# THREE CLASSES OF COMBINATORIAL SUMS INVOLVING CENTRAL BINOMIAL COEFFICIENTS AND HARMONIC NUMBERS

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**Abstract**

We study three classes of combinatorial sums involving central binomial coefficients and harmonic numbers, odd harmonic numbers, and even indexed harmonic numbers, respectively. In each case we use summation by parts to derive recursive expressions for these sums. In addition, we offer an alternative approach to express one class of sums and some related sums in closed form in terms of Stirling numbers and $r$-Stirling numbers of the second kind.

## 1. Introduction

The *central binomial coefficients* are defined for $n = 0, 1, \ldots,$ by $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$. These numbers are listed as A000984 in the OEIS database [14]. They are closely related to the famous Catalan numbers (entry A000108 in [14]) $C_n = \frac{1}{n+1}\binom{2n}{n}$ and possess many intriguing properties ([1, 2, 3, 4, 5, 10, 12, 13, 15, 17]).

In Gould's table [9, Entry 1.108, p. 14], we find the following combinatorial identity: for an integer $r \geq 0$ define

$$S_r(n) = \sum_{k=0}^{n} 2^{-2k} k^r \binom{2k}{k}, \qquad S_0(n) = \sum_{k=0}^{n} 2^{-2k} \binom{2k}{k}. \tag{1}$$

Then we have

$$S_r(n) = 2^{-2n}(2n+1)\binom{2n}{n} \sum_{k=0}^{r} \binom{n}{k} \left\{ {r \atop k} \right\} \frac{k!}{2k+1}, \tag{2}$$

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where

$$\left\{ {n \atop k} \right\} = \frac{(-1)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j j^n,$$

are Stirling numbers of the second kind. The special case $S_0(n)$ is

$$S_0(n) = 2^{-2n}(2n+1)\binom{2n}{n}, \tag{3}$$

which can be derived directly by setting $x = -1/2$ in the known identity (see, for example, [9, Entry 1.5, p. 1]):

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k} = (-1)^n \binom{x-1}{n} = (-1)^n \frac{x-n}{x} \binom{x}{n}, \quad x \in \mathbb{C},$$

and using the fact that

$$\binom{-1/2}{k} = (-1)^k 2^{-2k} \binom{2k}{k}.$$

**Remark 1.** Identity (3) was recently rediscovered, in a rather long calculation, by Guo and Lim [8, Theorem 4], using integration techniques.

A recursive evaluation of the sums in Equation (1) was derived in [3, Theorem 5.1] and equals

$$(2r+1)S_r(n) = n^r 2^{-2n}(2n+1)\binom{2n}{n} + 2\sum_{j=1}^{r-1} \binom{r}{j+1} (-1)^{j+1} S_{r-j}(n). \tag{4}$$

For $r = 1, 2, 3$, Equation (2) or the recursion (4) immediately produce the identities

$$\sum_{k=0}^{n} 2^{-2k} k \binom{2k}{k} = \frac{n}{3} 2^{-2n}(2n+1)\binom{2n}{n}, \tag{5}$$

$$\sum_{k=0}^{n} 2^{-2k} k^2 \binom{2k}{k} = \frac{n}{15} 2^{-2n}(3n+2)(2n+1)\binom{2n}{n}, \tag{6}$$

and

$$\sum_{k=0}^{n} 2^{-2k} k^3 \binom{2k}{k} = \frac{n}{105} 2^{-2n}(15n^2+18n+2)(2n+1)\binom{2n}{n}. \tag{7}$$

In this paper, we are concerned with the following similar, but new, three classes of combinatorial sums involving harmonic numbers and defined for all integers $r \geq 0$:

$$S_r^H(n) = \sum_{k=0}^{n} 2^{-2k} k^r \binom{2k}{k} H_k, \tag{8}$$

and its odd harmonic number and even-indexed harmonic number variants

$$S_r^O(n) = \sum_{k=0}^{n} 2^{-2k} k^r \binom{2k}{k} O_k \tag{9}$$

and

$$S_r^{2H}(n) = \sum_{k=0}^{n} 2^{-2k} k^r \binom{2k}{k} H_{2k}, \tag{10}$$

where we always adopt the convention that $0^0 = 1$. Here and throughout the paper, the numbers $O_n, n \geq 0$, are the *odd harmonic numbers,* and $H_n, n \geq 0$, are the *ordinary harmonic numbers*, i.e.,

$$H_0 = 0, \quad H_n = \sum_{m=1}^{n} \frac{1}{m} \qquad \text{and} \qquad O_0 = 0, \quad O_n = \sum_{m=1}^{n} \frac{1}{2m-1}.$$

We will show that there are close connections between each of the sums $S_r^H(n)$, $S_r^O(n)$, and $S_r^{2H}(n)$, and the numbers $S_r(n)$. More precisely, we will use summation by parts to evaluate these sums recursively for each $r$, hereby using the recursion for $S_r(n)$ as in Equation (4). Similar recursive relations are also known for sums of integer powers (see for instance [6, 7]). In addition, we will offer an alternative approach to express $S_r^O(n)$ and some related sums in closed form in terms of Stirling numbers and $r$-Stirling numbers of the second kind. Our particular identity for $S_r^O(n)$ is given by (see Equation (18) below)

$$\begin{aligned} S_r^O(n) = 2^{-2n} \sum_{k=0}^{r} (-1)^k k! \left\{ {r+n-k \atop n} \right\}_{n-k} 2^{2k} \binom{2n+1}{2k+1} \binom{2(n-k)}{n-k} \\ \times \binom{n}{k}^{-1} (O_{n+1} - O_{k+1}), \end{aligned}$$

from which we deduce the evaluations

$$S_0^O(n) = (2n+1)\binom{2n}{n} 2^{-2n} \left(O_{n+1} - 1\right)$$

and

$$S_1^O(n) = \frac{n(2n+1)}{3} \binom{2n}{n} 2^{-2n} \left(O_{n+1} - \frac{1}{3}\right).$$

## 2. The Sums $S_r^H(n)$ and $S_r^O(n)$

As will become obvious in the sequel, it is better to start with the sums $S_r^O(n)$. Our first main result in this paper is the next theorem.

**Theorem 1.** *Let $S_r^O(n)$ be defined as in Equation (9). Then*

$$S_0^O(n) = 2^{-2n}\binom{2n}{n}\left((2n+1)O_n - 2n\right)$$

*and for $r \geq 1$ we have the following recursion:*

$$(2r+1)S_r^O(n) = n^r 2^{-2n}\binom{2n}{n}\left((2n+1)O_n + 1\right) - S_r(n) \\ + 2\sum_{j=1}^{r-1}\binom{r}{j+1}(-1)^{j+1}S_{r-j}^O(n), \tag{11}$$

*where $S_r(n)$ is defined in Equation (1) and given recursively in Equation (4).*

*Proof.* Let us define two sequences $u_k = 2^{-2k}\binom{2k}{k}O_k$ and $w_k = 2^{-2k}\binom{2k}{k}$. First we note that

$$w_k = 2(k+1)(w_k - w_{k+1}). \tag{12}$$

Using this property we have

$$\begin{aligned} u_k - u_{k+1} &= 2^{-2k}\binom{2k}{k}O_k - 2^{-2(k+1)}\binom{2(k+1)}{k+1}O_k - 2^{-2(k+1)}\binom{2(k+1)}{k+1}\frac{1}{2k+1} \\ &= \frac{1}{2(k+1)}2^{-2k}\binom{2k}{k}O_k - \frac{1}{2(k+1)}2^{-2k}\binom{2k}{k}. \end{aligned}$$

Hence,

$$2(k+1)(u_k - u_{k+1}) = u_k - w_k = u_k - 2(k+1)(w_k - w_{k+1})$$

or

$$u_k = 2(k+1)(u_k + w_k - (u_{k+1} + w_{k+1})).$$

From here we can use telescoping and calculate

$$\begin{aligned} \sum_{k=0}^{n} u_k &= 2\left(\sum_{k=0}^{n}(k+1)(u_k - u_{k+1}) + \sum_{k=0}^{n}(k+1)(w_k - w_{k+1})\right) \\ &= 2\sum_{k=0}^{n}(u_k + w_k) - 2(n+1)(u_{n+1} + w_{n+1}) \end{aligned}$$

and, using Equation (3),

$$\sum_{k=0}^{n} u_k = -2\sum_{k=0}^{n} w_k + 2(n+1)(u_{n+1} + w_{n+1}) = 2^{-2n}\binom{2n}{n}\left((2n+1)O_n - 2n\right).$$

Now, suppose that $r \geq 1$. To calculate $S_r^O(n)$ we use summation by parts with $z_k = u_k + w_k$ and $a_k = k(k-1)^r$. We have

$$\begin{aligned}
S_r^O(n) &= 2\sum_{k=0}^{n} k^r(k+1)(z_k - z_{k+1}) \\
&= 2\sum_{k=0}^{n} a_{k+1}(z_k - z_{k+1}) \\
&= 2\sum_{k=0}^{n} z_k(a_{k+1} - a_k) - 2a_{n+1}z_{n+1} \\
&= 2\sum_{k=0}^{n} z_k\left((r+1)k^r - \sum_{j=2}^{r}\binom{r}{j}(-1)^j k^{r-j+1}\right) - 2z_{n+1}(n+1)n^r \\
&= (2r+2)\left(S_r^O(n) + S_r(n)\right) - 2(n+1)n^r(u_{n+1} + w_{n+1}) \\
&\quad - 2\sum_{j=2}^{r}\binom{r}{j}(-1)^j\left(S_{r-j+1}^O(n) + S_{r-j+1}(n)\right).
\end{aligned}$$

Shifting the summation index and simplifying yields

$$\begin{aligned}
(2r+1)S_r^O(n) &= n^r 2^{-2n}\binom{2n}{n}((2n+1)O_n + 2n + 2) - (2r+2)S_r(n) \\
&\quad + 2\sum_{j=1}^{r-1}\binom{r}{j+1}(-1)^{j+1}\left(S_{r-j}^O(n) + S_{r-j}(n)\right).
\end{aligned}$$

The final expression follows by inserting Equation (4) and again simplifying. □

**Corollary 1.** *We have*

$$\begin{aligned}
\sum_{k=0}^{n} 2^{-2k}k\binom{2k}{k}O_k &= \frac{n}{3}(2n+1)2^{-2n}\binom{2n}{n}\Big(O_{n+1} - \frac{1}{3}\Big), \\
\sum_{k=0}^{n} 2^{-2k}k^2\binom{2k}{k}O_k &= \frac{n}{15}(2n+1)2^{-2n}\binom{2n}{n}\Big((3n+2)O_{n+1} - \frac{1}{15}(9n+16)\Big),
\end{aligned}$$

*and*

$$\begin{aligned}
\sum_{k=0}^{n} 2^{-2k}k^3\binom{2k}{k}O_k = \frac{n}{105}(2n+1)2^{-2n}\binom{2n}{n}\Big(&(15n^2 + 18n + 2)O_{n+1} \\
&- \frac{1}{105}(225n^2 + 648n + 352)\Big).
\end{aligned}$$

*Proof.* Set $r = 1, 2, 3$ in Equation (11), combine with Equations (5)-(7), and simplify using $O_{n+1} = O_n + \frac{1}{2n+1}$. □

Our second theorem contains the results for the sums $S_r^H(n)$.

**Theorem 2.** *Let $S_r^H(n)$ be defined as in Equation (8). Then*

$$S_0^H(n) = (2n+1)2^{-2n}\binom{2n}{n}(H_n - 2) + 2 \tag{13}$$

*and for $r \geq 1$ we have the following recursion:*

$$\begin{aligned}(2r+1)S_r^H(n) &= n^r(2n+1)2^{-2n}\binom{2n}{n}(H_n+1) + 2(-1)^r - S_r(n)\\ &\quad + 2\sum_{j=1}^{r-1}\binom{r}{j+1}(-1)^{j+1}S_{r-j}^H(n) - 2\sum_{j=0}^{r}\binom{r+1}{j}(-1)^{r-j}S_j(n),\end{aligned} \tag{14}$$

*where $S_r(n)$ is defined in Equation (1) and given recursively in Equation (4).*

*Proof.* We apply the same ideas as in the proof of Theorem 1. We start with the two sequences $v_k = 2^{-2k}\binom{2k}{k}H_k$ and $w_k = 2^{-2k}\binom{2k}{k}$. Then

$$\begin{aligned}v_k - v_{k+1} &= 2^{-2k}\binom{2k}{k}H_k - 2^{-2(k+1)}\binom{2(k+1)}{k+1}H_k - 2^{-2(k+1)}\binom{2(k+1)}{k+1}\frac{1}{k+1}\\ &= \frac{1}{2(k+1)}2^{-2k}\binom{2k}{k}H_k - \frac{1}{2(k+1)}2^{-2k}\binom{2k}{k}\left(1+\frac{k}{k+1}\right).\end{aligned}$$

Hence, using Equation (12),

$$2(k+1)(v_k - v_{k+1}) = v_k - 2(k+1)(w_k - w_{k+1}) - 2k(w_k - w_{k+1})$$

or

$$v_k = 2(k+1)(v_k - v_{k+1} + w_k - w_{k+1}) + 2k(w_k - w_{k+1}).$$

An application of the telescoping property yields

$$\sum_{k=0}^n v_k = 2\sum_{k=0}^n v_k - 2(n+1)v_{n+1} + 4\sum_{k=0}^n w_k - 4(n+1)w_{n+1} - 2w_0 + 2w_{n+1}$$

and finally

$$\begin{aligned}S_0^H(n) = \sum_{k=0}^n v_k &= 2(n+1)v_{n+1} + (4n+2)w_{n+1} - 4\sum_{k=0}^n w_k + 2w_0\\ &= (2n+1)2^{-2n}\binom{2n}{n}\left(H_{n+1} + \frac{2n+1}{n+1} - 4\right) + 2.\end{aligned}$$

The expression for $S_0^H(n)$ as stated in Equation (13) follows.

Now assume that $r \geq 1$. To calculate $S_r^H(n)$ we use summation by parts twice. We set $Z_k = v_k + w_k$ and $a_k = k(k-1)^r$, and calculate

$$\begin{aligned} S_r^H(n) &= 2\sum_{k=0}^{n} k^r(k+1)(Z_k - Z_{k+1}) + 2\sum_{k=0}^{n} k^{r+1}(w_k - w_{k+1}) \\ &= 2\sum_{k=0}^{n} a_{k+1}(Z_k - Z_{k+1}) + 2\sum_{k=0}^{n} k^{r+1}(w_k - w_{k+1}) \\ &= 2\sum_{k=0}^{n} Z_k(a_{k+1} - a_k) - 2a_{n+1}Z_{n+1} + 2A_r(n), \end{aligned}$$

where we have set

$$A_r(n) = \sum_{k=0}^{n} k^{r+1}(w_k - w_{k+1}).$$

This gives

$$\begin{aligned} S_r^H(n) = (2r+2)\left(S_r^H(n) + S_r(n)\right) - n^r(2n+1)2^{-2n}\binom{2n}{n}(H_{n+1}+1) + 2A_r(n) \\ - 2\sum_{j=1}^{r-1}\binom{r}{j+1}(-1)^{j+1}\left(S_{r-j}^H(n) + S_{r-j}(n)\right) \end{aligned}$$

or

$$\begin{aligned} (2r+1)S_r^H(n) = n^r(2n+1)2^{-2n}\binom{2n}{n}H_{n+1} - S_r(n) - 2A_r(n) \\ + 2\sum_{j=1}^{r-1}\binom{r}{j+1}(-1)^{j+1}S_{r-j}^H(n). \end{aligned}$$

It remains to provide a link between $A_r(n)$ and $S_r(n)$. This is done by applying partial summation a second time. With $b_k = (k-1)^{r+1}$ we have

$$\begin{aligned} A_r(n) &= \sum_{k=0}^{n} b_{k+1}(w_k - w_{k+1}) \\ &= \sum_{k=0}^{n} w_k(b_{k+1} - b_k) - w_{n+1}b_{n+1} + w_0 b_0 \\ &= \sum_{k=0}^{n} w_k \sum_{j=0}^{r}\binom{r+1}{j}k^j(-1)^{r-j} - w_{n+1}b_{n+1} + w_0 b_0 \\ &= \sum_{j=0}^{r}\binom{r+1}{j}k^j(-1)^{r-j}S_j(n) - \frac{2n+1}{2(n+1)}2^{-2n}\binom{2n}{n}n^{r+1} + (-1)^{r+1} \end{aligned}$$

and the theorem is proved. $\square$

**Corollary 2.** *If $n$ is a nonnegative integer, then we have*

$$\sum_{k=0}^{n} 2^{-2k} k \binom{2k}{k} H_k = \frac{1}{3}(2n+1)2^{-2n}\binom{2n}{n}\Big(nH_n - \frac{2}{3}n + 2\Big) - \frac{2}{3},$$

$$\sum_{k=0}^{n} 2^{-2k} k^2 \binom{2k}{k} H_k = \frac{1}{5}(2n+1)2^{-2n}\binom{2n}{n}\Big(\frac{n}{3}(3n+2)H_n - \frac{2}{45}(9n^2-14n+15)\Big) + \frac{2}{15},$$

*and*

$$\sum_{k=0}^{n} 2^{-2k} k^3 \binom{2k}{k} H_k = \frac{1}{7}(2n+1)2^{-2n}\binom{2n}{n}\Big(\frac{n}{15}(15n^2+18n+2)H_n - \frac{2}{1575}(225n^3-297n^2+37n+105)\Big) + \frac{2}{105}.$$

*Proof.* Set $r = 1, 2, 3$ in Equation (14) and combine with Equations (5)-(7). $\square$

## 3. Further Observations

From the two main results derived in the previous section we can also deduce the third main result dealing with sums with even indexed harmonic numbers, $S_r^{2H}(n)$, given in Equation (10). For this class of combinatorial sums the following recursion is true.

**Theorem 3.** *We have*

$$S_0^{2H}(n) = \sum_{k=0}^{n} 2^{-2k}\binom{2k}{k} H_{2k} = 2^{-2n}\binom{2n}{n}\left((2n+1)H_{2n} - (4n+1)\right) + 1,$$

*and for $r \geq 1$ the following recursion holds:*

$$(2r+1)S_r^{2H}(n) = n^r 2^{-2n}\binom{2n}{n}\left((2n+1)H_{2n} + n + \frac{3}{2}\right) + (-1)^r - \frac{3}{2}S_r(n) + 2\sum_{j=1}^{r-1}\binom{r}{j+1}(-1)^{j+1}S_{r-j}^{2H}(n) - \sum_{j=0}^{r}\binom{r+1}{j}(-1)^{r-j}S_j(n),$$

*where $S_r(n)$ is defined in Equation (1) and given recursively in Equation (4). In particular,*

$$\sum_{k=0}^{n} 2^{-2k} k \binom{2k}{k} H_{2k} = \frac{1}{9}2^{-2n}\binom{2n}{n}\left(3n(2n+1)H_{2n} - 4n^2 + 7n + 3\right) - \frac{1}{3}.$$

*Proof.* The basic connection between $H_n$ and $O_n$,

$$O_n = H_{2n} - \frac{1}{2}H_n,$$

translates immediately to

$$S_r^{2H}(n) = S_r^O(n) + \frac{1}{2}S_r^H(n),$$

and the recursive expression is deduced from Theorems 1 and 2. $\square$

## 4. An Alternative Approach

In this section, based on a known polynomial identity, we express $S_r^O(n)$ and some related sums in closed form in terms of Stirling numbers and $r$-Stirling numbers of the second kind.

Recall that for integers $n$ and $k$, the Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}$, are the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k,$$

where $(x)_k$ is the falling factorial defined by $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1)$ for $n > 0$. We will often make use of the property

$$\left\{ {n \atop k} \right\} = 0, \text{if } k > n.$$

The exponential generating function of $\left\{ {n \atop k} \right\}$ equals

$$\sum_{n \geq k} \left\{ {n \atop k} \right\} \frac{z^n}{n!} = \frac{1}{k!} \left(e^z - 1\right)^k.$$

For any positive integer $r$, the *$r$-Stirling numbers of the second kind* $\left\{ {n+r \atop k+r} \right\}_r$ are defined by the exponential generating function

$$\sum_{n \geq k} \left\{ {n+r \atop k+r} \right\}_r \frac{z^n}{n!} = \frac{1}{k!} e^{rz} \left(e^z - 1\right)^k.$$

Details about these special numbers can be found in [11] and references therein.

We also recall the notion of generalized harmonic numbers. For $0 \neq z \in \mathbb{C} \setminus \mathbb{Z}^-$, generalized harmonic numbers $H_z$ and generalized odd harmonic numbers $O_z$ are defined by the recurrence relations

$$H_z = H_{z-1} + \frac{1}{z} \qquad \text{and} \qquad O_z = O_{z-1} + \frac{1}{2z-1},$$

with $H_0 = 0$ and $O_0 = 0$. Generalized harmonic numbers are connected with the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ through the fundamental relation [16]

$$H_z = \psi(z+1) + \gamma,$$

where $\gamma$ is the Euler–Mascheroni constant. This fundamental relation shows that generalized harmonic numbers must be viewed as digamma functions and can also be differentiated with respect to $z$. The known result for the digamma function at half-integer arguments [16, Equation (51)], namely,

$$\psi(n+1/2) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n} \frac{1}{2k-1},$$

provides useful relations between harmonic and odd harmonic numbers that will be employed at some places in this section. Finally, the recurrence relations for $H_z$ and $O_z$ imply that if $z = n$ is a nonnegative integer, then $H_n$ and $O_n$ are the ordinary (odd) harmonic numbers as defined in the introduction.

**Lemma 1.** *If $k$, $r$, and $v$ are nonnegative integers, then*

$$\frac{d^r}{dx^r}\left(1-e^x\right)^k\bigg|_{x=0} = \sum_{p=0}^{k}(-1)^p\binom{k}{p}p^r = (-1)^k k!\left\{ {r \atop k} \right\},$$

*and*

$$\frac{d^r}{dx^r}\left(1-e^x\right)^k e^{vx}\bigg|_{x=0} = \sum_{p=0}^{k}(-1)^p\binom{k}{p}(v+p)^r = (-1)^k k!\left\{ {r+v \atop k+v} \right\}_v, \tag{15}$$

*where $\left\{ {m \atop n} \right\}$ and $\left\{ {m+r \atop n+r} \right\}_r$ are, respectively, Stirling numbers of the second kind and $r$-Stirling numbers.*

*Proof.* Since

$$\left(1-e^x\right)^k e^{vx} = \sum_{p=0}^{k}(-1)^p\binom{k}{p}e^{(v+p)x},$$

we have

$$\frac{d^r}{dx^r}\Big(\left(1-e^x\right)^k e^{vx}\Big) = \sum_{p=0}^{k}(-1)^p\binom{k}{p}\left(v+p\right)^r e^{(v+p)x},$$

and hence Equation (15). □

**Theorem 4.** *If $n$ and $r$ are nonnegative integers, and $t$ is not an integer, then*

$$\sum_{k=0}^{n}(-1)^k k^r\binom{t}{k}(H_t - H_{t-k}) = \sum_{k=0}^{r}(-1)^k k!\left\{ {r+n-k \atop n} \right\}_{n-k}\binom{n-t}{n-k}(H_{k-t} - H_{n-t}). \tag{16}$$

*Proof.* Write $\exp(x)$ for $x$ in the known polynomial identity (variation on Gould [9, Identity (1.10), p. 2]):

$$\sum_{k=0}^{n}(-1)^k\binom{t}{k}x^k = \sum_{k=0}^{n}\binom{n-t}{k}(1-x)^{n-k}\,x^k, \tag{17}$$

and differentiate $r$ times with respect to $x$ to obtain

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k k^r\binom{t}{k}e^{kx} &= \sum_{k=0}^{n}\frac{d^r}{dx^r}\left((1-e^x)^{n-k}\,e^{kx}\right)\binom{n-t}{k}\\ &= \sum_{k=0}^{n}\frac{d^r}{dx^r}\left((1-e^x)^{k}\,e^{(n-k)x}\right)\binom{n-t}{n-k}.\end{aligned}$$

Evaluation at $x=0$ yields

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k k^r\binom{t}{k} &= \sum_{k=0}^{n}\frac{d^r}{dx^r}\left((1-e^x)^{k}\,e^{(n-k)x}\right)\binom{n-t}{n-k}\Bigg|_{x=0}\\ &= \sum_{k=0}^{n}\frac{d^r}{dx^r}\left((1-e^x)^{k}\,e^{(n-k)x}\right)\Bigg|_{x=0}\binom{n-t}{n-k}\\ &= \sum_{k=0}^{r}(-1)^k k!\left\{{r+n-k \atop n}\right\}_{n-k}\binom{n-t}{n-k},\end{aligned}$$

which gives Equation (16) upon differentiating with respect to $t$, since

$$\frac{d}{dt}\binom{t}{k} = \binom{t}{k}(H_t - H_{t-k}) \qquad \text{and} \qquad \frac{d}{dt}\binom{n-t}{k} = \binom{n-t}{k}(H_{n-t-k} - H_{n-t}).$$

∎

The four lowest explicit formulas for $r = 0, 1, 2, 3$ evaluations of Equation (16) are presented in Proposition 1.

**Proposition 1.** *If $n$ is a nonnegative integer and $t$ is not an integer, then*

$$\sum_{k=0}^{n}(-1)^k\binom{t}{k}(H_t - H_{t-k}) = \binom{n-t}{n}\left(H_{-t} - H_{n-t}\right),$$

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k k\binom{t}{k}(H_t - H_{t-k}) = &-\binom{n-t}{n-1}(H_{1-t} - H_{n-t})\\ &+ n\binom{n-t}{n}\left(H_{-t} - H_{n-t}\right),\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k k^2\binom{t}{k}(H_t - H_{t-k}) = 2&\binom{n-t}{n-2}\left(H_{2-t} - H_{n-t}\right)\\ &-(2n-1)\binom{n-t}{n-1}\left(H_{1-t} - H_{n-t}\right) + n^2\binom{n-t}{n}\left(H_{-t} - H_{n-t}\right),\end{aligned}$$

*and*

$$\sum_{k=0}^{n}(-1)^k k^3\binom{t}{k}(H_t-H_{t-k}) = -6\binom{n-t}{n-3}\left(H_{3-t}-H_{n-t}\right)$$
$$+6(n-1)\binom{n-t}{n-2}\left(H_{2-t}-H_{n-t}\right)-(3n^2-3n+1)\binom{n-t}{n-1}\left(H_{1-t}-H_{n-t}\right)$$
$$+n^3\binom{n-t}{n}\left(H_{-t}-H_{n-t}\right).$$

Theorem 4 also provides a closed form for the sums $S_r^O(n)$.

**Proposition 2.** *If $n$ and $r$ are nonnegative integers, then*

$$\begin{aligned} S_r^O(n) &= \sum_{k=0}^{n} 2^{-2k}k^r\binom{2k}{k}O_k \\ &= 2^{-2n}\sum_{k=0}^{r}(-1)^k k!\left\{ {r+n-k \atop n} \right\}_{n-k} 2^{2k}\binom{2n+1}{2k+1}\binom{2(n-k)}{n-k} \\ &\qquad \times\binom{n}{k}^{-1}(O_{n+1}-O_{k+1}), \end{aligned} \tag{18}$$

*In particular,*

$$S_0^O(n)=(2n+1)\binom{2n}{n}2^{-2n}\left(O_{n+1}-1\right)$$

*and*

$$S_1^O(n)=\frac{n(2n+1)}{3}\binom{2n}{n}2^{-2n}\left(O_{n+1}-\frac{1}{3}\right).$$

*Proof.* Set $t=-1/2$ in Equation (16). Use

$$\binom{-1/2}{p}=(-1)^p\binom{2p}{p}2^{-2p} \qquad \text{and} \qquad \binom{p+1/2}{q}=2^{-2q}\binom{2p+1}{2q}\binom{2q}{q}\binom{p}{q}^{-1},$$

and the fact that $H_{h-1/2}=2O_h-2\ln 2$. $\square$

**Proposition 3.** *If $n$ and $r$ are nonnegative integers, then*

$$\begin{aligned} &\sum_{k=1}^{n}\frac{2^{-2k}}{2k-1}k^r\binom{2k}{k}(1-O_{k-1}) \\ &\qquad = 2^{-2n}\binom{2n}{n}\sum_{k=0}^{r}(-1)^k k!\left\{ {r+n-k \atop n} \right\}_{n-k}\binom{n}{k}2^{2k}\binom{2k}{k}^{-1}(O_n-O_k). \end{aligned}$$

*In particular,*

$$\sum_{k=1}^{n}\frac{2^{-2k}}{2k-1}\binom{2k}{k}(1-O_{k-1})=2^{-2n}\binom{2n}{n}O_n.$$

*Proof.* Set $t = 1/2$ in Equation (16). Use

$$\binom{1/2}{k} = \frac{(-1)^{k+1}2^{-2k}}{2k-1}\binom{2k}{k} \quad \text{and} \quad \binom{n-1/2}{n-k} = 2^{2k-2n}\binom{2n}{n}\binom{2k}{k}^{-1}\binom{n}{k},$$

and the fact that $H_{1/2-k} = 2O_{k-1} - 2\ln 2$. $\square$

**Theorem 5.** *If $n$ and $r$ are nonnegative integers, and $t$ is not an integer, then*

$$\sum_{k=0}^{n}(-1)^k\binom{t}{n-k}k^r(H_t - H_{t-n+k}) = (-1)^n\sum_{k=0}^{r}k!\left\{{r \atop k}\right\}\binom{n-t}{n-k}(H_{k-t} - H_{n-t}). \tag{19}$$

*Proof.* Write $1/x$ for $x$ in Equation (17) to obtain

$$\sum_{k=0}^{n}(-1)^k\binom{t}{n-k}x^k = \sum_{k=0}^{n}(-1)^{n-k}\binom{n-t}{n-k}(1-x)^k. \tag{20}$$

Now write $\exp x$ for $x$, differentiate $r$ times with respect to $x$ and evaluate at $x = 0$ using Lemma 1. Finally differentiate the resulting expression with respect to $t$ using

$$\frac{d}{dt}\binom{t}{n-k} = \binom{t}{n-k}(H_t - H_{t-n+k}) \quad \text{and} \quad \frac{d}{dt}\binom{n-t}{n-k} = \binom{n-t}{n-k}(H_{k-t} - H_{n-t}).$$

$\square$

**Proposition 4.** *If $n$ and $r$ are nonnegative integers, then*

$$\sum_{k=0}^{n}2^{2k}\binom{2(n-k)}{n-k}k^r O_{n-k}$$
$$= (2n+1)\binom{2n}{n}\sum_{k=0}^{r}\frac{2^{2k}}{2k+1}k!\left\{{r \atop k}\right\}\binom{n}{k}\binom{2k}{k}^{-1}(O_{n+1} - O_{k+1}).$$

*Proof.* Set $t = -1/2$ in Equation (19). Use

$$\binom{-1/2}{n-k} = (-1)^{n-k}\binom{2(n-k)}{n-k}2^{-2(n-k)},$$
$$\binom{n+1/2}{n-k} = \frac{2n+1}{2k+1}\binom{2n}{n}\binom{2k}{k}^{-1}2^{-2(n-k)}\binom{n}{k},$$

and $H_{-1/2} - H_{-1/2-n+k} = -2O_{n-k}$ and $H_{k+1/2} - H_{n+1/2} = 2(O_{k+1} - O_{n+1})$. $\square$

**Proposition 5.** *If $n$ and $r$ are nonnegative integers, then*

$$\sum_{k=0}^{n-1}\frac{2^{2k}}{2(n-k)-1}\binom{2(n-k)}{n-k}k^r(1 - O_{n-k-1})$$
$$= \binom{2n}{n}\sum_{k=0}^{r}2^{2k}\binom{2k}{k}^{-1}k!\left\{{r \atop k}\right\}\binom{n}{k}(O_n - O_k).$$

*Proof.* Set $t = 1/2$ in Equation (19). Use

$$\binom{1/2}{n-k} = \frac{(-1)^{n-k+1}2^{-2(n-k)}}{2(n-k)-1}\binom{2(n-k)}{n-k},$$
$$\binom{n-1/2}{n-k} = 2^{2k-2n}\binom{2n}{n}\binom{2k}{k}^{-1}\binom{n}{k},$$

and $H_{1/2} - H_{1/2-n+k} = 2(1 - O_{n-k-1})$ and $H_{n-1/2} - H_{k-1/2} = 2(O_n - O_k)$. □

**Theorem 6.** *If $n$ and $r$ are nonnegative integers, and $t$ is not an integer, then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n-t}{n-k}k^r\left(H_{k-t} - H_{n-t}\right) = \sum_{k=0}^{r}k!\left\{{r \atop k}\right\}\binom{t}{n-k}\left(H_t - H_{t-n+k}\right). \tag{21}$$

*Proof.* This is a consequence of a symmetry relation of Equation (20). □

**Proposition 6.** *If $n$ and $r$ are nonnegative integers, then*

$$\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}2^{2k}\binom{2k}{k}^{-1}k^r\left(O_{n+1} - O_{k+1}\right)$$
$$= \frac{1}{2n+1}\binom{2n}{n}^{-1}\sum_{k=0}^{r}(-1)^k k!\left\{{r \atop k}\right\}2^{2k}\binom{2\left(n-k\right)}{n-k}O_{n-k}.$$

*Proof.* Set $t = -1/2$ in Equation (21). □

In particular, we have

$$\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}2^{2k}\binom{2k}{k}^{-1}O_{k+1} = \frac{1}{(2n+1)^2}, \tag{22}$$

since

$$\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}2^{2k}\binom{2k}{k}^{-1} = \frac{1}{2n+1}.$$

From Equation (22), we also have

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(2k+1)^2} = \frac{2^{2n}}{2n+1}\binom{2n}{n}^{-1}O_{n+1}.$$

**Proposition 7.** *If $n$ and $r$ are nonnegative integers, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{2k}\binom{2k}{k}^{-1}k^r\left(O_n - O_k\right)$$
$$= \binom{2n}{n}^{-1}\sum_{k=0}^{r}\frac{(-1)^k}{2\left(n-k\right)-1}k!\left\{{r \atop k}\right\}2^{2k}\binom{2\left(n-k\right)}{n-k}\left(1 - O_{n-k-1}\right).$$

*Proof.* Set $t = 1/2$ in Equation (21). □

In particular,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{2k}\binom{2k}{k}^{-1}(O_n - O_k) = \frac{1-O_{n-1}}{2n-1},$$

which can be simplified to give

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{2k}\binom{2k}{k}^{-1}O_k = -\frac{2n}{(2n-1)^2}.$$

In addition, by the binomial transformation we also get

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2k}{(2k-1)^2} = -2^{2n}\binom{2n}{n}^{-1}O_n.$$

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