



GENERALIZED BINOMIAL COEFFICIENTS AND THE MONOID OF LEXICOGRAPHICALLY MINIMAL EXTENSIONS

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Abstract

A sequence $(e_i)_{i \leq m}$ of nonnegative integers e_i , where $m \in \mathbb{N}$ or $m = \infty$, is called a *binomid index* if $\sum_{i=n-k+1}^n e_i \geq \sum_{i=1}^k e_i$ for all $k, n \in \mathbb{N}$ such that $1 \leq k \leq n \leq m$. Infinite binomid indices give rise to binomid sequences (also known as Raney sequences) and generalized binomial coefficients. A finite binomid index η can be extended to a unique lexicographically minimal infinite binomid index $\tilde{\eta}$. This binomid index $\tilde{\eta}$ is necessarily eventually periodic. In this research, we give a formula for the minimal period and provide an upper bound for the preperiod of $\tilde{\eta}$. We also show that the monoid of lexicographically minimal extensions of finite binomid indices is an inductive limit of finitely presented monoids.

1. Introduction

Let $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $(\mathbb{N}_0)^\mathbb{N} := \{(e_1, e_2, \dots) \mid e_i \in \mathbb{N}_0\}$. For $m \in \mathbb{N}$, we also let $(\mathbb{N}_0)^m := \{(e_1, e_2, \dots, e_m) \mid e_i \in \mathbb{N}_0\}$. A sequence $f = (f_n)_n = (f_1, f_2, \dots)$ of positive integers f_n is a *binomid sequence* (or Raney sequence) if, for all $n, k \in \mathbb{N}$ with $n \geq k$, the *f-nomid coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_f := \frac{f_n f_{n-1} \cdots f_{n-k+1}}{f_k f_{k-1} \cdots f_1}$$

is an integer (see [14, 1]). Clearly, the *f-nomid coefficients* generalize the usual binomial coefficients (which we obtain by taking $f = (1, 2, 3, \dots)$). There are several papers written on the topic of *f-nomid coefficients*. In [9], Gould considered the

binomid coefficients arising from the sequence of Fibonacci numbers. In this paper, Gould coined the term *Fontené-Ward generalized binomial coefficients* after tracing the genesis of the generalized definition from these two pioneering papers: [8, 15]. In [3], Bennett et al. considered the so-called *Lucasnomials*, which are generalized binomial coefficients arising from Lucas sequences. In [7], Edgar and Spivey considered the sequence of integers arising from multiplicative arithmetic functions. Many authors studied the fractal nature of the generalized Pascal's triangle from generalized binomial coefficients, see e.g. [5, 6].

Now, if f is a binomid sequence, then there is a factorization of f into prime factors $f(p) = (p^{\nu_p(f_n)})_n$, where $\nu_p(f_n)$ is the p -adic valuation of f_n , such that

$$f = \prod_{p \in \mathbb{P}} f(p)$$

with $\mathbb{P}$ denoting the set of prime numbers. Consequently, each $f(p)$ is binomid and

$$\begin{bmatrix} n \\ k \end{bmatrix}_f = \prod_{p \in \mathbb{P}} \begin{bmatrix} n \\ k \end{bmatrix}_{f(p)}.$$

Hence, it suffices to consider a sequence $f(p) = (p^{e_1}, p^{e_2}, \dots)$ for some $p \in \mathbb{P}$ and index $(e_1, e_2, \dots) \in (\mathbb{N}_0)^\mathbb{N}$. An index $(e_i)_{i \leq m}$, where $m \in \mathbb{N}$ or $m = \infty$, is called a *binomid index* if, for all k, n such that $1 \leq k \leq n \leq m$,

$$\sum_{i=n-k+1}^n e_i \geq \sum_{i=1}^k e_i.$$

We denote by $\mathbb{I}$ the set of all infinite binomid indices.

A finite binomid index η may always be extended into an infinite binomid index. Among the possible extensions in $\mathbb{I}$, we choose the unique lexicographically minimal one and denote this by $\tilde{\eta}$. We call $\tilde{\eta}$ the *lex-minimal extension* of η .

In [4], the authors showed that the lex-minimal extension $\tilde{\eta}$ is eventually periodic. However, not all eventually periodic binomid indices are lex-minimal. Meanwhile, every periodic binomid index is lex-minimal. Moreover, the set $\mathbb{L}$ of all lex-minimal extensions is a commutative monoid under componentwise addition.

For $l \in \mathbb{N}_0$, let σ^l be the (right) shift map that sends an infinite sequence $(e_1, e_2, \dots)$ to $(0^l, e_1, e_2, \dots)$. From [4], we know that the elements of

$$\left\{ \sum_{j=1}^r \sigma^{m_j}(\lambda_j) \mid r \in \mathbb{N}, m_1, \dots, m_r \in \mathbb{N}_0, \lambda_1, \dots, \lambda_r \in \mathbb{L} \right\}$$

are all eventually periodic binomid indices. We do not know whether the converse holds or not.

In this article, we reformulate the problem of computing the lex-minimal extension as an *unbounded knapsack problem*. We give a formula for the minimal period of $\tilde{\eta}$ and give an upper bound on the minimal preperiod. Finally, we introduce a notion of atomic binomid indices to establish a structure result (Theorem 3) on the set of lex-minimal extensions. The atomic decomposition of a lex-minimal extension may provide a new perspective on solving an unbounded knapsack problem.

2. Unbounded Knapsack Problem

We begin this section by introducing some notation. Let $\mathbf{Sum}(\eta) := (S_i(\eta))_i = (S_i)_i$ be the sequence of partial sums of $\eta = (e_i)_i \in (\mathbb{N}_0)^\mathbb{N}$; that is, $S_i = e_1 + e_2 + \cdots + e_i$ for all $i \in \mathbb{N}$. Let $\mathbf{Ave}(\eta) := (A_i(\eta))_i = (A_i)_i$ be the sequence of averages, where $A_i := S_i/i$. For $k \in \mathbb{N}$, let $\eta|_k$ denote the truncated sequence $(e_i)_{i \leq k}$.

We have the following useful result characterizing binomid indices and lex-minimal extensions.

Proposition 1 ([4]). *Let $\eta \in (\mathbb{N}_0)^\mathbb{N}$. For $r \in \mathbb{N}$ with $r \geq 2$, let*

$$\Delta_\eta(r) := \min\{\delta_\eta(i, j) \mid i + j = r\},$$

where

$$\delta_\eta(i, j) := S_{i+j} - S_i - S_j$$

for $i, j \in \mathbb{N}$. Then

1. η is binomid if and only if $\delta_\eta(i, j) \geq 0$ for all $i, j \geq 1$;
2. η is the lex-minimal extension of $\eta|_k$ if and only if $(\Delta_\eta(r))_{r > k} = 0^\infty$.

Remark 1. If $\eta \in \mathbb{I}$ and $k, n \in \mathbb{N}$, then $A_k \leq A_{kn}$. Hence, $\lim_{i \rightarrow \infty} A_i$ exists, possibly with value ∞ . Moreover, δ is additive: for binomid indices η and γ ,

$$\delta_{\eta+\gamma}(i, j) = \delta_\eta(i, j) + \delta_\gamma(i, j).$$

It follows that

$$\Delta_{\eta+\gamma}(r) \geq \Delta_\eta(r) + \Delta_\gamma(r).$$

for all $r \in \mathbb{N}$ with $r \geq 2$.

By Proposition 1, $\mathbf{Sum}(\eta) = (S_j)_j$ is superadditive for any binomid η , i.e., for any $(a_i)_{i \leq m} \in (\mathbb{N}_0)^m$, we have

$$\sum_{i=1}^m a_i S_i \leq S_{\sum_{i=1}^m i a_i}.$$

The process of computing the lex-minimal extension of a finite binomid index may be viewed as an *unbounded knapsack problem* (see [11]).

Definition 1. Let $v = (v_i)_{i \leq m}$ and $w = (w_i)_{i \leq m}$ be finite sequences with $v_i \in \mathbb{N}_0$ and $w_i \in \mathbb{N}$ for all $1 \leq i \leq m$. For $c \in \mathbb{N}_0$, the unbounded knapsack problem $\mathbf{UKP}[v; w; c]$ asks to find a sequence $(n_i)_{i \leq m}$ of nonnegative integers that maximizes the sum

$$\sum_{i=1}^m n_i v_i$$

subject to the constraint

$$\sum_{i=1}^m n_i w_i \leq c.$$

We denote the set of optimal solutions also by $\mathbf{UKP}[v; w; c]$ and the optimal value by $\phi(v; w; c)$. For $c \in \mathbb{N}_0$, the solution set $\mathbf{UKP}[v; w; c]$ is nonempty since the zero vector is a *feasible* solution. If $w_i = 1$ for some $1 \leq i \leq m$, then an optimal solution can be chosen so that

$$\sum_{i=1}^m n_i w_i = c.$$

Conversely, suppose that the sequences $v = (v_i)_{i \leq m}$ and $w = (w_i)_{i \leq m}$ are given, with $w_i = 1$ for some $1 \leq i \leq m$. Set

$$N := \max\{w_i \mid 1 \leq i \leq m\}.$$

With the convention $\phi(v; w; 0) = 0$, define

$$\eta = (\eta_j)_{j \leq N}$$

by

$$\eta_j := \phi(v; w; j) - \phi(v; w; j - 1).$$

Then

$$\mathbf{Sum}(\tilde{\eta}) = (\phi(v; w; c))_{c \geq 1}.$$

Thus, computing a lex-minimal extension is equivalent to computing the optimal values $\phi(v; w; c)$ of an associated unbounded knapsack problem as the capacity c varies. More precisely, we have the following result.

Proposition 2. Let η be a finite binomid index of length m and let $\mathbf{Sum}(\tilde{\eta}) = (S_j)_j$. For all $c \in \mathbb{N}$, there exists $(n_i)_{i \leq m} \in \mathbf{UKP}[(S_i)_{i \leq m}; (i)_{i \leq m}; c]$ such that

$$S_c = \sum_{i=1}^m n_i S_i$$

and

$$c = \sum_{i=1}^m n_i i.$$

Proof. We proceed by induction on $c \in \mathbb{N}$. If $c \leq m$, take $n_c = 1$ and $n_i = 0$ for $i \neq c$. This gives the desired equalities. We now show that this choice is optimal. Suppose $(a_i)_{i \leq m} \in (\mathbb{N}_0)^m$ such that $c = \sum_{i=1}^m a_i i$. By superadditivity of $\mathbf{Sum}(\tilde{\eta})$,

$$\sum_{i=1}^m a_i S_i \leq S_{\sum_{i=1}^m a_i i} = S_c,$$

so the choice is optimal. Assume the claim holds up to some $c \in \mathbb{N}$. We may suppose $c + 1 > m$. By Proposition 1, there exist $r, s \in \mathbb{N}$ such that

$$c + 1 = r + s, \quad S_{c+1} = S_r + S_s.$$

By induction, choose optimal solutions $(a_i)_{i \leq m}$ and $(b_i)_{i \leq m}$ for capacities r and s satisfying

$$S_r = \sum_{i=1}^m a_i S_i, \quad r = \sum_{i=1}^m i a_i,$$

and

$$S_s = \sum_{i=1}^m b_i S_i, \quad s = \sum_{i=1}^m i b_i.$$

Set $n_i = a_i + b_i$. Then

$$c + 1 = \sum_{i=1}^m i n_i$$

and

$$S_{c+1} = \sum_{i=1}^m n_i S_i.$$

Finally, let $(k_i)_{i \leq m} \in (\mathbb{N}_0)^m$ such that $\sum_{i=1}^m k_i i = c + 1$. By superadditivity of $\mathbf{Sum}(\tilde{\eta})$, we have

$$\sum_{i=1}^m k_i S_i \leq S_{c+1}.$$

Thus, the choice $(n_i)_{i \leq m}$ is optimal. $\square$

3. Preperiod and Period of $\tilde{\eta}$

The lex-minimal extension $\tilde{\eta}$ of a finite binomid index η is eventually periodic. Hence, we can write $\tilde{\eta}$ as $\eta_0(\eta_1)^\infty$. We call $|\eta_1|$ the *minimal period* of $\tilde{\eta}$ if this length is minimal. Likewise, we call $|\eta_0|$ the *minimal preperiod* if this length is minimal. Of course, any integer greater than $|\eta_0|$ may be taken as a preperiod and any multiple $k|\eta_1|$, where $k \in \mathbb{N}$, may be taken as a period.

In this section, we give an exact formula for the minimal period of $\tilde{\eta}$. Throughout, we assume that η is a finite binomid index of length m . We let $\mathbf{Sum}(\tilde{\eta}) = (S_i)_i$ and $\mathbf{Ave}(\tilde{\eta}) = (A_i)_i = (S_i/i)_i$. We set $A_{\max} := \max\{A_i \mid 1 \leq i \leq m\}$. From [4, Theorem 1], if $i \in \mathbb{N}$ such that $A_i = A_{\max}$, then i is a period of $\tilde{\eta}$. To proceed, we first prepare some lemmas.

Lemma 1. *For a lex-minimal extension $\tilde{\eta}$, we have $A_{\max} = \max\{A_i \mid i \in \mathbb{N}\}$.*

Proof. By Proposition 2, for $k > m$, there exists

$$(n_i)_{i \leq m} \in \mathbf{UKP}[(S_i)_{i \leq m}; (i)_{i \leq m}; k]$$

such that

$$kA_k = S_k = \sum_{i=1}^m n_i S_i = \sum_{i=1}^m n_i i A_i \leq \left(\sum_{i=1}^m n_i i \right) A_{\max} = kA_{\max}.$$

Hence, $A_k \leq A_{\max}$ for all $k \geq 1$. This implies that $A_{\max} = \max\{A_k \mid k \in \mathbb{N}\}$. $\square$

Lemma 2. *For a lex-minimal extension $\tilde{\eta}$,*

$$\mathbf{GCD}(\{i \leq m \mid A_{\max} = A_i\}) = \mathbf{GCD}(\{i \in \mathbb{N} \mid A_{\max} = A_i\}).$$

Proof. Let $g = \mathbf{GCD}(\{i \leq m \mid A_{\max} = A_i\})$. Let $k \in \mathbb{N}$ such that $A_k = A_{\max}$. If $k \leq m$, then $g \mid k$. Suppose $k > m$. From the proof of Lemma 1, we have

$$\sum_{i=1}^m n_i i A_i = \sum_{i=1}^m n_i i A_{\max} = kA_{\max}$$

since $kA_k = kA_{\max}$. Thus, if $A_i \neq A_{\max}$, then $n_i = 0$ necessarily because $A_{\max}$ is maximal. It follows that

$$k = \sum_{\substack{i=1 \\ A_i = A_{\max}}}^m n_i i.$$

Hence, $g \mid k$. That is, $g = \mathbf{GCD}(\{i \in \mathbb{N} \mid A_{\max} = A_i\})$. $\square$

Lemma 3. *The minimal period of $\tilde{\eta}$ is divisible by $\mathbf{GCD}(\{i \leq m \mid A_{\max} = A_i\})$.*

Proof. Since $\tilde{\eta}$ is a binomid index, it follows that $A_k \leq A_{kn}$ for all $k, n \in \mathbb{N}$. Therefore, if k is in the set $\{i \in \mathbb{N} \mid A_{\max} = A_i\}$, then, so is nk . Thus, we can choose $k \in \mathbb{N}$ sufficiently larger than both m and the minimal preperiod of $\tilde{\eta}$ such that $A_k = A_{\max}$. Let p be the minimal period of $\tilde{\eta}$ and let $l := k/\mathbf{GCD}(k, p)$. Since $k \mid lp$, we have $A_{k+lp} = A_{\max}$.

Let S be the sum of the p entries of the periodic part of $\tilde{\eta}$. Then $S_{k+lp} = S_k + lS$. Also,

$$\begin{aligned} S_{k+lp} &= (k + lp)A_{k+lp} \\ &= (k + lp)A_{\max} \\ &= kA_{\max} + lpA_{\max} \\ &= S_k + lpA_{\max}. \end{aligned}$$

Hence, $lS = lpA_{\max}$, i.e., $S = pA_{\max}$. It follows that

$$\begin{aligned} (k + p)A_{k+p} &= S_{k+p} \\ &= S_k + S \\ &= kA_{\max} + pA_{\max} \\ &= (k + p)A_{\max}. \end{aligned}$$

Thus, $A_{k+p} = A_{\max}$. So, $k, k + p \in \{i \in \mathbb{N} \mid A_{\max} = A_i\}$. Therefore, $\mathbf{GCD}(\{i \in \mathbb{N} \mid A_{\max} = A_i\})$ divides $p = k + p - k$. $\square$

Theorem 1. *The minimal period of $\tilde{\eta}$ is $\mathbf{GCD}(\{i \leq m \mid A_{\max} = A_i\})$.*

Proof. If $i, j \in \mathbb{N}$ are periods of $\tilde{\eta}$, then so is $\mathbf{GCD}(i, j)$. Thus, $\mathbf{GCD}(\{i \leq m \mid A_{\max} = A_i\})$ is a period of $\tilde{\eta}$. By Lemma 3, it follows that $\mathbf{GCD}(\{i \leq m \mid A_{\max} = A_i\})$ is the minimal period of $\tilde{\eta}$. $\square$

Theorem 2. *If $k \in \{i \leq m \mid A_i = A_{\max}\}$, then $km(m + 1)/2$ is a preperiod of $\tilde{\eta}$.*

Proof. Let $k \leq m$ such that $A_k = A_{\max}$. Let $q = km(m + 1)/2$. Then $A_q = A_{\max}$ and $S_q = qA_{\max}$. We show that

$$S_{q+nk+r} = nS_k + S_{q+r}$$

for all $n \in \mathbb{N}$ and $0 \leq r < k$. By Proposition 2, there exists

$$(l_i)_{i \leq m} \in \mathbf{UKP}[(S_i)_{i \leq m}; (i)_{i \leq m}; q + nk + r]$$

such that

$$S_{q+nk+r} = \sum_{i=1}^m l_i S_i$$

and

$$\sum_{i=1}^m il_i = q + nk + r.$$

For each $i \leq m$, there exist nonnegative integers l'_i and l''_i with $k|l'_i$ and $l''_i < k$ such that $l_i = l'_i + l''_i$. Then there exists $t \in \mathbb{N}_0$ such that

$$\sum_{i=1}^m il'_i = q + nk - tk$$

and

$$\sum_{i=1}^m il''_i = tk + r.$$

Since $l''_i < k$, we have

$$tk + r = \sum_{i=1}^m il''_i < \sum_{i=1}^m ik = km(m+1)/2 = q.$$

Thus, $tk \leq q$. By Proposition 2,

$$\begin{aligned} S_{q+nk+r} &= \sum_{i=1}^m l_i S_i \\ &= \sum_{i=1}^m l'_i S_i + \sum_{i=1}^m l''_i S_i \\ &\leq S_{q+nk-tk} + \sum_{i=1}^m l''_i S_i \\ &\leq S_{q+nk+r}. \end{aligned}$$

In particular,

$$S_{q+nk+r} = S_{q+nk-tk} + \sum_{i=1}^m l''_i S_i.$$

Meanwhile,

$$S_{q+nk-tk} = (q/k + n - t)S_k$$

because $k \mid q + nk - tk$. Then

$$S_{q+nk+r} = nS_k + (q/k - t)S_k + \sum_{i=1}^m l''_i S_i.$$

Define $(m_i)_{i \leq m}$ where $m_k = l''_k + \frac{q}{k} - t$ and $m_i = l''_i$ for $i \neq k$. Then

$$S_{q+nk+r} = nS_k + \sum_{i=1}^m m_i S_i$$

and

$$q + r = \sum_{i=1}^m im_i.$$

By Proposition 2,

$$\begin{aligned} S_{q+nk+r} &= nS_k + \sum_{i=1}^m m_i S_i \\ &\leq nS_k + S_{q+r} \\ &\leq S_{nk} + S_{q+r} \\ &\leq S_{q+nk+r}. \end{aligned}$$

Thus, $S_{q+nk+r} = nS_k + S_{q+r}$. This implies the periodicity of $(e_i)_{i>q}$. $\square$

The bound in Theorem 2 is generally not optimal. For example, take $\eta = (0, 1, 2, 1, 1)$. Then $(S_1, \dots, S_5) = (0, 1, 3, 4, 5)$ and $(A_1, \dots, A_5) = (0, 1/2, 1, 1, 1)$. Hence, $A_{\max} = 1$, the critical weights are 3, 4, 5, and the minimal period is $p = \mathbf{GCD}(3, 4, 5) = 1$. Indeed,

$$\tilde{\eta} = (0, 1, 2)(1)^\infty.$$

In contrast, Theorem 2, with $k = 3$ and $m = 5$, gives the upper bound $km(m + 1)/2 = 45$. For a fixed finite binomid index, a sharper preperiod can often be obtained by examining the numerical semigroup generated by the indices i of the densities A_i attaining the maximal density $A_{\max}$. The proof of Theorem 2 does not use this additional structure and instead gives a uniform worst-case estimate. For the purpose of establishing the main result Theorem 3, the uniform bound above is sufficient.

4. Atomic Binomid Indices

The set $\mathbb{L}$ of lex-minimal extensions is a monoid under componentwise addition. In this section, we provide a structure result for $\mathbb{L}$ by introducing the notion of atomic binomid indices. In particular, we show that the set of atomic lex-minimal extensions is the unique basis of the monoid $\mathbb{L}$.

We first prove the following result that asserts that if a lex-minimal extension is decomposable into binomid indices as a sum, then the summands, themselves, are lex-minimal extensions.

Proposition 3. *Let η be a lex-minimal extension. Let η_1 and η_2 be binomid indices such that $\eta = \eta_1 + \eta_2$. Then η_1 and η_2 are lex-minimal extensions. Moreover, if η is the lex-minimal extension of $\eta|_k$ for some $k \in \mathbb{N}$, then η_1 and η_2 are lex-minimal extensions of $\eta_1|_k$ and $\eta_2|_k$, respectively.*

Proof. Since η_1 and η_2 are binomid, it follows that $\delta_{\eta_1}(i, j) \geq 0$ and $\delta_{\eta_2}(i, j) \geq 0$ for all $i, j \geq 1$. Suppose η is the lex-minimal extension of $\eta|_k$ for some $k \in \mathbb{N}$. Then $\Delta_\eta(k') = 0$ for all $k' > k$. Moreover, for $k' > k$, there exist $i, j \geq 1$ such that

$\delta_\eta(i, j) = 0$ and $k' = i + j$. By additivity, $\delta_\eta = \delta_{\eta_1} + \delta_{\eta_2}$. Hence, $\delta_{\eta_1}(i, j) = 0$ and $\delta_{\eta_2}(i, j) = 0$. It follows that $\Delta_{\eta_1}(k') = 0$ and $\Delta_{\eta_2}(k') = 0$ for all $k' > k$. In other words, η_1 and η_2 are lex-minimal extensions of $\eta_1|_k$ and $\eta_2|_k$. $\square$

Definition 2. Let η be a binomid index. We call η *atomic* if there are no nonzero binomid indices η_1 and η_2 such that $\eta = \eta_1 + \eta_2$. We denote by **Atom** the set of all nonzero atomic binomid indices.

Let us consider some examples.

Example 1. Let $m \in \mathbb{N}$. Clearly, m^∞ is not atomic as $m^\infty = 1^\infty + \cdots + 1^\infty$ (m summands of 1^∞). Let $E_m := (0^{m-1}, 1)$. It follows that $\widetilde{E_m} = (E_m)^\infty$ since E_m is monotonically increasing ([4, Proposition 3]). Moreover, it is atomic. Indeed, if E_m is a sum of two binomid indices, then one is necessarily zero.

Let $\eta = (0, 1, 1)$. Then $\tilde{\eta} = (\eta)^\infty$. If $\tilde{\eta}$ is not atomic, then we decompose η as $\eta = (0, 0, 1) + (0, 1, 0)$ necessarily. Now, $(0, 0, 1) = (0, 0, 1)^\infty$ and $(0, 1, 0) = (0, 1, 0)^\infty$. So, $(0, 0, 1) + (0, 1, 0) = (0, 1, 1, 1, 0, 2)^\infty \neq \tilde{\eta}$. Hence, $\tilde{\eta}$ is atomic. Likewise, $(0^m, 1, 1)^\infty$ is an atomic lex-minimal extension for all $m \in \mathbb{N}$. Thus,

$$(0^{m-1}, 1)^\infty, (0^m, 1, 1)^\infty \in \mathbf{Atom} \cap \mathbb{L}.$$

Example 2. Let $\eta_1 = (0, 1, 1)$, $\eta_2 = (0, 1, 2, 1, 1)$, $\eta_3 = (0, 2, 1)$ and $\eta_4 = (0, 0, 2, 0, 1)$. Then

$$\begin{aligned} \tilde{\eta}_1 &= (0, 1, 1)^\infty & \tilde{\eta}_3 &= (0, 2)(1, 1)^\infty \\ \tilde{\eta}_2 &= (0, 1, 2)(1)^\infty & \tilde{\eta}_4 &= (0, 0, 2)(0, 1, 1)^\infty \end{aligned}$$

and each of these is atomic. Moreover, $\tilde{\eta}_1 + \tilde{\eta}_2 = \tilde{\eta}_3 + \tilde{\eta}_4$. Thus, the decomposition of a lex-minimal extension into a sum of atomic sequences is not unique.

To proceed, we first define two maps on the set of all finite binomid indices of a given length.

Definition 3. For $k \in \mathbb{N}$, let $\mathbb{L}_k$ be the set of all finite binomid indices of length k . Define the map $L_k : \mathbb{L}_k \rightarrow \mathbb{L}$ by $L_k(\eta) = \tilde{\eta}$. For $j, k \in \mathbb{N}$ with $j \leq k$, define the map $L_{j,k} : \mathbb{L}_j \rightarrow \mathbb{L}_k$ by

$$L_{j,k}(\eta) = L_j(\eta)|_k = \tilde{\eta}|_k.$$

Remark 2. As illustrated by Example 1, $L_{j,k}$ is not usually a homomorphism.

Given a subset A of a monoid M with identity element 0, we denote by $\langle A \rangle$ the submonoid of M given by the span

$$\langle A \rangle := \{\mu_1 + \mu_2 + \cdots + \mu_n \mid \mu_i \in A, n \in \mathbb{N}\} \cup \{0\}.$$

The subset A is called a *basis* of M if $M = \langle A \rangle$ and for all finite sequences $(\beta_i)_{i \leq n}$ over A , with the β_i 's pairwise distinct, and $(m_i)_{i \leq n}$ over $\mathbb{N}_0$, if $\beta_j = \sum_{i=1}^n m_i \beta_i$ for some $1 \leq j \leq n$, then $m_j \geq 1$. Example 2 justifies this definition of the basis.

We adapt the definition of atomic binomid index in Definition 2 to present the notion of an atomic element of a general monoid M , which is employed in the following useful result.

Proposition 4. *Let M be a submonoid of $(\mathbb{N}_0)^m$ for some $m \in \mathbb{N}$ or $m = \mathbb{N}$. Let A be the set of atomic elements of M . If every nonzero element of M can be written as a finite sum of elements of A , then A is the unique basis of M .*

Proof. We first demonstrate that A is a basis of M . By assumption, A spans M . No element of A can be decomposed into a nontrivial linear combination of elements in A with nonnegative integer coefficients. Hence, A constitutes a basis for M .

Suppose M has another basis B distinct from A . If there exists $\nu \in A \setminus B$, then ν must be expressible as a nontrivial linear combination of elements in B . This, however, contradicts the atomic nature of ν . Hence, $A \subseteq B$. If there exists $\omega \in B \setminus A$, then ω can be written as $\omega = \sum_{i=1}^n m_i \alpha_i$ where $\omega \neq \alpha_i \in A \subseteq B$ and $m_i \in \mathbb{N}_0$. Let $(\beta_i)_{i \leq n+1}$ be a sequence in B such that $\beta_1 = \omega$ and $\beta_{i+1} = \alpha_i$ for all $i \leq n$. Then $\beta_1 = 0 \cdot \beta_1 + \sum_{i=1}^n m_i \beta_{i+1}$. This contradicts the fact that B is a basis. So, $A = B$. $\square$

Proposition 5. *The set $\mathbf{Atom} \cap \mathbb{L}$ is the unique basis of $\mathbb{L}$. Moreover, $\mathbf{Atom} \cap L_k(\mathbb{I}_k)$ is the unique basis of $\langle L_k(\mathbb{I}_k) \rangle$ for $k \in \mathbb{N}$.*

Proof. Let η be an element of $\mathbb{L}$ such that $\eta \in L_k(\mathbb{I}_k)$ for some $k \in \mathbb{N}$. If $\eta = \sum_i \eta_i$ where $\eta_i \in \mathbb{L}$ for all i , then $\eta_i \in L_k(\mathbb{I}_k)$ by Proposition 3. Since the elements of $\mathbb{L}$ are sequences of nonnegative integers, the number of summands in the decomposition $\eta = \sum_i \eta_i$ is less than or equal to $S_k(\eta)$ and hence, the number of summands is finite. If we take the decomposition $\eta = \sum_i \eta_i$ with the maximal number of summands, then we see that all summands η_i are in $\mathbf{Atom} \cap L_k(\mathbb{I}_k) \subseteq \mathbf{Atom} \cap \mathbb{L}$. Thus, the first statement follows from Proposition 4.

To show the second statement, we prove that $\mathbf{Atom} \cap L_k(\mathbb{I}_k)$ spans $\langle L_k(\mathbb{I}_k) \rangle$. Indeed, let $\eta \in \langle L_k(\mathbb{I}_k) \rangle$. Then η can be decomposed as a sum $\sum_i \eta_i$ where η_i are elements of $L_k(\mathbb{I}_k)$. By the above argument, each η_i can be written as a finite sum of elements of $\mathbf{Atom} \cap L_k(\mathbb{I}_k)$. This implies $\langle L_k(\mathbb{I}_k) \rangle$ satisfies the assumption of Proposition 4. Thus, $\mathbf{Atom} \cap L_k(\mathbb{I}_k)$ is the unique basis of $\langle L_k(\mathbb{I}_k) \rangle$. $\square$

In the latter part of this section, we show that $\mathbf{Atom} \cap L_k(\mathbb{I}_k)$ is finite. First, we mention the following remark.

Remark 3. For $k \in \mathbb{N}$, the space $\mathbb{I}_k$ is a polyhedral monoid defined by the conditions $\{S_{i+j} - S_i - S_j \geq 0 \mid i+j \leq k\}$. Thus, $\mathbb{I}_k$ has a finite basis, which may be computed by using Integer Programming (see Bachem's method in [2]).

Example 3. The set $B = \{(1, 1, 1), (0, 1, 0), (0, 0, 1)\}$ is a basis of $\mathbb{I}_3$. However, $L_3(B) = \{(1)^\infty, (0, 1)^\infty, (0, 0, 1)^\infty\}$ does not span $L_3(\mathbb{I}_3)$. Indeed, the lex-minimal

extension $\widetilde{(0, 1, 1)} = (0, 1, 1)^\infty$ cannot be written as a sum $x(1)^\infty + y(0, 1)^\infty + z(0, 0, 1)^\infty$, where $x, y, z \in \mathbb{N}_0$.

From the above example, we see that $L_k(B)$ does not necessarily span $L_k(\mathbb{I}_k)$ when B is a basis of $\mathbb{I}_k$.

In what follows, we show that the monoid $\langle L_k(\mathbb{I}_k) \rangle$ is finitely presented. For $k \in \mathbb{N}$, set $(k, l(k)] := \{i \in \mathbb{N} \mid k < i \leq l(k)\}$. Let **Path** be the set of all maps

$$\pi : (k, l(k)] \longrightarrow \{i \in \mathbb{N} \mid i < l(k)\}$$

such that $\pi(i) < i$ for all $i \in (k, l(k)]$. By Proposition 2, we have the following lemma.

Lemma 4. *If $\eta \in L_{k, l(k)}(\mathbb{I}_k)$, then there is a map $\pi \in \mathbf{Path}$ such that*

$$S_i - S_{\pi(i)} - S_{i-\pi(i)} = 0$$

for all $i \in (k, l(k)]$.

Lemma 5. *The monoid $\langle L_k(\mathbb{I}_k) \rangle$ is finitely generated for all $k \in \mathbb{N}$. In particular, its basis $\mathbf{Atom} \cap L_k(\mathbb{I}_k)$ is finite.*

Proof. For $k \in \mathbb{N}$, define $l(k) := k^3 + \mathbf{LCM}(\{i \in \mathbb{N} \mid i \leq k\})$. Consider the restriction map

$$\text{Res}_{l(k)} : \langle L_k(\mathbb{I}_k) \rangle \longrightarrow \mathbb{I}_{l(k)}$$

given by

$$\text{Res}_{l(k)}(\eta) = \eta|_{l(k)}.$$

We show that $\text{Res}_{l(k)}$ is an injective homomorphism. By Theorems 1 and 2, every element of $L_k(\mathbb{I}_k)$ has a common preperiod k^3 and common period $\mathbf{LCM}(\{i \in \mathbb{N} \mid i \leq k\})$. Consequently, every element of $\langle L_k(\mathbb{I}_k) \rangle$ has the same common preperiod and period. Therefore, each element of $\langle L_k(\mathbb{I}_k) \rangle$ is uniquely determined by its first $l(k) = k^3 + \mathbf{LCM}(\{i \in \mathbb{N} \mid i \leq k\})$ entries. Thus, if $\text{Res}_{l(k)}(\eta) = \text{Res}_{l(k)}(\gamma)$ for $\eta, \gamma \in \langle L_k(\mathbb{I}_k) \rangle$, then $\eta = \gamma$. In other words, $\text{Res}_{l(k)}$ is injective. Since the restriction map commutes with componentwise addition, it is a monoid homomorphism. Thus, we have that $\text{Res}_{l(k)}(\langle L_k(\mathbb{I}_k) \rangle) \cong \langle L_k(\mathbb{I}_k) \rangle$ as monoids.

Let $\eta \in L_{k, l(k)}(\mathbb{I}_k) = \text{Res}_{l(k)}(L_k(\mathbb{I}_k))$. By Lemma 4, there exists a map $\pi \in \mathbf{Path}$ such that, for all $i \in (k, l(k)]$,

$$S_i - S_{\pi(i)} - S_{i-\pi(i)} = 0.$$

Moreover, if $\eta \in \mathbb{I}_{l(k)}$ satisfies the equation $S_i - S_{\pi(i)} - S_{i-\pi(i)} = 0$ for all $i \in (k, l(k)]$ for some $\pi \in \mathbf{Path}$, then there exists $\mu \in \mathbb{I}_k$ such that

$$\eta = L_{k, l(k)}(\mu).$$

Thus, for $\pi \in \mathbf{Path}$, the set

$$\text{Poly}(\pi) := \{\eta \in \mathbb{I}_{l(k)} \mid S_i - S_{\pi(i)} - S_{i-\pi(i)} = 0 \text{ for all } i \in (k, l(k)]\},$$

which is clearly closed under addition, is a submonoid of $\text{Res}_{l(k)}(\langle L_k(\mathbb{I}_k) \rangle)$. Since the condition $S_i - S_{\pi(i)} - S_{i-\pi(i)} = 0$ is equivalent to the following system of inequalities

$$\begin{cases} S_i - S_{\pi(i)} - S_{i-\pi(i)} \geq 0 \\ -(S_i - S_{\pi(i)} - S_{i-\pi(i)}) \geq 0, \end{cases}$$

the set $\text{Poly}(\pi)$ is a polyhedral submonoid of $\mathbb{I}_{l(k)} \subset (\mathbb{N}_0)^{l(k)}$. By a generalized Minkowski's theorem of Hilbert (see [10, 2, 12]), $\text{Poly}(\pi)$ is finitely generated. Likewise, $\sum_{\pi \in \mathbf{Path}} \text{Poly}(\pi)$ is finitely generated because $\mathbf{Path}$ is finite. Recall that $\text{Res}_{l(k)}(L_k(\mathbb{I}_k)) \subset \bigcup_{\pi \in \mathbf{Path}} \text{Poly}(\pi)$. Hence, we have

$$\sum_{\pi \in \mathbf{Path}} \text{Poly}(\pi) = \text{Res}_{l(k)}(\langle L_k(\mathbb{I}_k) \rangle).$$

It follows that $\text{Res}_{l(k)}(\langle L_k(\mathbb{I}_k) \rangle)$ is finitely generated. Thus, $\langle L_k(\mathbb{I}_k) \rangle$ is finitely generated. $\square$

Rédei [13] proved that when a commutative monoid is finitely generated, it is also finitely presented. Hence, we have the following result.

Lemma 6. *The monoid $\langle L_k(\mathbb{I}_k) \rangle$ of $\mathbb{L}$ is finitely presented for all $k \in \mathbb{N}$.*

Finally, we have the following result.

Theorem 3. *The set $\mathbb{L}$ of lex-minimal extensions is an inductive limit of the increasing system $(\langle L_k(\mathbb{I}_k) \rangle)_{k \in \mathbb{N}}$ of finitely presented monoids. Moreover, $(\mathbf{Atom} \cap L_k(\mathbb{I}_k))_{k \in \mathbb{N}}$ is an increasing system of generators of $(\langle L_k(\mathbb{I}_k) \rangle)_{k \in \mathbb{N}}$.*

Proof. Let

$$M_k := \langle L_k(\mathbb{I}_k) \rangle.$$

Since every element of $L_k(\mathbb{I}_k)$ is also the lex-minimal extension of its prefix of length $k+1$, we have

$$L_k(\mathbb{I}_k) \subseteq L_{k+1}(\mathbb{I}_{k+1}),$$

and consequently, $M_k \subseteq M_{k+1}$. Moreover, every element of $\mathbb{L}$ is the lex-minimal extension of some finite binomid index. Hence,

$$\mathbb{L} = \bigcup_{k \geq 1} M_k.$$

Thus, $\mathbb{L}$ is the inductive limit of the increasing system $(M_k)_{k \geq 1}$. By Lemma 6, each M_k is finitely presented. Finally,

$$A_k := \mathbf{Atom} \cap L_k(\mathbb{I}_k)$$

is the basis of M_k . Since $L_k(\mathbb{I}_k) \subseteq L_{k+1}(\mathbb{I}_{k+1})$, we have $A_k \subseteq A_{k+1}$. Therefore, $(A_k)_{k \geq 1}$ is an increasing system of generators for $(M_k)_{k \geq 1}$. $\square$

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