



## MEASURES FOR THE SUMMATION OF IRWIN SERIES

Jean-François Burnol

*Département de Mathématiques, Université de Lille, France*

`jean-francois.burnol@math.univ-lille.fr`

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### Abstract

We consider the series of reciprocals of those positive integers with exactly  $k$  occurrences of a given  $b$ -ary digit  $d$  (Irwin series), and obtain geometrically convergent representations for their sums. They are expressed in terms of the moments and Stieltjes transforms of some measures on the unit interval. The moments obey linear recurrences allowing straightforward numerical implementations. This framework also allows to study the large  $k$  limit, and, in further works by the author, the large  $b$  asymptotics.

### 1. Introduction

The Irwin series [22] are sub-series of the harmonic series with conditions on the number of occurrences of digits in the denominators. For example, only those positive integers having at most 77 occurrences of the digit 3, and/or at most 125 occurrences of the digit 5, are kept. Irwin showed that such series converge. This generalized an earlier contribution by Kempner [23] where the convergence was shown for those series whose terms have no occurrence of a given digit.

Hardy and Wright establish the Kempner result as Theorem 144 of [20]. Their statement “The sum of the reciprocals of the numbers which miss a given digit is convergent” is proven for an arbitrary radix  $r$ , whereas Kempner had considered only decimal figures. It is found in the section “The representations of numbers as decimals” which discusses topics such as sets of measure zero and the theorem of Borel stating that almost every real number is normal in any base [20, Thm. 148].

The Kempner and Irwin series have been the object of various studies and generalizations [1–9, 17–19, 21–33]. A characteristic of Theorems 1 and 2 established in the present work is that they provide theoretically *exact* formulas which can be converted straightforwardly into efficient numerical algorithms<sup>1</sup>. They are thus an alternative to the (also efficient!) numerical algorithms of Baillie [8, 9]. In

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<sup>1</sup>See <https://burnolmath.gitlab.io/irwin/#sagemath-implementation>.

all examples computed by the author, the results matched those returned by the Baillie algorithms for Kempner and Irwin series. In a later work [12], we started the extension to counting occurrences of multi-digit strings  $d_1 \dots d_p$ . Here too, the numerical results in the case of no occurrence matched what is provided by the Schmelzer–Baillie [30] algorithm. For  $p = 2$  and one occurrence, the approach of [12] is currently the only published algorithm available.

We shall consider here, for any triple of integers  $(b, d, k)$  with  $b > 1$ ,  $0 \leq d < b$ , and  $k \geq 0$ , the associated Irwin series, i.e., the subsum of the harmonic series where only those terms are kept whose denominators have exactly  $k$  occurrences of the  $b$ -ary digit  $d$ . The case  $b = 2$ ,  $d = 1$ ,  $k = 0$  is special: it gives an empty series, and the value of the sum is thus zero. We explain how to generalize our earlier work [11] which handled  $k = 0$ . Here as in [11], the core objects are moments and Stieltjes transforms of suitable measures on the unit interval.

For  $b = 10$ , Farhi [17] proved that the limit as  $k \rightarrow \infty$  is  $10 \log(10)$ . We prove that, in general, it equals  $b \log(b)$ . Farhi’s method could be extended to general  $b > 1$ , but our approach is very different. We actually give two proofs, one a corollary of the convergence of certain measures toward the Lebesgue measure. We extend this latter method of proof in [16] to the case of counting occurrences of a multi-digit string  $d_1 \dots d_p$ , recovering the recent theorem by Allouche, Hu, and Morin [6], which says that the limit as  $k \rightarrow \infty$  in that case is  $b^p \log(b)$ .

In further works, which were also based upon the present one, we obtained various new results on Kempner–Irwin series and their generalizations, in particular regarding their large  $b$  asymptotics [13–15].

## 2. Notation and Terminology

Let  $b > 1$  be an integer, which is kept fixed throughout the paper. The set  $\mathbb{N}$  is defined as  $\mathbb{Z}_{\geq 0}$  (i.e., it contains zero). We define  $b$ -imal numbers as the elements of  $\cup_{l \geq 0} b^{-l} \mathbb{Z}$ . For such a  $b$ -imal number  $x$ , the smallest  $l \geq 0$  with  $x \in b^{-l} \mathbb{Z}$  is called its *depth*.

Let  $\mathbf{D} = \{0, \dots, b-1\}$  be the set of  $b$ -ary digits. We pick a  $d \in \mathbf{D}$  which will be fixed throughout the paper. The case  $d = 0$  has some specific features and will require special consideration at some places. We let  $\mathbf{A} = \mathbf{D} \setminus \{d\}$ . We let  $N = \#\mathbf{A} = b-1$  and  $N_1 = \#(\mathbf{A} \setminus \{0\})$ .

The space of *strings*  $\mathbf{D}$  is defined to be the union of all Cartesian products  $\mathbf{D}^l$  for  $l \geq 0$ . Notice that for  $l = 0$  one has  $\mathbf{D}^0 = \{\emptyset\}$ , i.e., the set with one element. We will call this special element the *none-string*. The length  $|X| = l$  of a string  $X$  is the integer  $l$  such that  $X \in \mathbf{D}^l$ . For  $l = 1$  a string is thus the same thing as a digit, although for the sake of clarity it is better to treat them as separate things. There is a map from strings to integers which to  $X = (d_l, \dots, d_1)$  assigns

$n(X) = d_l b^l + \cdots + d_1$ , and sends the none-string to zero. This map is many-to-one, and each integer  $n$  has a unique minimal-length representation  $X(n)$  which is called the (minimal)  $b$ -representation of  $n$ .

The length of an integer  $n$  is denoted  $l(n) = |X(n)|$ . Thus,  $l(n)$  is the smallest non-negative integer such that  $n < b^{l(n)}$ . Note that  $l(0) = 0$ , not 1. The letter  $l$  is also used to refer to a specific integer, e.g., we may say “let  $l$  be an integer, and consider the integers  $m$  such that  $l(m) = l$ ”. We hope that this notation will not create confusion.

Define, for  $k \geq 0$ :

$$\mathcal{X}^{(k)} = \{X \in \mathbf{D} \mid X \text{ contains the digit } d \text{ exactly } k \text{ times}\}.$$

The set  $\mathcal{X}^{(k)}$  contains the none-string if and only if  $k = 0$ . We write  $\mathcal{X}_l^{(k)}$  for its subsets of strings having length  $l$ , for  $l \geq 0$ . Let  $\mathcal{A}^{(k)}$  be the set of non-negative integers with minimal  $b$ -imal representation in  $\mathcal{X}^{(k)}$ , and  $\mathcal{A}_l^{(k)}$  the subset of those having length  $l$ . We obtain a partition  $\mathbb{N} = \mathbb{Z}_{\geq 0} = \mathcal{A}^{(0)} \cup \mathcal{A}^{(1)} \cup \mathcal{A}^{(2)} \cup \cdots$ . We write  $k_{b,d}(n)$  (or often simply  $k(n)$ ) for the number of occurrences of the digit  $d$  in the minimal  $b$ -representation of the integer  $n$ . With this notation, the Irwin sums considered in this paper are:

$$H_{b,d}^{(k)} = \sum_{n>0, k_{b,d}(n)=k} \frac{1}{n}.$$

As we are handling only positive terms, the order of summation is irrelevant. The cardinality of the set of strings of  $l$   $b$ -ary digits,  $k$  among them equal to  $d$ , is  $\binom{l}{k}(b-1)^{l-k}$ . Terms in  $H_{b,d}^{(k)}$  with denominators of the same length  $l$  constitute “blocks”  $S_{b,d;l}^{(k)}$ :

$$S_{b,d;l}^{(k)} = \sum_{l(n)=l, k_{b,d}(n)=k} \frac{1}{n}.$$

If  $k > l$ , no integer with  $l$  digits can have  $k$  occurrences of  $d$ , and  $S_{b,d;l}^{(k)} = 0$ . Note also that  $H_{2,1}^{(0)} = 0$  because the series is empty. Taking into account that each  $n^{-1}$  with  $b^{l-1} \leq n < b^l$  contributes at most  $b^{-(l-1)}$ , we see that  $S_{b,d;l}^{(k)} = O_{l \rightarrow \infty}(l^k(1-1/b)^l)$ . So,  $H_{b,d}^{(k)} < \infty$ .

As the base  $b$  and the digit  $d$  are fixed throughout the paper, we shall usually drop them from the subscripts and abbreviate  $H_{b,d}^{(k)}$  and  $S_{b,d;l}^{(k)}$  to, respectively,  $H^{(k)}$  and  $S_l^{(k)}$ . The notation  $S_l^{(\leq k)}$  means  $\sum_{0 \leq j \leq k} S_l^{(j)}$ , and  $S_l^{(< k)} = \sum_{0 \leq j < k} S_l^{(j)}$ .

Regarding integration against measures  $\mu$ , we will usually write  $d\mu(x)$ , and sometimes  $\mu(dx)$  with the same meaning. The letter  $d$  here is not a  $b$ -ary digit...

### 3. The New Series Representing Irwin Sums

Here, we gather together the main results.

**Theorem 1.** Let  $b \in \mathbb{N}$ ,  $b > 1$ , and  $d \in \{0, \dots, b-1\}$ . Define, for each  $j \geq 1$ ,  $\gamma_j = \sum_{\substack{a \neq d \\ 0 \leq a < b}} a^j$ . Let  $u_{0;0} = b$  and let  $(u_{0;m})_{m \geq 0}$  satisfy the recurrence

$$m \geq 1 \implies (b^{m+1} - b + 1)u_{0;m} = \sum_{j=1}^m \binom{m}{j} \gamma_j u_{0;m-j}.$$

Let  $u_{j;m}$  ( $j \geq 1$ ,  $m \geq 0$ ) satisfy the recurrence

$$m \geq 0 \implies (b^{m+1} - b + 1)u_{j;m} = \sum_{j=1}^m \binom{m}{j} \gamma_j u_{j;m-j} + \sum_{j=0}^m \binom{m}{j} d^j u_{j-1;m-j}.$$

In particular  $u_{j;0} = u_{j-1;0} = \dots = u_{0;0} = b$ .

Let  $k \geq 0$  and  $l \geq 1$ . The Irwin sum  $H^{(k)}$  can be expressed using integers of length at most  $l$  and having at most  $k$  occurrences of the digit  $d$  (as indicated below by the superscript  $(\leq k)$ ):

$$H^{(k)} = \sum_{0 < n < b^{l-1}} \binom{(k)}{n} \frac{1}{n} + b \cdot \sum_{b^{l-1} \leq n < b^l} \binom{(\leq k)}{n} \frac{1}{n} + \sum_{m=1}^{\infty} (-1)^m \sum_{b^{l-1} \leq n < b^l} \binom{(\leq k)}{n} \frac{u_{k-k(n);m}}{n^{m+1}}. \quad (1)$$

The quantities  $u_{j;m}$  ( $j \geq 0$ ,  $m \geq 1$ ) have the following properties:

- They are non-negative and vanish only if  $j = 0$  and  $b = 2$  and  $d = 1$ .
- They decrease (strictly if not zero) for increasing  $m$ .
- They increase strictly for increasing  $j$  and converge to  $b/(m+1)$ .

*Proof.* This uses most everything from the present paper. In brief: the validity of the series is established in Proposition 5, which is a corollary to the integral formulas from Proposition 4, themselves being variants of the  $H^{(k)} = \int_{[b^{-1}, 1)} \frac{d\mu_k(x)}{x}$  log-like expression of Proposition 3. The coefficients  $u_{j;m}$  are the moments of measures  $\mu_j$  which are the main topic of this paper. The recurrence relations are established in Proposition 6. The bounds on the  $u_{j;m}$ , and their limits as  $j \rightarrow \infty$ , are obtained in Proposition 7. The  $b/(m+1)$  limit for  $j \rightarrow \infty$  is also a corollary of Proposition 9 which establishes the convergence of the measures  $\mu_k$  to  $b dx$ .  $\square$

**Remark 1.** We examine when the alternating series in Equation (1) has only vanishing contributions. The involved coefficients  $u_{j;m}$  must vanish, which can happen only for  $(b, d, j) = (2, 1, 0)$ . Assume thus  $b = 2$  and  $d = 1$ . The contributing  $n$  with  $l$  digits are constrained by  $k(n) \leq k$ . If  $k = 0$ , there is no such  $n$ , and

the alternating series is empty. If  $k = 1$ , only  $k(n) = 1$  is realized (as a positive integer must have its leading digit equal to 1), and only by  $n = 2^{l-1}$ . But then  $j = k - k(n) = 0$  and thus  $u_{j;m} = 0$  and the contribution vanishes. If  $k > 1$ , the contribution of  $n = 2^{l-1}$  is  $(-1)^m u_{k-1;m} / 2^{(l-1)(m+1)}$  and is non-zero. The alternating series is thus either identically vanishing ( $b = 2$ ,  $d = 1$ ,  $k \leq 1$ ) or its terms decrease strictly in absolute value.

For  $l = 1$  the series is built with inverse powers of the digits. If  $k \geq 1$  all digits contribute. The first contributions to Equation (1) (left of the alternating series) add up to  $b(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{b-1})$ , which is thus an upper bound. Also with  $l = 1$ , if  $k = 0$  we have the same upper bound if  $d = 0$ , and if  $d > 0$  we omit  $b/d$  as the integer  $n = d$  gives no contribution. This upper bound  $b \sum_{1 \leq n < b} \frac{1}{n} - b \mathbf{1}_{k=0, d>0}(k, d) \frac{1}{d}$  is strict, except with  $b = 2$ ,  $d = 1$ , and  $k \leq 1$  all true, in which case the upper bound is the exact value of  $H^{(k)}$ . In all other cases, keeping only the first from the alternating series in Equation (1), we obtain a strict lower bound of  $H^{(k)}$ . See also Propositions 1 and 2.

**Remark 2.** One can use the theorem with  $l = 1$ , but if  $k \geq 1$ , the one-digit number  $n = 1$  will then always contribute, and as all coefficients  $u_{j;m}$  are bounded below if  $d \neq b - 1$  by  $1/(m + 1)$  (see Proposition 7), we obtain for level 1,  $k \geq 1$ ,  $d \neq b - 1$ , a series as poorly converging as the one for  $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ .

It is preferable for numerical implementations to use at least  $l = 2$ . The finite sum of the reciprocal powers  $1/n^{m+2}$  is bounded above by  $b^l / (b^{l-1})^{m+2}$ . Combined with the  $0 \leq u_{l;m} \leq b/(m + 1)$  bounds this gives an upper bound  $b^{-m(l-1)+3-l} / (m + 2)$ . Hence, each additional term of the series will give about  $l - 1$  new places of precision, in radix  $b$  representation, for the approximation of the Irwin sum. Using  $l = 3$  has the advantage of dividing by two the needed range of  $m$  for the same target precision. It does induce additional cost in computing the inverse power sums, as they have more contributions. In our initial **SAGEMATH** implementation from 2024<sup>2</sup> we observed for  $b = 10$  and  $d = 9$  that  $l = 3$  was beneficial at about 1200+ decimal digits for  $H^{(0)}$  and already at 600+ digits for  $H^{(1)}$  and 400+ digits for  $H^{(2)}$  compared to using  $l = 2$ . But this depends on the actual implementation and on the numerical libraries used. For small bases, the benefit of choosing  $l = 3$  — and even  $l = 4$  in the cases  $b = 2$  and  $b = 3$  — becomes evident already at substantially lower target precision.

Theorem 2 below writes  $H^{(k)}$  as a series of non-negative terms that obey a linear recurrence relation and decay geometrically to zero. The terms are positive except for  $(b, d, k) = (2, 1, 0)$ .

**Theorem 2.** Let  $b > 0$ ,  $d \in \{0, \dots, b - 1\}$ . For each  $j \geq 1$ , let  $\gamma'_j$  be  $\sum_{\substack{a \neq b-1-d \\ 0 \leq a < b}} a^j$ .

<sup>2</sup>Available at <https://arxiv.org/src/2402.09083v1/anc>. A more sophisticated version, applying parallelism to some extent, is now available at <https://gitlab.com/burnolmath/irwin>.

Let  $(v_{0;m})_{m \geq 0}$  satisfy the recurrence

$$m \geq 0 \implies (b^{m+1} - b + 1)v_{0;m} = b^{m+1} + \sum_{j=1}^m \binom{m}{j} \gamma'_j v_{0;m-j}.$$

In particular,  $v_{0;0} = b$ . Let  $v_{j;m}$  ( $j \geq 1, m \geq 0$ ) satisfy the recurrence

$$m \geq 0 \implies (b^{m+1} - b + 1)v_{j;m} = \sum_{j=1}^m \binom{m}{j} \gamma'_j v_{j;m-j} + \sum_{j=0}^m \binom{m}{j} (b-1-d)^j v_{j-1;m-j}.$$

In particular,  $v_{j;0} = v_{j-1;0} = \dots = v_{0;0} = b$  for all  $j \geq 0$ .

Let  $l \geq 1$  and  $k \geq 0$ . One has

$$H^{(k)} = \sum_{0 < n < b^{l-1}} \binom{k}{n} \frac{1}{n} + b \cdot \sum_{b^{l-1} \leq n < b^l} \binom{k}{n} \frac{1}{n+1} + \sum_{m=1}^{\infty} \sum_{b^{l-1} \leq n < b^l} \binom{k}{n} \frac{v_{k-k(n);m}}{(n+1)^{m+1}}.$$

The superscript  $\binom{k}{n}$  means to restrict to integers  $n$  having in base  $b$  at most  $k$  occurrences of the digit  $d$ .

The quantities  $v_{j;m}$  ( $j \geq 0, m \geq 1$ ) have the following properties:

- They are positive and bounded above by  $b$ .
- They decrease strictly for increasing  $m$ , except for  $b = 2, d = 1$ , and  $j = 0$ , in which case  $v_{0;m} = 2$  for all  $m$ .
- They decrease strictly for increasing  $j$  and converge to  $b/(m+1)$ .

*Proof.* The series are established in Proposition 5. The recurrences are Equations (7) and (8). The value  $b/(m+1)$  of the limit of  $(v_{j;m})_{j \geq 0}$ , and the monotonicity, are proven in Proposition 10. Another proof follows from Proposition 9 about the convergence  $\mu_j \rightarrow bdx$  and the definition of the  $v_{j;m}$  as complementary power moments.  $\square$

The next statement was proven by Farhi [17] for  $b = 10$ . We obtain it here for all  $b > 1$ .

**Proposition 1.** *With  $d \neq 0$  the sequence  $(H^{(k)})$  is strictly decreasing for  $k \geq 1$ . With  $d = 0$  it is strictly decreasing already starting at  $k = 0$ . In both cases the sequence converges to  $b \log b$ .*

*Proof.* We apply Theorem 2 with  $l = 1$ . Starting with  $k = 1$ , there are no restrictions on the single-digit integers  $n$  intervening in the inverse power sums. The theorem says in particular that the coefficients  $v_{j;m}$  decrease when  $j$  increases. This gives the decrease  $(H^{(k)})_{k \geq 1}$ . The theorem also gives their limits as  $j \rightarrow \infty$ . We recognize after taking these limits term by term the Taylor series of  $-\log(1-h)$

evaluated at  $h = (n + 1)^{-1}$ , for  $1 \leq n < b$ . Adding up these logarithms gives a telescopic finite sum:

$$\lim H^{(k)} = b \sum_{1 \leq n < b} -\log\left(1 - \frac{1}{n+1}\right) = b \log(b).$$

For  $d = 0$  all non-zero digits contribute to the series already for  $k = 0$ . So the decrease starts already at  $k = 0$ .  $\square$

**Proposition 2.** For  $b > 2$  and  $d \neq 0$  one has  $H^{(0)} > b \log(b) - b \log(1 + \frac{1}{d})$ . Hence  $H^{(0)} > b \log(b/2)$ .

*Proof.* Same proof as for the previous proposition using  $l = 1$ . The sole difference is that only integers  $n$  in  $\{1, \dots, b-1\} \setminus \{d\}$  contribute to the series.  $\square$

**Remark 3.** Hence, except for the sole case  $b = 2$ ,  $d = 1$ ,  $k = 0$ , we have  $H^{(k)} > \min(3 \log(3/2), 2 \log(2)) = 3 \log(3/2) > 1.2$ . This means that in a floating point context, we can decide of how many terms to keep in the series, solely on the basis of fixed point estimates, i.e., absolute comparison to 1.

#### 4. Irwin Sums as Integrals

Recall that, given an enumerable subset  $\{x_1, x_2, \dots\}$  of the real line, and a series with non-negative terms  $\sum_{n \geq 1} c_n$ , possibly diverging, one can define the set-function  $\mu : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  which assigns to any subset  $G$  of the real line the quantity  $\mu(G) = \sum_{n=1}^{\infty} c_n \mathbf{1}_G(x_n) \in [0, \infty]$ . This set-function is  $\sigma$ -additive and is a (non-negative, discrete, possibly infinite) measure. Integrability of a complex-valued function  $f$  on the real line means in this context  $\sum_{n=1}^{\infty} c_n |f(x_n)| < \infty$ . Then  $\int_{\mathbb{R}} f(x) d\mu(x)$  is defined as  $\sum_{n=1}^{\infty} c_n f(x_n)$ , and is invariant under any permutation of the indexing of the set  $\{(x_n, c_n), n \geq 1\}$ . We will make free use of the notation of measures and integrals in the following. We write  $\mu = \sum_{i \in \mathcal{I}} c_i \delta_{x_i}$ , where the countable index set  $\mathcal{I}$  does not have to be  $\mathbb{N}$ . In this paper we consider only measures supported in  $[0, 1)$  and having finite total mass. The *support* is defined to be the set  $\{x_i, c_i > 0\}$  (not its closure in the usual topology). A *Dirac mass* is a measure  $c\delta_x$  having a single real number  $x$  in its support. The *weight* is  $c = \mu(\{x\}) > 0$ .

**Definition 1.** The measure  $\mu_{b,d}^{(k)}$ , or for short in the sequel  $\mu_k$ , is the (infinite) sum of Dirac masses at the rational numbers  $x = n(X)/b^{|X|}$ , for  $X \in \mathcal{X}^{(k)}$ , with respective weights  $1/b^{|X|}$ :

$$\mu_k = \sum_{X \in \mathcal{X}^{(k)}} b^{-|X|} \delta_{n(X)/b^{|X|}}.$$

It is supported in  $[0, 1)$ .

**Remark 4.** In our previous work [11] of which the present paper is a continuation in the single-digit case, we defined a measure on  $\mathbb{R}_{\geq 0}$  but the proofs of the main Theorem used only its restrictions to  $[0, 1)$ . We thus here define our measures to only have support on  $[0, 1)$  and leave aside considerations relative to what the “correct” extension to  $\mathbb{R}_{\geq 0}$  is.

**Remark 5.** Suppose  $d \neq 0$ . Then the strings with  $n(X) = 0$ , i.e., the none-string and those containing only 0’s, belong to  $\mathcal{X}^{(k)}$  if and only if  $k = 0$ . For  $k = 0$  there is a total weight of  $1 + b^{-1} + b^{-2} + \dots = b/(b - 1)$  assigned to the Dirac at the origin. But for  $k \geq 1$  and  $d \neq 0$ , there is no such Dirac mass.

If  $d = 0$ ,  $n(X) = 0$  for  $X \in \mathcal{X}^{(k)}$  happens if and only if  $X$  is the string consisting of  $k$  zeros. So, in this case, there is always a Dirac mass at the origin, which has weight  $b^{-k}$ .

In the following calculation displaying the binomial series, one has  $N = b - 1$ :

$$\begin{aligned} \mu_k([0, 1)) &= \sum_{l \geq 0} b^{-l} \# \{X \in \mathcal{X}^{(k)} \cap \mathcal{D}^l\} = \sum_{l \geq k} b^{-l} \binom{l}{k} N^{l-k} \\ &= \sum_{p=0}^{\infty} b^{-k-p} \binom{k+p}{k} N^p = b^{-k} \frac{1}{(1 - N/b)^{k+1}} = \frac{b}{(b - N)^{k+1}} = b. \end{aligned}$$

Hence, the measure  $\mu_k$  is finite for any  $k \geq 0$  and its total mass is  $b$ .

We can express Irwin numbers as log-like quantities. In the next proposition, recall that  $\mu_k$  is short for  $\mu_{b,d}^{(k)}$ , i.e., it also depends on  $b$  and  $d$ .

**Proposition 3.** For  $b \geq 2$ ,  $d$  a  $b$ -ary digit, and  $k \geq 0$ , we have

$$H_{b,d}^{(k)} = \int_{[b^{-1}, 1)} \frac{d\mu_k(x)}{x}.$$

*Proof.* The integral is the sum of  $b^{-|X|}/(n(X)/b^{|X|}) = n(X)^{-1}$  over all strings  $X$  containing the digit  $d$  exactly  $k$  times, and having a non-zero leading digit. Such strings are in one-to-one correspondance with positive integers, so this is  $H^{(k)}$ .  $\square$

We need some additional notation for the next proposition: for  $n > 0$  of length  $q \geq l$ , we let  $\text{ld}_l(n)$  be the integer  $m$  of length  $l$  which is “at start” of  $n$ , i.e.,  $m$  is the floor of  $n/b^{q-l}$ .

**Proposition 4.** Let  $n > 0$  be an integer of length  $l$ . Let  $k \geq 0$ . One has

$$\int_{[0, 1)} \frac{d\mu_k(x)}{n + x} = \sum_{\text{ld}_l(m)=n}^{(k(n)+k)} \frac{1}{m}.$$

*Proof.* The measure  $\mu_k$  is defined as an infinite weighted sum of Dirac masses indexed by strings  $X$  having exactly  $k$  occurrences of the digit  $d$ . We obtain

$$\int_{[0,1)} \frac{d\mu_k(x)}{n+x} = \sum_{X \in \mathcal{X}^{(k)}} \frac{1}{b^{|X|}(n + n(X)/b^{|X|})} = \sum_{X \in \mathcal{X}^{(k)}} \frac{1}{n \cdot b^{|X|} + n(X)}.$$

The set of denominators present in this last sum is exactly the set of positive integers with  $\text{ld}_l(m) = n$  and  $k(m) = k(n) + k$ .  $\square$

If  $d \neq 0$ , the contributions of any  $x$  in the support of  $\mu_k$  are of the type  $1/m$ ,  $1/(bm)$ ,  $1/(b^2m)$ ,  $\dots$ , as 0 can be appended as the trailing element of a string  $X$  without modifying the number of occurrences of  $d$  nor the  $b$ -imal number  $x = n(X)/b^{|X|}$ . If we index rather by the strings as done here, there is a one-to-one correspondance, which proves more convenient. The explanations in our earlier work [11] would have been a bit simplified by this language, which however requires the additional notation defined here.

As the positive measure with support in the unit interval is finite, it has moments of all orders and these moments are the key quantities in our analysis. We define, for  $k \geq 0$  and  $m \geq 0$ :

$$u_{k;m} = \int_{[0,1)} x^m d\mu_k(x).$$

Except for the sole case of  $k = 0$ ,  $b = 2$ ,  $d = 1$ , for which  $\mu_0$  is a Dirac mass of weight 2 at the origin, and  $u_{0;m} = 0$  for  $m \geq 1$ ,  $(u_{k;m})_{m \geq 0}$  is a strictly decreasing sequence converging to zero. This follows by dominated convergence from  $\mu_k$  being supported in  $[0, 1)$ . One can also argue elementarily as in [11, Proof of Theorem 4].

**Corollary 1.** *Let  $k \geq 0$  and let  $n$  be a positive integer having  $k(n) \leq k$  occurrences of the digit  $d$ . The contribution to  $H_{b,d}^{(k)}$  from the denominators “starting with (identical digits as in)  $n$ ” can be computed as an alternating series:*

$$\sum_{\text{ld}_l(m)=n}^{(k)} \frac{1}{m} = \sum_{m=0}^{\infty} (-1)^m \frac{u_{k-k(n);m}}{n^{m+1}}.$$

*Proof.* We use the formula of Proposition 4 with  $k - k(n)$ . For  $n > 1$ , we have that  $1/(n+x) = 1/n - x/n^2 + x^2/n^3 - \dots$  converges absolutely and uniformly with respect to  $x \in [0, 1)$  (even inclusive of  $x = 1$ ) and we can thus integrate term per term. For  $n = 1$ , the remainders after integration will be, up to sign, the integrals of  $x^{m+1}/(1+x)$  and thus their absolute values are bounded by  $\mu_{k-k(n);m+1}$ , and they converge to zero. Interchanging summation and integration is thus valid in that case too.  $\square$

**Remark 6.** For  $k = 0$ ,  $b = 2$ ,  $d = 1$ , there is no positive integer with 0 occurrence of the digit 1, so the statement is empty.

We define *complementary moments*  $v_{j;m} = \int_{[0,1)} (1-x)^m d\mu_j(x)$  for  $j, m$  non-negative integers.

**Corollary 2.** *Let  $k \geq 0$  and let  $n$  be a positive integer of length  $l$  and having at most  $k$  occurrences of the digit  $d$ , i.e.,  $k(n) \leq k$ . Then*

$$\sum_{\text{ld}_l(m)=n}^{(k)} \frac{1}{m} = \sum_{m=0}^{\infty} \frac{v_{k-k(n);m}}{(n+1)^{m+1}}.$$

*The right-hand side is a positive series with geometric convergence.*

*Proof.* This follows from

$$\frac{1}{n+x} = \frac{1}{n+1-(1-x)} = \sum_{m=0}^{\infty} \frac{(1-x)^m}{(n+1)^{m+1}},$$

and term-by-term integration on  $[0,1)$  against the measure  $\mu_k$ .  $\square$

**Proposition 5.** *Let  $k \geq 0$  and  $l \geq 1$ . Then*

$$H^{(k)} = \sum_{1 \leq j < l} S_j^{(k)} + bS_l^{(\leq k)} + \sum_{m=1}^{\infty} (-1)^m \sum_{0 \leq i \leq k} u_{k-i;m} \sum_{\substack{l(n)=l \\ k(n)=i}} \frac{1}{n^{m+1}}. \quad (2)$$

*Equivalently*

$$H^{(k)} = \sum_{1 \leq j < l} S_j^{(k)} + bS_l^{(\leq k)} + \sum_{m=1}^{\infty} (-1)^m \sum_{\substack{l(n)=l \\ k(n) \leq k}} \frac{u_{k-k(n);m}}{n^{m+1}}.$$

*One has similarly*

$$H^{(k)} = \sum_{1 \leq j < l} S_j^{(k)} + b \sum_{\substack{l(n)=l \\ k(n) \leq k}} \frac{1}{n+1} + \sum_{m=1}^{\infty} \sum_{\substack{l(n)=l \\ k(n) \leq k}} \frac{v_{k-k(n);m}}{(n+1)^{m+1}}.$$

*Proof.* Just apply Corollary 1, or Corollary 2, to each integer  $n$  of length  $l$  and such that  $k(n) \leq k$ , then use  $u_{k-i;0} = b = v_{k-i;0}$  to handle the contributions from  $m = 0$ .  $\square$

## 5. Integral Identities, Recurrence, and Asymptotics of Moments

The next lemma will allow us to obtain recurrence formulas for the moments.

**Lemma 1.** *Let  $f$  be a bounded function on  $[0, b)$ . Let  $k \geq 1$ . One has*

$$\int_{[0,1)} f(bx) d\mu_k(x) = \frac{1}{b} \sum_{a \neq d} \int_{[0,1)} f(a+x) d\mu_k(x) + \frac{1}{b} \int_{[0,1)} f(d+x) d\mu_{k-1}(x),$$

where the first summation is over all digits distinct from  $d$ . For  $k = 0$ , one has

$$\int_{[0,1)} f(bx) d\mu_0(x) = f(0) + \frac{1}{b} \sum_{a \neq d} \int_{[0,1)} f(a+x) d\mu_0(x). \quad (3)$$

**Remark 7.** Equation (3) was already stated in [11, Lemma 7] except for  $b = 2$  and  $d = 1$ : indeed, reference [11] has a set  $A \subset D$  of so-called admissible digits and assumes that this set is not reduced to the singleton  $\{0\}$ , which is however what happens for  $(b, d, k) = (2, 1, 0)$ . As the measure  $\mu_0$  then equals twice the Dirac at the origin, Equation (3) reads in that special case  $2f(0) = f(0) + \frac{1}{2}(2f(0))$ .

*Proof of Lemma 1.* Suppose  $k \geq 1$ . Each string  $X$  of length  $l$  and having exactly  $k$  occurrences of  $d$  contributes  $b^{-l}f(n(X)/b^{l-1})$ . For  $l = 0$ ,  $X$  is the none-string and does not contribute anything as we have supposed  $k \geq 1$ . For  $l \geq 1$  let  $a$  be the leading digit of  $X$ . If  $a \neq d$ , then the  $l - 1$  remaining digits of  $X$  give a string  $Y$  which again has exactly  $k$  occurrences of  $d$ . So the strings with initial digit  $a \neq d$  contribute the sum of the  $b^{-1}b^{1-l}f(a+n(Y)/b^{l-1})$  over all  $Y$  with  $k$  occurrences of  $d$ . This is  $b^{-1} \int_{[0,1)} f(a+x) d\mu_k(x)$ . If  $a = d$ , then  $Y$  (which may be the none-string) has  $k - 1$  occurrences of  $d$ . Hence, this contributes  $b^{-1} \int_{[0,1)} f(d+x) d\mu_{k-1}(x)$ .

We also consider  $k = 0$ . Here the none-string contributes  $f(0)$  to the integral on the left-hand side. The strings  $X$  of length  $l \geq 1$  in  $\mathcal{X}^{(0)}$  do not contain the digit  $d$ . So here the possible  $a$  are distinct from  $d$ , and the tail string  $Y$ , possibly the none-string, automatically also belongs to  $\mathcal{X}^{(0)}$ . Hence, we have Equation (3).  $\square$

Recall from [11, Proposition 8] the recurrence (obtained as a corollary to Equation (3)):

$$m \geq 1 \implies (b^{m+1} - b + 1)u_{0;m} = \sum_{j=1}^m \binom{m}{j} \left( \sum_{a \neq d} a^j \right) u_{0;m-j}. \quad (4)$$

We note that this formula requires  $m \geq 1$ . It also works in the  $k = 0$ ,  $b = 2$ ,  $d = 1$  case, as all moments for  $m \geq 1$  then vanish and the power sum for  $j = m \geq 1$  is  $0^m = 0$ . We obtain recurrences for the moments of the measures  $\mu_k$ ,  $k \geq 1$ .

**Proposition 6.** *Let  $k \geq 1$ . The moments  $u_{k;m}$  of the  $k$ -th measure  $\mu_k$  are related to those of  $\mu_{k-1}$  via the recurrence*

$$(b^{m+1} - b + 1)u_{k;m} = \sum_{j=1}^m \binom{m}{j} \left( \sum_{a \neq d} a^j \right) u_{k;m-j} + \sum_{j=0}^m \binom{m}{j} d^j u_{k-1;m-j}. \quad (5)$$

One has  $u_{k;0} = b$  and the above identity also holds for  $m = 0$ .

*Proof.* We apply Lemma 1 to the function  $f(x) = x^m$ ,  $m \geq 1$ . After multiplying by  $b$  we have on the left-hand side  $b^{m+1}u_{k;m}$ . On the right-hand side, we apply the binomial formula (which requires the convention  $0^0 = 1$ ) and separate the  $j = 0$  contribution from the first sum, obtaining

$$\begin{aligned} & \sum_{a \neq d} \sum_{j=0}^m \binom{m}{j} a^j u_{k;m-j} + \sum_{j=0}^m \binom{m}{j} d^j u_{k-1;m-j} \\ &= (b-1)u_{k;m} + \sum_{j=1}^m \binom{m}{j} \left( \sum_{a \neq d} a^j \right) u_{k;m-j} + \sum_{j=0}^m \binom{m}{j} d^j u_{k-1;m-j}. \end{aligned}$$

This gives the stated formula.  $\square$

The following result is important both for theory and practice.

**Proposition 7.** *For each  $m \geq 1$  the sequence  $(u_{k;m})_{k \geq 0}$  is strictly increasing and converges to  $b/(m+1)$ :*

$$u_{0;m} < u_{1;m} < \cdots < u_{k;m} \xrightarrow{k \rightarrow \infty} \frac{b}{m+1}.$$

Let  $f = \max(\mathbb{D} \setminus \{0, d\})$ . The case  $f = 0$  happens only if  $b = 2$  and  $d = 1$ . Suppose  $f > 0$ , then for all  $m \geq 1$  one has

$$\frac{1}{m+1} \left( \frac{f}{b-1} \right)^m < u_{0;m} < \frac{b}{m+1} \left( \frac{f}{b-1} \right)^m. \quad (6)$$

If  $b = 2$  and  $d = 1$  one has  $u_{0;m} = 0$  for  $m \geq 1$  and  $u_{1;m} = 2/(2^{m+1} - 1)$  for  $m \geq 0$ .

*Proof.* If  $b = 2$  and  $d = 1$  the measure  $\mu_0$  is  $2\delta_0$ . The values of  $u_{1;m}$  are directly given by the recurrence from Equation (5) whose right-hand side in that case only has a single non-zero contribution, which is  $u_{0;0} = 2$  hence the value for  $u_{1;m}$ .

The estimate from Equation (6) of  $u_{0;m}$  (for either  $b > 2$  or  $d \neq 1$ ) is from [11, Proposition 10].

We prove  $u_{1;m} > u_{0;m}$  for all  $m \geq 1$ . This is already known for  $b = 2$  with  $d = 1$  as  $u_{0;m} = 0$  so we exclude this case in the next paragraph.

We compare the recurrence of the  $(u_{1;m})$  sequence (Equation (5)) with the one of the  $(u_{0;m})$  sequence (Equation (4)). They look the same apart from the fact that  $k = 1$  has more contributions, all non-negative. So  $u_{1;m} \geq u_{0;m}$  by induction on  $m$  (as it is an equality for  $m = 0$ ). Reexamining Equation (5) we see that the second sum on its right-hand side always contains the  $j = 0$  contribution  $u_{0;m}$  which is positive. So in fact in the previous argument we had  $(b^{m+1} - b + 1)u_{1;m} > (b^{m+1} - b + 1)u_{0;m}$  for  $m \geq 1$ , hence  $u_{1;m} > u_{0;m}$ .

Let  $k \geq 1$  and suppose we have shown already  $u_{k;m} > u_{k-1;m}$  for all  $m \geq 1$ .

We consider Equation (5) for  $k + 1$ . We can suppose inductively that  $u_{k+1;n} \geq u_{k;n}$  for  $0 \leq n < m$  as this holds for  $n = 0$ . And we know  $u_{k;n} \geq u_{k-1;n}$  for all  $n$ . Using this we obtain a lower bound  $(b^{m+1} - b + 1)u_{k;m}$  for the right-hand side of Equation (5) with  $k + 1$ . Hence,  $u_{k+1;m} \geq u_{k;m}$ . Thus, this holds for all  $m$  by induction on  $m$ . Reexamining Equation (5) for  $k + 1$  we see that the last sum has the contribution for  $j = 0$  which is  $u_{k;m}$  which is known to be greater than  $u_{k-1;m}$ . So in fact our lower bound is strict and  $u_{k+1;m} > u_{k;m}$  for all  $m \geq 1$ . Hence, the conclusion by induction on  $k$ .

So, for each  $m \geq 1$ , there holds  $u_{0;m} < u_{1;m} < \dots < u_{k;m} < \dots$ . Further, all are bounded above by  $b$  as  $u_{k;m} \leq u_{k;0} = b$ . So, there exists a finite limit  $w_m = \lim_{k \rightarrow \infty} u_{k;m}$ . Letting  $k \rightarrow \infty$  in Equation (5) we obtain for all  $m \geq 0$ :

$$(b^{m+1} - b + 1)w_m = \sum_{j=1}^m \binom{m}{j} \left( \sum_{a=0}^{b-1} a^j \right) w_{m-j} + w_m.$$

As  $u_{k;0} = b$  for all  $k \geq 0$ ,  $w_0 = \lim u_{k;0} = b$ . We prove by induction that  $w_m = b/(m+1)$  holds for all  $m \geq 0$ . Assume it is true up to  $m = M - 1$  for some  $M \geq 1$ . Substituting this into the recurrence relation above (after having removed from both sides one copy of  $w_M$ ) leads to

$$(b^{M+1} - b)w_M = \sum_{j=1}^M \binom{M}{j} \left( \sum_{a=0}^{b-1} a^j \right) \frac{b}{M-j+1}.$$

Note that

$$\begin{aligned} \sum_{j=1}^M \binom{M}{j} \left( \sum_{a=0}^{b-1} a^j \right) \frac{M+1}{M+1-j} &= \sum_{j=1}^M \binom{M+1}{j} \left( \sum_{a=0}^{b-1} a^j \right) \\ &= \sum_{a=0}^{b-1} ((a+1)^{M+1} - a^{M+1} - 1) = b^{M+1} - b. \end{aligned}$$

So  $(M+1)(b^{M+1} - b)w_M$  is equal to  $b(b^{M+1} - b)$ . Hence,  $w_M = b/(M+1)$  and this completes the proof.  $\square$

**Remark 8.** Defining for all  $k \geq 0$  and all  $m \geq 0$

$$\sigma_{k;m} = (m+1)u_{k;m},$$

we obtain from Equation (5) ( $k \geq 1$ )

$$(b^{m+1} - b + 1)\sigma_{k;m} = \sum_{j=1}^m \binom{m+1}{j} \sum_{a \neq d} a^j \sigma_{k;m-j} + \sum_{j=0}^m \binom{m+1}{j} d^j \sigma_{k-1;m-j}.$$

We note that this is a barycentric equality with non-negative coefficients:

$$\begin{aligned} \sum_{j=1}^m \binom{m+1}{j} \left( \sum_{a \neq d} a^j \right) + \sum_{j=0}^m \binom{m+1}{j} d^j \\ = \sum_{a=0}^{b-1} \left( (a+1)^{m+1} - a^{m+1} - 1 \right) + 1 = b^{m+1} - b + 1. \end{aligned}$$

An alternative proof of  $\frac{1}{m+1} \left( \frac{f}{b-1} \right)^m < u_{k;m} < \frac{b}{m+1}$  can be based upon this.

## 6. Convergence to Lebesgue Measure and Farhi Theorem

In terms of  $g(x) = f(bx)$ , the integral formula of Lemma 1 becomes

$$\int_{[0,1)} g(x) d\mu_k(x) = \frac{1}{b} \sum_{a \neq d} \int_{[0,1)} g\left(\frac{a}{b} + \frac{x}{b}\right) d\mu_k(x) + \frac{1}{b} \int_{[0,1)} g\left(\frac{d}{b} + \frac{x}{b}\right) d\mu_{k-1}(x).$$

This motivates a closer examination of the restrictions of  $\mu_k$  to sub-intervals such as  $[i/b, (i+1)/b)$  for  $0 \leq i < b$ .

**Lemma 2.** *Let  $x \in [0, 1)$  be a  $b$ -imal number of depth  $l$ . Let  $X$  be the string of length  $l$  such that  $n(X)/b^l = x$ . Let  $j$  be the number of occurrences of  $d$  in  $X$ . Let  $l' \geq l$ . Set  $j' = j$  if  $d > 0$ , and  $j' = j + l' - l$  if  $d = 0$ . Let  $U$  be any subset of the open interval  $(x, x + b^{-l'})$ . Finally, let  $k \in \mathbb{N}$ .*

- if  $k < j'$  then  $\mu_k(U) = 0$ ,
- if  $k \geq j'$  then  $\mu_k(U) = b^{-l'} \mu_{k-j'}(b^{l'}U - b^{l'}x)$ .

*Proof.* If  $d > 0$ ,  $j$  is also the number of occurrences of  $d$  in the integer  $b^l x$ , but for  $d = 0$ ,  $j$  will be greater than that if  $x < b^{-1}$  (due to leading zeros in  $X$ ).

Any string  $Y$  such that  $n(Y)/b^{|Y|} \in (x, x + b^{-l'})$  has the shape

$$Y = X \underbrace{0 \dots 0}_{l'-l \text{ zeros}} Z,$$

where  $Z = z_1 \dots z_p$  and at least one  $z_i$  is not zero. The number of occurrences of  $d$  in  $Y$  is the sum of  $j'$  (which was defined depending on whether  $d > 0$  or  $d = 0$ ) with the number of occurrences in  $Z$ . So if  $k < j'$ , then no such string  $Y$  has exactly  $k$  occurrences of  $d$  and  $\mu_k$  is restricted to the zero measure on  $(x, x + b^{-l'})$ . Suppose  $k \geq j'$ . Set  $y = n(Y)/b^{|Y|}$  and  $z = n(Z)/b^{|Z|}$ . Thus,  $y = x + z/b^{l'}$  with  $0 < z < 1$ , and  $z = b^{l'}(y - x)$ . Conversely, any string  $Z$  having at least one non-zero digit can be extended as above to give  $Y$  such that  $n(Y)/b^{|Y|}$  is in  $(x, x + b^{-l'})$

and  $k(Y) = k(Z) + j'$  ( $k(T)$  is the number of occurrences of  $d$  in a string  $T$ ). Summing over all  $Y$  with  $n(Y)/b^{|Y|} = y$  and having  $k$  occurrences of  $d$  (if  $d = 0$ , there is only one such  $Y$  for each  $y$ ), and over all  $Z$  giving the same  $z$  and having  $k - j'$  occurrences of  $d$ , we get  $\mu_k(\{y\}) = b^{-l'} \mu_{k-j'}(\{z\})$ . Finally, summing over all strings  $Y$  such that  $n(Y)/b^{|Y|} \in U \subset (x, x + b^{-l'})$ , we obtain the stated formula for  $\mu_k(U)$ .  $\square$

For some half-open intervals, a simple formula showing the behavior of  $\mu_k$  is obtained next.

**Proposition 8.** *Let  $x \in [0, 1)$  be of depth  $l$  and let  $j$  be defined as in the previous lemma. For  $k < j$  the restriction of  $\mu_k$  to the half-open interval  $[x, x + b^{-l})$  vanishes. For  $k \geq j$  and any subset  $U \subset [x, x + b^{-l})$  one has  $\mu_k(U) = b^{-l} \mu_{k-j}([b^l U - b^l x])$ .*

*Proof.* We use the Lemma 2 with  $l' = l$ ,  $j' = j$ . For  $k < j$  one thus has  $\mu_k((x, x + b^{-l})) = 0$ . And  $\mu_k(\{x\})$  is also zero because there are already  $j > k$  occurrences of the digit  $d$  in  $x$  (this includes leading zeros located after the radix separator). So  $\mu_k([x, x + b^{-l})) = 0$ .

Suppose  $k \geq j$ . We know from Lemma 2 that for any  $U \subset (x, x + b^{-l})$ ,  $\mu_k(U) = b^{-l} \mu_{k-j}(b^l U - b^l x)$ . There remains to examine what happens for the singleton  $\{x\}$ . Let  $c(k - j) = \mu_{k-j}(\{0\})$ . We need to check that  $\mu_k(\{x\}) = b^{-l} c(k - j)$ . The value of  $\mu_k(\{x\})$  depends on whether  $d = 0$  or  $d > 0$ . In the former case,  $\mu_k(\{x\}) = b^{-l-(k-j)}$  (recall  $k \geq j$ ). In the latter case, it is equal to  $b^{-l}(1 - 1/b)^{-1}$  if  $k = j$ , and vanishes if  $k > j$ . Using Remark 5, we obtain  $\mu_k\{x\} = b^{-l} c(k - j)$  in all cases, which completes the proof.  $\square$

**Remark 9.** In particular the total mass  $\mu_k([x, x + b^{-l}))$  is 0 for  $k < j$  and  $b^{1-l}$  (i.e.,  $b$  times Lebesgue measure) for  $j \geq k$ . The sequence  $(\mu_k([x, x + b^{-l})))_{k \geq 0}$  is thus non-decreasing.

This is compatible with the moments being increasing as  $k$  increases (cf. Proposition 7). But the sequences  $(\mu_k([t, u]))_{k \geq 0}$  associated with half-open intervals  $[t, u)$  can not possibly all be non-decreasing: if they were, Equation 3 from Proposition 3 expressing  $H^{(k)}$  as a log-like integral would cause  $(H^{(k)})$  to also be a non-decreasing sequence. But as first proven by Farhi [17] for  $b = 10$ , and generally here in Proposition 1, they actually decrease strictly for  $k \geq 1$ .

This apparent paradox is explained by the fact that for  $l' > l$ ,  $\mu_k([x, x + b^{-l'}))$  has a less simple behavior when  $k$  varies, than the one which is valid for  $l' = l$  and described in Proposition 8. The details can be deduced from Lemma 2. We will only need that starting with  $k = j'$ , the sequence  $(\mu_k([x, x + b^{-l'})))$  becomes constant, equal to  $b^{1-l'}$ . This follows from the next lemma.

**Lemma 3.** *Let  $x \in [0, 1)$  be a  $b$ -imal number of depth  $l$  for some  $l \in \mathbb{N}$ . Let  $l' \geq l$  and let  $j$  and  $j'$  be defined as in Lemma 2. Let  $k \geq j'$ . Let  $U$  be any subset of the half-open interval  $[x, x + b^{-l'})$ . Then  $\mu_k(U) = b^{-l'} \mu_{k-j'}(b^{l'} U - b^{l'} x)$ .*

*Proof.* We know this already from Lemma 2 if  $U$  does not contain  $x$ . It remains to consider the case of  $U = \{x\}$ .

- If  $d = 0$ : for  $k \geq j$ , there is only one string  $X$  containing  $k$  occurrences of the digit 0 and such that  $n(X)/b^{|X|} = x$ . It is obtained by adding  $k - j$  trailing zeros to the string of length  $l$  representing  $b^l x$ . So  $\mu_k(\{x\}) = b^{-l-(k-j)}$ . On the other hand  $j' = j + l' - l$  so  $k - j' = k - j + l - l'$  and the weight of  $\delta_0$  in  $\mu_{k-j'}$  is  $b^{-(k-j')}$  by Remark 5. So  $b^{-l'} \mu_{k-j'}(\{0\}) = b^{-l'-k+j'} = b^{-k+j-l}$  which matches  $\mu_k(\{x\})$ .
- If  $d \neq 0$ : for  $k = j$ , strings with trailing zeros contribute to the weight at  $x$ , and the total weight is  $b^{-l} \sum_{i \geq 0} b^{-i}$ . On the other hand  $j' = j$  so  $k - j' = k - j = 0$  and the weight of  $\delta_0$  in  $\mu_0$  is  $\sum_{i \geq 0} b^{-i}$ . Multiplying this by  $b^{-l'}$  we obtain indeed  $\mu_j(\{x\})$ . For  $k > j$ ,  $\mu_k(\{x\}) = 0$ . And also  $b^{-l'} \mu_{k-j'}(\{0\}) = 0$ . Again the values match.  $\square$

**Proposition 9.** Let  $t < u$  be any two  $b$ -imal numbers in  $[0, 1]$ . Let  $l'$  be large enough for  $b^{l'} t$  and  $b^{l'} u$  to be integers. For  $k \geq l'$  there holds

$$\mu_k([t, u]) = bu - bt.$$

Let generally  $I$  be any sub-interval of  $[0, 1]$ . Then (with  $|I|$  defined as  $\sup I - \inf I$ , i.e., the Lebesgue measure of  $I$ )

$$\lim \mu_k(I) = b|I|.$$

*Proof.* Let  $t = n/b^{l'}$ ,  $u = m/b^{l'}$  for some integers  $0 \leq n < m \leq b^{l'}$ . It is enough to consider the case  $m = n + 1$ , by additivity. Lemma 3 says in particular that  $\mu_k([t, t + b^{-l'}]) = b^{-l'} \mu_{k-j'}([0, 1])$  for  $k \geq j'$ . The quantity  $j'$  is here some integer at most equal to  $l'$ . And  $\mu_i([0, 1]) = b$  for all  $i \in \mathbb{N}$ . This gives the result for  $\mu_k([t, u])$ . Note that here perhaps  $l' > l$ , where  $l$  is the smallest integer such that  $b^l t \in \mathbb{N}$ , and Proposition 8 would not have been enough to conclude in that case.

Let  $I$  be any sub-interval of  $[0, 1]$ . If  $I$  is a singleton the statement is known from evaluations of  $\mu_k(x)$  (which is zero if  $x$  is not  $b$ -imal). If  $I$  is not a singleton, then  $\liminf \mu_k(I) \geq \lim \mu_k([t, u]) = bu - bt$  for any choice of  $b$ -imal numbers  $t < u$  in the interior  $\overset{\circ}{I}$ . So  $\liminf \mu_k(I) \geq b|I|$  and the upper bound for  $\limsup \mu_k(I)$  is shown similarly.  $\square$

**Theorem 3.** Let  $n > 0$  be an integer. Then

$$\lim_{k \rightarrow \infty} \sum_{m \text{ starts with } n}^{(k)} \frac{1}{m} = b \log\left(1 + \frac{1}{n}\right).$$

*Proof.* The condition on  $m$  is  $\text{ld}_l(m) = n$ , where  $l$  is the number of digits in  $n$ . Using Proposition 4, we are thus looking at

$$\lim_{k \rightarrow \infty} \int_{[0,1]} \frac{\mu_k(dx)}{n+x}.$$

According to the previous proposition and familiar arguments from measure theory this limit exists and its value is

$$\int_{[0,1)} \frac{b dx}{n+x} = b \log\left(1 + \frac{1}{n}\right). \quad \square$$

**Remark 10.** Adding these formulas for  $n = 1$  to  $b - 1$  we recover Farhi theorem [17]:  $\lim H^{(k)} = b \log b$ . This can also be obtained from taking the limit in Equation (3). Expanding using the power series for the logarithm function we obtain a series which is the limit term per term of the one from Corollary 1.

## 7. Complementary Moments

Let  $E_k(t) = \int_{[0,1)} e^{tx} d\mu_k(x)$  be the exponential generating function of the moments. Using Lemma 1, one obtains:

$$E_k(bt) = \begin{cases} \frac{1}{b} \left( \sum_{a \neq d} e^{at} E_k(t) + e^{dt} E_{k-1}(t) \right) & (k > 0), \\ 1 + \frac{1}{b} \sum_{a \neq d} e^{at} E_0(t) & (k = 0). \end{cases}$$

Define  $F_k(t) = e^t E_k(-t)$ . This is the exponential generating function of the  $v_{k;m}$  ( $m \geq 0$ ) which are defined right before Corollary 2. Setting  $d' = b - 1 - d$ , one has:

$$F_k(bt) = \frac{1}{b} \left( \sum_{a \neq d'} e^{at} F_k(t) + e^{d't} F_{k-1}(t) \right),$$

$$F_0(bt) = e^{bt} + \frac{1}{b} \sum_{a \neq d'} e^{at} F_0(t).$$

Thus,  $v_{k;m}$  obey for  $k \geq 1$  the same recurrences stated in Proposition 6 for  $u_{k;m}$ , except for the replacement of  $d$  by  $d' = b - 1 - d$ . In other terms, setting  $\gamma'_j = \sum_{a \neq d', 0 \leq a < b} a^j$ , we have:

$$(b^{m+1} - b + 1)v_{k;m} = \sum_{j=1}^m \binom{m}{j} \gamma'_j v_{k;m-j} + \sum_{j=0}^m \binom{m}{j} (d')^j v_{k-1;m-j}. \quad (7)$$

And for  $k = 0$  we get:

$$(b^{m+1} - b + 1)v_{0;m} = b^{m+1} + \sum_{j=1}^m \binom{m}{j} \gamma'_j v_{0;m-j}. \quad (8)$$

The previous formula is also valid for  $m = 0$ : it gives  $v_{k;0} = b$ .

**Proposition 10.** For each  $m \geq 1$ , the sequence  $(v_{k;m})_{k \in \mathbb{N}}$  (which is bounded above by  $b = v_{k;0} = u_{k;0}$ ) is strictly decreasing and converges to  $b/(m+1)$ .

*Proof.* First of all,  $v_{k;m} = \int_{[0,1)} (1-x)^m d\mu_k(x) \leq v_{k;0} = b$  with equality possible for  $m \geq 1$  only if all the mass of  $\mu_k$  is concentrated at the origin. This happens if and only if  $b = 2$  and  $d = 1$  and  $k = 0$ . Using the inequality we get

$$\sum_{j=0}^m \binom{m}{j} (d')^j v_{k-1;m-j} \leq \sum_{j=0}^m \binom{m}{j} (b-1)^j b = b^{m+1}.$$

For  $d' = 0$ , the left-hand side is  $v_{k-1;m}$  and it is less than  $b^{m+1}$  for  $m \geq 1$ . For  $d' > 0$ , we are not in the case  $b = 2$  and  $d = 1$ , so  $v_{k-1;m-j} < b$  for  $j < m$ . Consequently, and looking in particular at the  $j = 0$  contribution, we have, if  $m \geq 1$ , a strict inequality in the above equation.

Hence, for  $k \geq 1$ , and  $m \geq 1$ , from Equation (7):

$$(b^{m+1} - b + 1)v_{k;m} < \sum_{j=1}^m \binom{m}{j} \gamma'_j v_{k;m-j} + b^{m+1}.$$

If we had equality here, this would be the same recurrence with the same starting point as for  $v_{0;m}$ . As we have an inequality, we get by induction on  $m$   $v_{k;m} \leq v_{0;m}$ . But then, the right-hand side is  $\leq (b^{m+1} - b + 1)v_{0;m}$ , so we get  $v_{k;m} < v_{0;m}$  for  $m \geq 1$ .

Let  $k = 2$ . We obtain, from Equation (7), the upper bound

$$(b^{m+1} - b + 1)v_{2;m} \leq \sum_{j=1}^m \binom{m}{j} \gamma'_j v_{2;m-j} + \sum_{j=0}^m \binom{m}{j} (d')^j v_{0;m-j}.$$

If the less-than-or-equal sign was replaced with an equality, it would be the recurrence which applies to  $(v_{1,m})$ . As the two sequences have the same  $m = 0$  value we get  $v_{2;m} \leq v_{1;m}$  for all  $m$ . But in the second sum in Equation (7) with  $k = 2$  we have the  $j = 0$  term which is  $v_{1;m}$ , and it is known to be less than  $v_{0;m}$  if  $m \geq 1$ . So the above displayed inequality is strict for  $m \geq 1$ . Then using  $v_{2;m-j} \leq v_{1;m-j}$ , and Equation (7) with  $k = 1$ , in the right-hand side, we get  $(b^{m+1} - b + 1)v_{2;m} < (b^{m+1} - b + 1)v_{1;m}$ , and hence  $v_{2;m} < v_{1;m}$  for all  $m \geq 1$ .

This argument can be repeated inductively and establishes that  $(v_{k;m})_{k \geq 0}$  is strictly decreasing for each fixed  $m \geq 1$ .

The value of the limit as  $k \rightarrow \infty$  can be established as in the proof of Proposition 7. Alternatively, the second paragraph of Proposition 9 says that the sequence of probability measures  $(b^{-1}\mu_k)_{k \geq 0}$  converges weakly to the Lebesgue measure on the interval  $[0, 1)$ . This implies the convergence of the complementary moments to those of the Lebesgue measure:  $\lim_{k \rightarrow \infty} b^{-1}v_{k;m} = (m+1)^{-1}$ .  $\square$

**Remark 11.** For the notion of weak convergence of measures, especially of probability measures on the real line or some interval, see [10, Section 25].

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