



ABOUT SOME NESTED SUMS OF RECIPROCAL SQUARED NUMBERS

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Abstract

Closed forms are obtained for two families of nested sums built from reciprocal squares: one over all positive integers, the other over odd positive integers only, together with their strictly ordered duals. The method interprets the finite truncations as complete homogeneous symmetric polynomials, expressed via the cycle index of the symmetric group. A combinatorial corollary of the Faà di Bruno formula then yields explicit evaluations in terms of Bernoulli numbers, Euler numbers, and powers of π .

1. Introduction

In this paper, we are interested in computing the values of nested multiple sums of the form

$$R(k) = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \cdots \sum_{i_k=1}^{i_{k-1}} x_{i_1} x_{i_2} \cdots x_{i_k} = \sum_{1 \leq i_k \leq i_{k-1} \leq \dots \leq i_1} x_{i_1} x_{i_2} \cdots x_{i_k}$$

and in the "dual" nested sums

$$A(k) = \sum_{i_1=1}^{\infty} \sum_{i_2=i_1+1}^{\infty} \cdots \sum_{i_k=i_{k-1}+1}^{\infty} x_{i_1} x_{i_2} \cdots x_{i_k} = \sum_{1 \leq i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

The sums $R(k)$ and $A(k)$ are formally related by an inclusion-exclusion relation. Indeed, with the convention $A(0) = R(0) = 1$ and with an obvious meaning for the

coefficients $\sigma_i^{(j)}$, we have

$$\begin{aligned} A(k) &= \sum_{i_1=1}^{\infty} x_{i_1} \sigma_{i_1}^{(k)} = \sum_{i_1=1}^{\infty} x_{i_1} \left(A(k-1) - \sum_{i_2=0}^{i_1} x_{i_2} \sigma_{i_2}^{(k-1)} \right) \\ &= R(1)A_1(k-1) - \left(\sum_{i_1=1}^{\infty} x_{i_1} \sum_{i_2=1}^{i_1} x_{i_2} \sigma_{i_2}^{(k-1)} \right) \\ &= \dots = \sum_{l=1}^k (-1)^{l+1} R(l)A(k-l) \end{aligned}$$

and thus

$$\sum_{\ell=0}^k (-1)^{\ell+1} R(\ell)A(k-\ell) = 0. \quad (1)$$

This expresses the fundamental relationship between the elementary symmetric polynomials and the complete homogeneous ones when the sums are truncated to n . This paper focuses on the special cases in which the weights are squared reciprocals

$$x_i = \frac{1}{(2i-1)^2} \quad \text{and} \quad x_i = \frac{1}{i^2}.$$

These nested sums arise naturally in combinatorics, number theory, and the theory of multiple zeta values. They are related to multiple zeta values [3] and Euler sums; specific instances appear in Hoffman's study of multiple harmonic series [4].

1.1. Preliminaries

Let $\mathcal{P}(k)$ denote the set of all partitions of the integer k . For $\boldsymbol{\pi} \in \mathcal{P}(k)$ we write $\boldsymbol{\pi} = [1^{\pi_1}, \dots, k^{\pi_k}]$ where π_l denotes the multiplicity of the part l in the partition of k . With this convention, we have trivially, for any $\boldsymbol{\pi} \in \mathcal{P}(k)$, $k = \sum_{l=1}^k l\pi_l$ and we define the length of the partition as $l(\boldsymbol{\pi}) = \sum_{l=1}^k \pi_l$.

The first tool of our analysis is the recognition that for finite n , the sum $R(k)$, truncated at n , equals the complete elementary symmetric polynomial $h_k(x_1, \dots, x_n)$. It is a classical fact, going back to Pólya's work on the cycle index polynomial [6], that h_k can be expressed as a polynomial in the power sums $p_l = \sum_{j=1}^n x_j^l$ using the cycle index polynomial $Z(S_k)$ of the symmetric group S_k . A more direct formulation of this identity can be found in Clarke [2]. Concretely,

$$h_k(x_1, \dots, x_n) = \sum_{\boldsymbol{\pi} \in \mathcal{P}(k)} a(\boldsymbol{\pi}) \prod_{l=1}^k \left(\sum_{j=1}^n x_j^l \right)^{\pi_l}, \quad (2)$$

where $a(\boldsymbol{\pi})$ has the explicit expression

$$a(\boldsymbol{\pi}) = \prod_{l=1}^k \frac{1}{\pi_l! l^{\pi_l}}, \quad \boldsymbol{\pi} \in \mathcal{P}(k).$$

Substituting our specific weights and taking the limit $n \rightarrow \infty$ yields explicit formulas for infinite nested sums in terms of products of zeta or beta values. The same symmetric function machinery, combined with the convolution identity (1), then provides closed-form evaluations for the dual sums $A(k)$.

1.2. Main Results

We now introduce our main results. For $k \geq 1$, we define the nested series

$$R_1(k) \stackrel{\text{def}}{=} \sum_{i_1=1}^{\infty} \frac{1}{(2i_1-1)^2} \sum_{i_2=1}^{i_1} \frac{1}{(2i_2-1)^2} \cdots \sum_{i_k=1}^{i_{k-1}} \frac{1}{(2i_k-1)^2},$$

$$R_0(k) \stackrel{\text{def}}{=} \sum_{i_1=1}^{\infty} \frac{1}{i_1^2} \sum_{i_2=1}^{i_1} \frac{1}{i_2^2} \cdots \sum_{i_k=1}^{i_{k-1}} \frac{1}{i_k^2},$$

and by $A_0(k)$ and $A_1(k)$ the dual nested series

$$A_1(k) \stackrel{\text{def}}{=} \sum_{i_1=1}^{\infty} \frac{1}{(2i_1-1)^2} \sum_{i_2=i_1+1}^{\infty} \frac{1}{(2i_2-1)^2} \cdots \sum_{i_k=i_{k-1}+1}^{\infty} \frac{1}{(2i_k-1)^2},$$

$$A_0(k) \stackrel{\text{def}}{=} \sum_{i_1=1}^{\infty} \frac{1}{i_1^2} \sum_{i_2=i_1+1}^{\infty} \frac{1}{i_2^2} \cdots \sum_{i_k=i_{k-1}+1}^{\infty} \frac{1}{i_k^2}.$$

We remark that $R_0(k) = \zeta^*(\{2\}^k)$ and $A_0(k) = \zeta(\{2\}^k)$ (multiple zeta values with all twos) in the theory of multiple zeta values. We have the following result.

Theorem 1. *For any $k \geq 0$, we have*

$$R_1(k) = \left(\frac{\pi}{2}\right)^{2k} \frac{E_{2k}^*}{(2k)!} = \frac{\pi}{4} \beta(2k+1),$$

with E_{2k}^* denoting the Euler (zig) numbers¹ and $\beta(2k+1)$ the Dirichlet beta-function.

We have a similar theorem for the series $R_0(k)$.

Theorem 2. *For any $k \geq 0$, we have*

$$R_0(k) = 2\pi^{2k} \frac{|(2^{2k-1}-1)B_{2k}|}{(2k)!} = 2\eta(2k),$$

with B_{2k} denoting the Bernoulli numbers and $\eta(2k)$ the Dirichlet eta function.

¹<https://oeis.org/A000364> (OEIS A000364), even powers only of $\sec(x) = 1/\cos(x)$

2. Proofs

To prove the theorems, we employ a corollary of Vella ([8]), which is based on the Faà di Bruno formula (1855). We believe that this usage is new. For completeness, we reproduce the corollary here.

Corollary 1 (Corollary 4 from [8]). *Let $T_n(f; a)$ denote the n^{th} Taylor coefficient of the function $f(x)$ expanded about $x = a$, so $T_n(f; a) = \frac{f^{(n)}(a)}{n!}$. If both f and g have n^{th} derivatives, then:*

$$T_n(f \circ g; a) = \sum_{\pi \in \mathcal{P}(n)} \binom{l(\pi)}{\pi_1, \dots, \pi_n} \cdot T_{l(\pi)}(f; g(a)) \cdot \prod_{i=1}^n [T_i(g; a)]^{\pi_i},$$

where $\binom{n}{\alpha_1, \dots, \alpha_n}$ denotes the multinomial coefficient.

Proof of Theorem 1. Observe first that for any $l \geq 1$, we have

$$\sum_{i=1}^{\infty} \frac{1}{(2i-1)^{2l}} = \sum_{i=1}^{\infty} \frac{1}{i^{2l}} - \frac{1}{2^{2l}} \sum_{i=1}^{\infty} \frac{1}{i^{2l}} = \frac{2^{2l}-1}{2^{2l}} \zeta(2l) = (2^{2l}-1) \frac{|B_{2l}| \pi^{2l}}{2(2l)!}. \quad (3)$$

For $n \geq 1$, let $x_n = 1/(2n-1)^2$. Using Equation (2) and Equation (3), we get

$$\begin{aligned} R_1(k) &= \sum_{\pi \in \mathcal{P}(k)} a(\pi) \prod_{l=1}^k \left[(2^{2l}-1) \frac{|B_{2l}| \pi^{2l}}{2(2l)!} \right]^{\pi_l} \\ &= \pi^{2k} \sum_{\pi \in \mathcal{P}(k)} \prod_{l=1}^k \frac{1}{\pi_l!} \left[(2^{2l}-1) \frac{|B_{2l}|}{2l(2l)!} \right]^{\pi_l} \\ &= \pi^{2k} \sum_{\pi \in \mathcal{P}(k)} \binom{l(\pi)}{\pi_1, \dots, \pi_k} \frac{1}{l(\pi)!} \prod_{l=1}^k \left[(2^{2l}-1) \frac{|B_{2l}|}{2l(2l)!} \right]^{\pi_l}. \end{aligned}$$

Using the generating function of the Bernoulli numbers, we find that

$$\frac{z}{2} \cot\left(\frac{z}{2}\right) - z \cot(z) = \sum_{n=1}^{\infty} (2^{2n}-1) |B_{2n}| \frac{z^{2n}}{(2n)!}.$$

Consider the entire series

$$g(z) = \sum_{n=1}^{\infty} (2^{2n}-1) |B_{2n}| \frac{z^{2n}}{(2n)(2n)!}.$$

Differentiating with respect to z , we get

$$zg'(z) = \frac{z}{2} \cot\left(\frac{z}{2}\right) - z \cot(z),$$

and thus

$$g(z) = \log\left(\frac{\sin(z/2)}{\sin(z)}\right) + \log(2) = \log\left(\frac{1}{\cos(z/2)}\right), \text{ for } 0 < z < \pi,$$

is the generating function of the sequence $(2^{2n} - 1) \frac{|B_{2n}|}{(2n)(2n)!}$. We apply Corollary 1 with g given above, $f(z) = \exp(z)$, and $a = 0$. Recognizing that $1/\cos(z/2) = \sec(z/2)$ is the generating function of $E_{2k}^*/2^k$, $k \geq 1$, we arrive at the conclusion by observing that

$$R_1(k) = \pi^{2k} T_k(\sec(z/2); 0).$$

□

Proof of Theorem 2. The arguments are very similar to those in the proof of Theorem 1, so we give only a sketch of the proof. Using the same methodology, we arrive at the expression

$$R_0(k) = \sum_{\pi \in \mathcal{P}(k)} a(\pi) \prod_{l=1}^k \zeta(2l)^{\pi_l}.$$

Simplifying, we get

$$\begin{aligned} R_0(k) &= (2\pi)^{2k} \sum_{\pi \in \mathcal{P}(k)} \prod_{l=1}^k \frac{1}{\pi_l!} \left(\frac{|B_{2l}|}{2l(2l)!} \right)^{\pi_l} \\ &= (2\pi)^{2k} \sum_{\pi \in \mathcal{P}(k)} \binom{l(\pi)}{\pi_1, \dots, \pi_k} \frac{1}{l(\pi)!} \prod_{l=1}^k \left[\frac{|B_{2l}|}{2l(2l)!} \right]^{\pi_l}. \end{aligned}$$

Next, we find that

$$g(z) = \log\left(\frac{z}{2\sin(z/2)}\right), \quad 0 < z < \pi,$$

is the generating function of the sequence $\frac{|B_{2l}|}{(2l)!2^{2l}}$, $l \geq 1$. Taking $f(z) = \exp(z)$, and $a = 0$ again (as $2\sin(z/2)/z$ can be extended by continuity at 0), we get the conclusion. □

3. Some Consequences

From the previous results, we obtain the following corollary.

Corollary 2. *For any $k > 0$, the following holds*

$$A_1(k) = \left(\frac{\pi}{2}\right)^{2k} \frac{1}{(2k)!}.$$

Proof. The result is proved by induction using equation (1). For $k = 0$, the identity holds by the convention $A_1(0) = 1$. For $k = 1$, using Equation (3) and Theorem 1, we have $A_1(1) = R_1(1) = \pi^2/8$, which matches the claimed formula.

Assume now that the formula for $A_1(l)$ holds for $0 \leq l \leq k - 1$. Substituting the induction hypothesis into the Equation (1), we obtain

$$\sum_{l=0}^k (-1)^{l+1} R_1(l) A_1(k-l) = \left(\frac{\pi}{2}\right)^{2k} \frac{1}{(2k)!} \sum_{l=0}^k (-1)^{l+1} \binom{2k}{2l} E_{2l}^*. \quad (4)$$

Thus, the formula holds for $l = k$ if and only if the right-hand side of Equation 4 vanishes. This identity is established in [1, Equation 3.27]². Hence the induction is complete. $\square$

The evaluation of $A_0(k) = \zeta(\{2\}^k) = \frac{\pi^{2k}}{(2k+1)!}$ is already proven in [3, Corollary 2.3] and can also be found in [5]. This shows that we have (thanks again to equation (1))

$$\frac{2\pi^{2k}}{(2k+1)!} \sum_{l=0}^k \binom{2k+1}{2l} (2^{2l-1} - 1) B_{2l} = 0.$$

This identity can be verified directly using the properties of the Bernoulli numbers (see [7], pages 108 and 123).

4. Conclusion

We have shown that the nested sums $R_0(k)$ and $R_1(k)$, built from reciprocal squares over all integers or over odd integers only, admit elegant closed forms involving Bernoulli numbers, Euler numbers, and powers of π . The main ingredient was the interpretation of the finite- n versions as complete homogeneous symmetric polynomials h_k , whose expression through the cycle index polynomial of S_k transforms the multiple sum into a combination of ordinary power sums.

The method employed here, which combines symmetric group combinatorics, generating functions, and classical special-function identities, is versatile and can be extended to other weight sequences x_i (higher powers $1/i^{2m}$ or $1/(2k-1)^{2m}$, for example).

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²See also https://proofwiki.org/wiki/Sum_of_Euler_Numbers_by_Binomial_Coefficients_Vanishes (Sum of Euler Numbers by Binomial Coefficients Vanishes).

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