SCHATUNOWSKY-TYPE THEOREMS FOR IMAGINARY QUADRATIC INTEGER RINGS OF CLASS NUMBER ONE

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Abstract

In this paper, we generalize Iwata's refinement of Schatunowsky's theorem to the rings of integers of imaginary quadratic fields $\mathbb{Q}(\sqrt{m})$ with class number one and prove that the corresponding sets $Q_{m,\ell}$ of quadratic integers whose quadratic totatives have at most ℓ prime factors are finite. In particular, we refine Dalezman's result for the Gaussian integers. We also provide explicit descriptions in the Eisenstein case.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers and $a \in \mathbb{N}$ be greater than 1. Denote the set of *totatives* of a except 1 by

$$T(a) := \{b \in \mathbb{N} \mid 1 < b < a,\ \gcd(a,b) = 1\}.$$

We also denote the set of rational prime numbers by $\mathbb{P}$. For any $p \in \mathbb{P}$, $T(p)$ is simply the set of natural numbers from 2 to $p-1$. In particular, $T(2) = \varnothing$. On the other hand, when a is a composite number, the set $T(a)$ is likely to contain many primes. For example, we have

$$T(4) = \{3\}, \qquad T(6) = \{5\}, \qquad T(8) = \{3,5,7\}, \qquad T(12) = \{5,7,11\}.$$

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The natural question is: "For what values of a, does $T(a)$ become a set consisting only of prime numbers?" The following result was first proved by Schatunowsky.

Theorem 1 ([5, p.281])**.** *The subset of* $\mathbb{N}$ *defined by*

$$S := \{a \in \mathbb{N} \mid p \in T(a) \text{ implies } p \in \mathbb{P}\}$$

is finite. Specifically, we observe that $S = \{2, 3, 4, 6, 8, 12, 18, 24, 30\}$.

This result can be generalized as follows: Suppose $b \in \mathbb{N}$ has the prime factorization

$$b = q_1^{e_1} q_2^{e_2} \cdots q_u^{e_u} \quad (q_i \in \mathbb{P}),$$

and set $\Omega(b) = e_1 + e_2 + \cdots + e_u$. The following result is due to Iwata.

Theorem 2 ([4])**.** *For each* $\ell \in \mathbb{N}$*, the set*

$$S_\ell := \{a \in \mathbb{N} \mid b \in T(a) \text{ implies } \Omega(b) \leq \ell\}$$

is finite. Note that S_1 *coincides with* S *in Theorem 1.*

Furthermore, Theorem 1 admits an extension to the Gaussian integers $\mathbb{G}$. For $\alpha \in \mathbb{G}$, let $N(\alpha)$ denote its norm, and set

$$T_{\mathbb{G}}(\alpha) := \{\beta \in \mathbb{G} \mid 1 < N(\beta) < N(\alpha),\ \gcd(\alpha, \beta) = 1\}.$$

Let $\mathbb{GP}$ be the set of Gaussian primes. Dalezman proved the following result.

Theorem 3 ([3])**.** *Up to multiplication by a unit,* $5 + 5i$ *is the element of*

$$S_{\mathbb{G}} := \{\alpha \in \mathbb{G} \mid \beta \in T_{\mathbb{G}}(\alpha) \text{ implies } \beta \in \mathbb{GP}\}$$

having maximal norm.

In this paper, we investigate analogues of Theorem 2 in the ring of integers of imaginary quadratic fields with class number one. We also refine Theorem 3 and establish a Schatunowsky-type theorem for the Gaussian integers and the Eisenstein integers.

Although the finiteness of sets $Q_{m,\ell}$ defined in Equation (4) may be anticipated from general arguments in algebraic number theory, such arguments are typically non-constructive and do not yield explicit descriptions. This paper provides a uniform Bonse-type framework for quadratic integer rings of class number one, together with complete explicit bounds and maximal elements in the Gaussian and Eisenstein cases.

2. $\mathcal{O}_m$ and $\mathcal{N}_m$

Let m be a square-free negative integer. The ring of integers of $\mathbb{Q}(\sqrt{m})$ is given by

$$\mathcal{O}_m = \mathbb{Z} + \mathbb{Z}\omega, \qquad \text{where} \quad \omega = \begin{cases} \sqrt{m} & (m \equiv 2, 3 \pmod 4), \\ \frac{1+\sqrt{m}}{2} & (m \equiv 1 \pmod 4). \end{cases}$$

The *norm* of an element $\alpha \in \mathcal{O}_m$ is defined as the product with its complex conjugate $\overline{\alpha} \in \mathcal{O}_m$, that is, $N(\alpha) = \alpha\overline{\alpha}$.

In general, $\mathcal{O}_m$ is an integral domain, and every prime element is automatically irreducible. It is also known that the converse holds if $\mathcal{O}_m$ is a unique factorization domain (UFD). The following result is well known (cf. [5], p. 210).

Proposition 1. *Let $m < 0$. Then, the ring of integers $\mathcal{O}_m$ is a UFD if and only if*

$$m \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}.$$

As in the case of $\mathbb{Z}$, every element in a UFD admits a unique factorization into prime elements. It should be noted that $\mathcal{O}_m$ is not a Euclidean domain for $m = -19, -43, -67, -163$, and hence a division algorithm is not available in these cases. Nevertheless, since this paper only requires the existence of prime factorizations and greatest common divisors in $\mathcal{O}_m$, we restrict our attention to the values of m listed in Proposition 1.

The units in $\mathcal{O}_m$ are well known.

Proposition 2. *Let $m < 0$. Then, the following holds:*

$$\mathcal{O}_m^\times = \begin{cases} \{\pm 1, \pm i\} \cong C_4 & \text{if } m = -1, \\ \{\pm 1, \pm\zeta, \pm\zeta^2\} \cong C_6 & \text{if } m = -3, \\ \{\pm 1\} \cong C_2 & \text{if } m \notin \{-1, -3\}, \end{cases}$$

where $i = \sqrt{-1}$ and $\zeta = \frac{1+\sqrt{-3}}{2}$.

As an analogue of $\mathbb{N} = \mathbb{Z}/\{-1, 1\}$, we consider prime factorizations in the quotient set

$$\mathcal{N}_m := \mathcal{O}_m/\mathcal{O}_m^\times.$$

Throughout this paper, representatives $\alpha \in \mathcal{N}_m$ are taken to satisfy $-\theta_m < \arg(\alpha) \leq \theta_m$, where

$$\theta_m := \begin{cases} \pi/4 & \text{if } m = -1, \\ \pi/6 & \text{if } m = -3, \\ \pi/2 & \text{if } m \notin \{-1, -3\}. \end{cases}$$

As is well known from class field theory, the set of rational primes $\mathbb{P}$ decomposes into three disjoint subsets with respect to $\mathcal{N}_m$: the set $\mathbb{P}_{m,1}$ of primes that ramify

in $\mathcal{O}_m$, the set $\mathbb{P}_{m,2}$ of primes that split in $\mathcal{O}_m$, and the set $\mathbb{P}_{m,3}$ of primes that remain inert in $\mathcal{O}_m$. If we denote by $\mathcal{P}_{m,i}$ $(i = 1, 2, 3)$ the corresponding sets of prime elements of $\mathcal{N}_m$ lying over $\mathbb{P}_{m,i}$, then $z \in \mathcal{P}_{m,1}$ satisfies $\arg z = \theta_m$, $z \in \mathcal{P}_{m,2}$ satisfies $|\arg z| < \theta_m$, and $z \in \mathcal{P}_{m,3}$ satisfies $\arg z = 0$.

The following result is straightforward.

Proposition 3. *Let m be one of the integers listed in Proposition 1. Then, every element of $\mathcal{N}_m$ admits a unique factorization, up to units and the ordering of the factors, as a product of elements from the set $\mathcal{P}_m := \mathcal{P}_{m,1} \sqcup \mathcal{P}_{m,2} \sqcup \mathcal{P}_{m,3}$.*

Example 1. In the case $m = -1$, the set of rational primes $\mathbb{P}$ is partitioned as follows:

$$\mathbb{P}_{-1,1} = \{2\},\ \ \mathbb{P}_{-1,2} = \{p \in \mathbb{P} \mid p \equiv 1 \pmod 4\},\ \ \mathbb{P}_{-1,3} = \{p \in \mathbb{P} \mid p \equiv 3 \pmod 4\}.$$

Accordingly, the set $\mathcal{P}_{-1}$ of Gaussian primes is the disjoint union of the following subsets:

$$\mathcal{P}_{-1,1} = \{1 + i\},\ \ \mathcal{P}_{-1,2} = \{\alpha \in \mathcal{N}_{-1} \mid N(\alpha) \in \mathbb{P}_{-1,2}\},\ \ \mathcal{P}_{-1,3} = \mathbb{P}_{-1,3}.$$

In the case $m = -3$, the set $\mathbb{P}$ is partitioned as follows:

$$\mathbb{P}_{-3,1} = \{3\},\ \ \mathbb{P}_{-3,2} = \{p \in \mathbb{P} \mid p \equiv 1 \pmod 3\},\ \ \mathbb{P}_{-3,3} = \{p \in \mathbb{P} \mid p \equiv 2 \pmod 3\}.$$

Accordingly, the set $\mathcal{P}_{-3}$ of Eisenstein primes is the disjoint union of the following subsets:

$$\mathcal{P}_{-3,1} = \{1 + \zeta\},\ \ \mathcal{P}_{-3,2} = \{\alpha \in \mathcal{N}_{-3} \mid N(\alpha) \in \mathbb{P}_{-3,2}\},\ \ \mathcal{P}_{-3,3} = \mathbb{P}_{-3,3}.$$

Remark 1. In the general case of $\mathcal{N}_m$, if $\rho \in \mathcal{P}_{m,i}$ for $i = 1, 2$, then $N(\rho) \in \mathbb{P}_{m,i}$; whereas if $\rho \in \mathcal{P}_{m,3}$, then $\sqrt{N(\rho)} \in \mathbb{P}_{m,3}$. Therefore, we obtain

$$N(\mathcal{P}_m) := \{N(\rho) \mid \rho \in \mathcal{P}_m\} = \mathbb{P}_{m,1} \sqcup \mathbb{P}_{m,2} \sqcup \mathbb{P}_{m,3}^2, \quad \text{where } \mathbb{P}_{m,3}^2 = \{p^2 \mid p \in \mathbb{P}_{m,3}\}.$$

This distinction plays a crucial role in defining an appropriate notion of quadratic primorials.

3. Numbering the Primes in $\mathcal{N}_m$

To prove a Schatunowsky-type theorem in $\mathcal{N}_m$, we establish a Bonse-type inequality in Section 4. For this purpose, it is necessary to impose an appropriate ordering on the elements of $\mathcal{P}_m$.

Definition 1. Define an ordering on the set $\mathcal{P}_m$ by the following two rules:

(a). If $N(\pi') < N(\pi)$, then $\pi' \prec \pi$.

(b). If $N(\pi') = N(\pi)$ and $\arg(\pi') < \arg(\pi)$, then $\pi' \prec \pi$.

We denote the resulting ordered sequence by $\{\rho_n\}_{n\geq 1}$.

Furthermore, let $\mathcal{P}^+_{m,2}$ be the subset of $\mathcal{P}_{m,2}$ consisting of elements with $\arg z > 0$, and let $\mathcal{P}^-_{m,2}$ be the subset consisting of those with $\arg z < 0$. We then have the ordered subset

$$\mathcal{P}^+_m := \mathcal{P}_{m,1} \sqcup \mathcal{P}^+_{m,2} \sqcup \mathcal{P}_{m,3},$$

and denote the resulting ordered sequence by $\{\pi_n\}_{n\geq 1}$. For notational convenience, we formally set $\pi_0 = 1$. Note that the ordering on $\mathcal{P}^+_m$ is determined solely by rule (a).

Hereafter, the value of m relevant to the prime sequences $\{\rho_n\}$ or $\{\pi_n\}$ will be clear from the context.

Example 2. The ordering of the elements of $\mathcal{P}_{-1}$ (Gaussian primes) is as follows:

$$\rho_1 = 1 + i = \pi_1, \quad \rho_2 = 2 - i = \overline{\pi_2}, \quad \rho_3 = 2 + i = \pi_2, \quad \rho_4 = 3 = \pi_3,$$

$$\rho_5 = 3 - 2i = \overline{\pi_4}, \quad \rho_6 = 3 + 2i = \pi_4, \quad \rho_7 = 4 - i = \overline{\pi_5}, \quad \rho_8 = 4 + i = \pi_5,$$

$$\rho_9 = 5 - 2i = \overline{\pi_6}, \quad \rho_{10} = 5 + 2i = \pi_6, \quad \ldots.$$

On the other hand, the ordering of the elements of $\mathcal{P}_{-3}$ (Eisenstein primes) is as follows:

$$\rho_1 = 1 + \zeta = \pi_1, \quad \rho_2 = 2 = \pi_2, \quad \rho_3 = 3 - \zeta = \overline{\pi_3}, \quad \rho_4 = 2 + \zeta = \pi_3,$$

$$\rho_5 = 4 - \zeta = \overline{\pi_4}, \rho_6 = 3 + \zeta = \pi_4, \quad \rho_7 = 5 - 2\zeta = \overline{\pi_5}, \quad \rho_8 = 3 + 2\zeta = \pi_5,$$

$$\rho_9 = 5 = \pi_6, \quad \ldots.$$

Note that the complex conjugate of $\pi = a + b\zeta \in \mathcal{N}_{-3}$ is given by $\overline{\pi} = (a + b) - b\zeta$. See also Tables 1 and 2 in the Appendix.

4. Bonse-Type Inequalities

Let p_n be the n-th rational prime. To reprove Theorem 1, Bonse [1] utilized the inequality

$$p_n\# := \prod_{k=1}^{n} p_k > p_{n+1}^2 \quad (n \geq 4). \tag{1}$$

We call $p_n\#$ the n-th *rational primorial*. In this section, we prove some inequalities analogous to Inequality (1) in $\mathcal{N}_m$. First, we introduce the following definition.

Definition 2. Let m be one of the integers listed in Proposition 1. The n-th *quadratic primorial* $\pi_n\#$ in $\mathcal{N}_m$ is defined inductively as follows:

$$\pi_0\# := 1, \quad \pi_n\# := \begin{cases} (\pi_{n-1}\#)\pi_n & (\pi_n \in \mathcal{P}_{m,1} \text{ or } \pi_n \in \mathcal{P}_{m,3}), \\ (\pi_{n-1}\#)\overline{\pi_n}\pi_n & (\pi_n \in \mathcal{P}^+_{m,2}) \end{cases} \quad \text{for } n \geq 1.$$

Example 3. The first few Gaussian primorials in $\mathcal{N}_{-1}$ are given as follows:

$$\pi_1\# = \pi_1 = 1+i, \quad \pi_2\# = (\pi_1\#)\overline{\pi_2}\pi_2 = 5(1+i), \quad \pi_3\# = (\pi_2\#)\pi_3 = 15(1+i),$$

$$\pi_4\# = (\pi_3\#)\overline{\pi_4}\pi_4 = 195(1+i), \quad \pi_5\# = (\pi_4\#)\overline{\pi_5}\pi_5 = 3315(1+i), \quad \ldots.$$

On the other hand, the first few Eisenstein primorials in $\mathcal{N}_{-3}$ are given as follows:

$$\pi_1\# = \pi_1 = 1+\zeta, \quad \pi_2\# = (\pi_1\#)\pi_2 = 2(1+\zeta), \quad \pi_3\# = (\pi_2\#)\overline{\pi_3}\pi_3 = 14(1+\zeta),$$

$$\pi_4\# = (\pi_3\#)\overline{\pi_4}\pi_4 = 182(1+\zeta), \quad \pi_5\# = (\pi_4\#)\overline{\pi_5}\pi_5 = 3458(1+\zeta), \quad \ldots.$$

For the sequence of rational primes $\{p_n\}$, it is well-known that the equality

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1 \tag{2}$$

holds. The following result is an analogue of this fact in $\mathcal{N}_m$.

Lemma 1. *For the sequences $\{\rho_n\}$ and $\{\pi_n\}$, the following holds:*

$$\lim_{n\to\infty} \frac{N(\rho_{n+1})}{N(\rho_n)} = 1 = \lim_{n\to\infty} \frac{N(\pi_{n+1})}{N(\pi_n)}.$$

Proof. From the ordering of $\mathcal{P}_m$ defined in Definition 1, we have $N(\rho_m) \leq N(\rho_n)$ whenever $m < n$, and together with Equality (2) and Remark 1, we have the left-hand equality. Since $\{\pi_n\}$ is a subsequence of $\{\rho_n\}$, the right-hand equality also holds. □

Proposition 4. *For each fixed non-negative integer ℓ, there exists a natural number $n(\ell)$ such that for all $n \geq n(\ell)$, the following inequality holds:*

$$N(\pi_{n+1}^{\ell+1}) < N(\pi_n\#).$$

Proof. For a fixed natural number k, we have

$$\lim_{n\to\infty} \frac{N(\pi_{n+1})}{N(\pi_{n-k})} = \lim_{n\to\infty} \frac{N(\pi_{n+1})}{N(\pi_n)} \cdot \frac{N(\pi_n)}{N(\pi_{n-1})} \cdot \cdots \cdot \frac{N(\pi_{n-k+1})}{N(\pi_{n-k})} = 1.$$

For sufficiently large n, we obtain

$$0 < \frac{N(\pi_{n+1}^{\ell+1})}{N(\pi_n\#)} \leq \frac{1}{N(\pi_{n-\ell-1}\#)} \prod_{k=0}^{\ell} \frac{N(\pi_{n+1})}{N(\pi_{n-k})}.$$

As $n \to \infty$, the right-hand side converges to 0. In particular, $N(\pi_{n+1}^{\ell+1})/N(\pi_n\#)$ becomes less than 1. □

4.1. Bonse-Type Inequality for Gaussian Integers

Here we determine the values of $n(\ell)$ appearing in Proposition 4 for specific values of ℓ in the Gaussian integer case. The following result was first proved by Dalezman [3].

Lemma 2 (Bertrand's Postulate for Gaussian Integers)**.** *For each $\pi_n \in \mathcal{P}_{-1}$ with $n \geq 2$, there exists a Gaussian prime ρ satisfying*

$$N(\pi_n) < N(\rho) \leq 2N(\pi_n).$$

In particular, we have $N(\pi_{n+1}) \leq 2N(\pi_n)$.

Proposition 5. *For each $\ell \in \mathbb{N}$, define $n(\ell) \in \mathbb{N}$ as follows:*

$$n(1) = 3,\ \ n(2) = 4,\ \ n(3) = 5,\ \ n(4) = 5,\ \ n(5) = 6,\ \ n(6) = 7,\ \ n(7) = 7,\ \ \ldots.$$

Then, for all $n \geq n(\ell)$, we have $N(\pi_{n+1}^{\ell+1}) < N(\pi_n\#)$.

Proof. We prove the statement by induction on n. For $\ell \leq 3$, we have $N(\pi_{n(\ell)+1}) > 2^{\ell+1}$. Assume that the inequality

$$N(\pi_{n+1}^{\ell+1}) < N(\pi_n\#)$$

holds for some $n \geq n(\ell)$. By Lemma 2, we obtain

$$\begin{aligned} N(\pi_{n+1}\#) &\geq N(\pi_n\#)N(\pi_{n+1}) > N(\pi_{n+1}^{\ell+1})N(\pi_{n(\ell)+1}) \\ &> 2^{\ell+1}N(\pi_{n+1}^{\ell+1}) \geq N(\pi_{n+2}^{\ell+1}). \end{aligned} \tag{3}$$

For $\ell \geq 4$, define $N(\ell)$ to be the smallest natural number such that $N(\pi_{N(\ell)+1}) > 2^{\ell+1}$ (see also Tables 1 and 2 in the Appendix). The first few values are

$$N(4) = 6, \quad N(5) = 11, \quad N(6) = 18, \quad N(7) = 28, \quad \ldots.$$

The inequality $N(\pi_{n+1}^{\ell+1}) < N(\pi_n\#)$ can be verified individually for $n \in \mathbb{N}$ satisfying $n(\ell) \leq n \leq N(\ell)$. If we assume that the inequality $N(\pi_{n+1}^{\ell+1}) < N(\pi_n\#)$ holds for some $n \geq N(\ell)$, then we obtain similar inequalities to those in Inequality (3). □

Remark 2. We only explicitly provide values of $n(\ell)$ for $\ell \leq 7$ due to the limited range of the data in Tables 1 and 2 in the Appendix. If necessary, values for larger ℓ can be obtained by computer calculation (the same applies to the case of Eisenstein integers below).

4.2. Bonse-Type Inequality for Eisenstein Integers

Here we determine the values of $n(\ell)$ for specific values of ℓ for Eisenstein integers.

Lemma 3 (Bertrand's Postulate for Eisenstein Integers)**.** *For each $\pi_n \in \mathcal{P}_{-3}$ with $n \geq 1$, there exists an Eisenstein prime ρ satisfying*

$$N(\pi_n) < N(\rho) \leq 2N(\pi_n).$$

In particular, we have $N(\pi_{n+1}) \leq 2N(\pi_n)$.

Proof. It suffices to prove that for each $k \geq 3$, there exists an Eisenstein prime ρ with $k < N(\rho) \leq 2k$. For small k, we verify directly: if $k = 3$, then $k < N(2) = 4 \leq 2k$; if $4 \leq k \leq 6$, then $k < N(2+\zeta) = 7 \leq 2k$. By a result of Breusch [2], for $k \geq 7$ there exists a rational prime p such that $p \equiv 1 \pmod 3$ and $k < p \leq 2k$. Since such a prime p can be written as $p = a^2 + ab + b^2$ for some $a, b \in \mathbb{N}$, we may take $\rho = a + b\zeta$. Thus, in all cases such a prime ρ exists. □

The following proposition can be proved by the same method as in Proposition 5.

Proposition 6. *For each $\ell \in \mathbb{N}$, define $n(\ell) \in \mathbb{N}$ by*

$$n(1) = 3,\ \ n(2) = 4,\ \ n(3) = 5,\ \ n(4) = 5,\ \ n(5) = 6,\ \ n(6) = 7,\ \ n(7) = 8,\ \ \ldots.$$

Then, for all $n \geq n(\ell)$, we have $N(\pi_{n+1}^{\ell+1}) < N(\pi_n\#)$.

5. Schatunowsky-Type Theorems for Quadratic Integers

Let m be one of the integers listed in Proposition 1. If an element $\beta \in \mathcal{N}_m$ has a prime factorization of the form $\beta = \varepsilon\tau_1^{e_1}\tau_2^{e_2}\cdots\tau_u^{e_u}$ with $\tau_\bullet \in \mathcal{P}_m$ and $\varepsilon \in \mathcal{O}_m^\times$, we define

$$\Omega(\beta) = e_1 + \cdots + e_u.$$

Note that if $\Omega(\beta) = 1$, then $\beta \in \mathcal{P}_m$.

For $\alpha, \beta \in \mathcal{N}_m$, let $\gcd(\alpha, \beta)$ denote the greatest common divisor of α and β, and define the set of *quadratic totatives of* α by

$$T_m(\alpha) := \{\beta \in \mathcal{N}_m \mid 1 < N(\beta) < N(\alpha) \text{ and } \gcd(\alpha, \beta) = 1\}.$$

For each $\ell \in \mathbb{N}$, define the set $Q_{m,\ell}$ by

$$Q_{m,\ell} := \{\alpha \in \mathcal{N}_m \mid N(\alpha) > 1,\ \beta \in T_m(\alpha) \text{ implies } \Omega(\beta) \leq \ell\}. \tag{4}$$

Lemma 4. *For $\alpha \in Q_{m,\ell}$, if $N(\pi_n^{\ell+1}) < N(\alpha)$, then $(\pi_n\#) \mid \alpha$.*

Proof. Let $\pi \in \mathcal{P}_m$ be such that $N(\pi^{\ell+1}) < N(\alpha)$. Since $\pi^{\ell+1}$ has more than ℓ prime factors, it follows from $\alpha \in Q_{m,\ell}$ that $\pi \mid \alpha$. □

Theorem 4. *Let $\ell \in \mathbb{N}$ be fixed, and let $n(\ell)$ be the natural number given by Proposition 4. If $N(\pi_{n(\ell)}^{\ell+1}) < N(\alpha)$, then $\alpha \notin Q_{m,\ell}$. In particular, the set $Q_{m,\ell}$ is finite.*

Proof. Assume that $N(\pi_{n(\ell)}^{\ell+1}) < N(\alpha)$. Then there exists $n \geq n(\ell)$ such that

$$N(\pi_n^{\ell+1}) < N(\alpha) \leq N(\pi_{n+1}^{\ell+1}).$$

By the definition of $n(\ell)$, we have $N(\pi_{n+1}^{\ell+1}) < N(\pi_n\#)$. Hence $N(\alpha) < N(\pi_n\#)$. However, if $\alpha \in Q_{m,\ell}$, then by Lemma 4 it follows that $N(\pi_n\#) \leq N(\alpha)$, which is a contradiction. □

Theorem 5. *Let $\ell \in \mathbb{N}$ be fixed, and let $n(\ell)$ be the natural number given by Proposition 4. Then*

$$Q_{m,\ell} = \bigcup_{n=1}^{n(\ell)} Q_{m,\ell,n},$$
$$\text{where} \quad Q_{m,\ell,n} = \{\alpha \in \mathcal{N}_m \mid N(\pi_{n-1}^{\ell+1}) < N(\alpha) \leq N(\pi_n^{\ell+1}),\ (\pi_{n-1}\#) \mid \alpha\}.$$

In particular, the element of $Q_{m,\ell}$ with the largest norm satisfies $(\pi_{n(\ell)-1}\#) \mid \alpha$ and $N(\alpha) \leq N(\pi_{n(\ell)}^{\ell+1})$.

Proof. If $1 < N(\alpha) \leq N(\pi_1^{\ell+1})$, then it is clear that $\alpha \in Q_{m,\ell}$. Suppose $\alpha \in Q_{m,\ell}$ and $N(\pi_{n-1}^{\ell+1}) < N(\alpha) \leq N(\pi_n^{\ell+1})$ for $n = 2, \ldots, n(\ell)$. Then, by Lemma 4, we have $(\pi_{n-1}\#) \mid \alpha$.

If $N(\pi_{n(\ell)}^{\ell+1}) < N(\alpha)$, then by Theorem 4 we conclude that $\alpha \notin Q_{m,\ell}$. Hence, for $n \geq n(\ell) + 1$, we have $Q_{m,\ell,n} = \varnothing$. □

In what follows, we explicitly determine the set $Q_{m,\ell}$ for the Gaussian and Eisenstein integer rings.

Corollary 1. *The sets $Q_{-1,\ell}$ for $\ell = 1, 2, 3$ are as follows:*

$$\begin{aligned}
Q_{-1,1} &= \{\alpha \mid 1 < N(\alpha) \leq 4\} \cup \{\gamma\pi_1 \mid 3 \leq N(\gamma) \leq 12\} \cup \{\pi_2\#\},\\
Q_{-1,2} &= \{\alpha \mid 1 < N(\alpha) \leq 8\} \cup \{\gamma\pi_1 \mid 5 \leq N(\gamma) \leq 62\} \cup \{\gamma(\pi_2\#) \mid 3 \leq N(\gamma) \leq 14\}\\
&\qquad \cup \{\gamma(\pi_3\#) \mid 2 \leq N(\gamma) \leq 4\},\\
Q_{-1,3} &= \{\alpha \mid 1 < N(\alpha) \leq 16\} \cup \{\gamma\pi_1 \mid 9 \leq N(\gamma) \leq 312\}\\
&\qquad \cup \{\gamma(\pi_2\#) \mid 13 \leq N(\gamma) \leq 131\} \cup \{\gamma(\pi_3\#) \mid 15 \leq N(\gamma) \leq 63\} \cup \{\pi_4\#\}.
\end{aligned}$$

In particular, the element of $Q_{-1,1}$ ($= S_{\mathbb{G}}$ in Theorem 3) with the largest norm is $\pi_2\# = 5(1+i)$, for $Q_{-1,2}$ it is $2(\pi_3\#) = 30(1+i)$, and for $Q_{-1,3}$ it is $\pi_4\# = 195(1+i)$.

The elements of largest norm in the first two cases, $Q_{-1,1}$ and $Q_{-1,2}$, were determined by Dalezman [3], although he did not explicitly describe the structure of these sets.

Corollary 2. *The sets $Q_{-3,\ell}$ for $\ell = 1, 2, 3$ are as follows:*

$$\begin{aligned}
Q_{-3,1} &= \{\alpha \mid 1 < N(\alpha) \le 9\} \cup \{\gamma\pi_1 \mid 4 \le N(\gamma) \le 5\} \cup \{\gamma(\pi_2\#) \mid 2 \le N(\gamma) \le 4\},\\
Q_{-3,2} &= \{\alpha \mid 1 < N(\alpha) \le 27\} \cup \{\gamma\pi_1 \mid 10 \le N(\gamma) \le 21\}\\
&\qquad \cup \{\gamma(\pi_2\#) \mid 6 \le N(\gamma) \le 28\} \cup \{\gamma(\pi_3\#) \mid 1 \le N(\gamma) \le 3\},\\
Q_{-3,3} &= \{\alpha \mid 1 < N(\alpha) \le 81\} \cup \{\gamma\pi_1 \mid 28 \le N(\gamma) \le 85\}\\
&\qquad \cup \{\gamma(\pi_2\#) \mid 22 \le N(\gamma) \le 200\} \cup \{\gamma(\pi_3\#) \mid 5 \le N(\gamma) \le 48\} \cup \{\pi_4\#\}.
\end{aligned}$$

In particular, the element of $Q_{-3,1}$ with the largest norm is $2(\pi_2\#) = 4(1+\zeta)$, for $Q_{-3,2}$ it is $(2-\zeta)(\pi_4\#) = 42$, and for $Q_{-3,3}$ it is $\pi_4\# = 182(1+\zeta)$.

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Appendix. Tables of Gaussian and Eisenstein Primes

n	1	2	3	4	5	6	7	8	9	10
π_n	$1+i$	$2+i$	3	$3+2i$	$4+i$	$5+2i$	$6+i$	$5+4i$	7	$7+2i$
$N(\pi_n)$	2	5	3^2	13	17	29	37	41	7^2	53
n	11	12	13	14	15	16	17	18	19	20
π_n	$6+5i$	$8+3i$	$8+5i$	$9+4i$	$10+i$	$10+3i$	$8+7i$	11	$11+4i$	$10+7i$
$N(\pi_n)$	61	73	89	97	101	109	113	11^2	137	149
n	21	22	23	24	25	26	27	28	29	30
π_n	$11+6i$	$13+2i$	$10+9i$	$12+7i$	$14+i$	$15+2i$	$13+8i$	$15+4i$	$16+i$	$13+10i$
$N(\pi_n)$	157	173	181	193	197	229	233	241	257	269

Table 1: Gaussian primes in $\mathcal{P}^+_{-1}$, listed in increasing order of their norms.

n	1	2	3	4	5	6	7	8	9	10
π_n	$1+\zeta$	2	$2+\zeta$	$3+\zeta$	$3+2\zeta$	5	$5+\zeta$	$4+3\zeta$	$6+\zeta$	$5+4\zeta$
$N(\pi_n)$	3	2^2	7	13	19	5^2	31	37	43	61
n	11	12	13	14	15	16	17	18	19	20
π_n	$7+2\zeta$	$8+\zeta$	$7+3\zeta$	$8+3\zeta$	$9+2\zeta$	$7+5\zeta$	11	$7+6\zeta$	$10+3\zeta$	$9+5\zeta$
$N(\pi_n)$	67	73	79	97	103	109	11^2	127	139	151
n	21	22	23	24	25	26	27	28	29	30
π_n	$12+\zeta$	$11+3\zeta$	$11+4\zeta$	$9+7\zeta$	$13+2\zeta$	$14+\zeta$	$11+6\zeta$	$12+5\zeta$	$15+\zeta$	$10+9\zeta$
$N(\pi_n)$	157	163	181	193	199	211	223	229	241	271

Table 2: Eisenstein primes in $\mathcal{P}^+_{-3}$, listed in increasing order of their norms.