist of a single element and the bijections are trivial.

To prove Conjecture 4.1 in the general cases, it would suffice to construct injective maps. Indeed, it is well known that the coefficient of q^w in the q-binomial coefficient $\binom{n+m}{m}_q$ counts the number of partitions of size w with at most m parts, each not exceeding n. Then, using the Pascal identity for q-binomial coefficients, one easily shows that the coefficient of $q^{\,w-m}$ in $\binom{n+m-1}{m}_q$ counts the number of partitions of size w with exactly m parts, each not exceeding n. Now, define the polynomials $\mathcal{P}_{n,p,j}$ and $\tilde{\mathcal{P}}_{n,p,j}$ such that

$$\mathcal{P}_{n,p,j}(q)\coloneqq\binom{n+p+j-1}{n}_q\binom{p+j-1}{p}_q\binom{n+p+j}{n+p}_q$$

and

$$\tilde{\mathcal{P}}_{n,p,j}(q)\coloneqq\binom{n+p+j}{n}_q\binom{p+j}{p}_q\binom{n+p+j-1}{n+p}_q.$$

By the multiplication rule for generating functions, the coefficient of $q^{\,w-2(n+p)}$ in $\mathcal{P}_{n,p,j}(q)$ (respectively in $\tilde{\mathcal{P}}_{n,p,j}(q)$) is precisely $|\mathcal{H}^w_{n,p,j}|$ (respectively $|\tilde{\mathcal{H}}^w_{n,p,j}|$).

Hence, $|\mathcal{H}^w_{n,p,j}| = |\tilde{\mathcal{H}}^w_{n,p,j}|$ since $\mathcal{P}_{n,p,j}(q) = \tilde{\mathcal{P}}_{n,p,j}(q)$, as observed in Section 3, prior to the statement of Conjecture 3.1. A similar argument applies in the elementary case; the details are omitted.

5. Two Final Remarks: q-Analogs and Comparison with Log-concavity

We propose q-analogs of the above conjectures. Recall the (partial) order $\le_q$ [7] on polynomials in $\mathbb{R}[q]$, which is a q-analog of the usual order $\le$ on $\mathbb{R}$. We write $f \le_q g$ if all coefficients of the polynomial $g - f$ are non-negative, in which case we may also write $g - f \in \mathbb{R}^+[q]$. *Strongly q-log-concave* sequences of such polynomials $x_k(q)$ are such that $x_i x_j \ge_q x_{i-1}x_{j+1}$ for all $j \ge i-1 \ge 0$ and then, two monomials X_λ and X_μ built from such polynomials satisfy $X_\lambda \ge_q X_\mu$ if and only if $\lambda \preceq \mu$. Consequently, Conjecture 4.1 would imply the following q-analog of Conjecture 3.1.

Conjecture 5.1. Let n, p, and j be integers satisfying $1 \le j \le p \le n$, and let x_k ($k \ge 1$) be a strongly q-log-concave sequence of polynomials in $\mathbb{R}[q]$. Then

$$E_x^{n,p,j}(q) := e_{n-p}(\mathbf{x_n})e_{p-j}(\mathbf{x_p})e_{n-j}(\mathbf{x_{n-1}}) - e_{n-p}(\mathbf{x_{n-1}})e_{p-j}(\mathbf{x_{p-1}})e_{n-j}(\mathbf{x_n}) \in \mathbb{R}^+[q]$$

and

$$H_x^{n,p,j}(q) := h_{n-p}(\mathbf{x_p})h_{p-j}(\mathbf{x_j})h_{n-j}(\mathbf{x_{j+1}}) - h_{n-p}(\mathbf{x_{p+1}})h_{p-j}(\mathbf{x_{j+1}})h_{n-j}(\mathbf{x_j}) \in \mathbb{R}^+[q].$$

In particular, the sequence $x_k := 1 + q + \cdots + q^{k-1}$ is a strongly q-log-concave sequence of polynomials and in this case, $e_{n-p}(\mathbf{x_{n-1}}) = \begin{bmatrix} n \\ p \end{bmatrix}_q$ is the *q-Stirling number of the first kind*, while $h_{n-p}(\mathbf{x_p}) = \begin{Bmatrix} n \\ p \end{Bmatrix}_q$ is the *q-Stirling number of the second kind* [7]. We then have the following q-analog of Conjecture 2.1.

Conjecture 5.2. Let n, p, and j be integers such that $1 \le j \le p \le n$. Then

$$S_2^{n,p,j}(q) := \begin{Bmatrix} p \\ j \end{Bmatrix}_q \begin{Bmatrix} n \\ p \end{Bmatrix}_q \begin{Bmatrix} n+1 \\ j+1 \end{Bmatrix}_q - \begin{Bmatrix} p+1 \\ j+1 \end{Bmatrix}_q \begin{Bmatrix} n+1 \\ p+1 \end{Bmatrix}_q \begin{Bmatrix} n \\ j \end{Bmatrix}_q \in \mathbb{N}[q]$$

and

$$S_1^{n,p,j}(q) := \begin{bmatrix} p+1 \\ j+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ p+1 \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q - \begin{bmatrix} p \\ j \end{bmatrix}_q \begin{bmatrix} n \\ p \end{bmatrix}_q \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}_q \in \mathbb{N}[q].$$

A similar conjecture can also be made for the *q-Stirling numbers in type B* which correspond to the case $x_i = 1 + q^2 \cdots + q^{2i}$. In [8], Sagan conjectures that the sequences of coefficients of the q-Stirling numbers are log-concave. One may further conjecture that the sequences of coefficients of the polynomials $S_2^{n,p,j}$ and $S_1^{n,p,j}$ are log-concave for each particular choice of n, p, and j. We verified this numerically for $n \le 20$. Numerical evidence also suggests that this property might hold in type B as well, contrary to the coefficients of the q-Stirling numbers in type B, which are conjectured to be only parity log-concave [8]. More generally, one may ask whether it holds that if $(x_k)_{k\geq 1}$ is a strongly q-log-concave sequence of polynomials in $\mathbb{R}[q]$, then, for each particular choice of n, p, and j, the sequences of coefficients of the polynomials $E_x^{n,p,j}(q)$ and $H_x^{n,p,j}(q)$ are themselves log-concave.

We have seen that the inequality involving the Stirling numbers of the second kind in Conjecture 2.1 entails their row log-concavity. However, we have not been able to derive an analogous result from the inequality involving Stirling numbers of the first kind. Sagan [7] proved the row log-concavity of the sequence $h_{n-p}(\mathbf{x_p})$, for fixed n and for x_k a log-concave sequence of positive real numbers, via the Lindström–Gessel–Viennot involution on certain lattice paths. This raises the question whether the inequality involving complete homogeneous polynomials in Conjecture 3.1 also implies the row log-concavity of the sequence $h_{n-p}(\mathbf{x_p})$ under the same assumptions. A derivation from this triple-product inequality, similar to the derivation leading to Inequality (2.4), yields

$$h_{n-p+1}(\mathbf{x_{p-1}})h_{n-p-1}(\mathbf{x_{p+1}})\Big(\frac{x_{p+1}}{x_p}+\frac{h_{n-p}(\mathbf{x_{p+1}})}{h_1(\mathbf{x_{p-1}})h_{n-p-1}(\mathbf{x_{p+1}})}\Big)\leq h_{n-p}(\mathbf{x_p})^2. \tag{5.1}$$

Hence, Conjecture 3.1 in the case of the complete homogeneous polynomials, together with the additional assumption that the sequence x_k of positive real numbers is non-decreasing, is sharper than Sagan's result in [7]. On the other hand, the sequence $x_k := e^{-k^2}$ is an example of a decreasing log-concave sequence of positive real numbers. In this case, Conjecture 3.1 may be weaker than Sagan's result, since when $x_k = e^{-k^2}$, numerical calculations suggest that the simple row log-concavity of $h_{n-p}(\mathbf{x_p})$ is sharper than Inequality (5.1).

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