



DIVISORS OF AN INTEGER IN A SHORT INTERVAL

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Abstract

Let $\mathcal{D}_n \subset \mathbb{N}$ denote the set of the $\tau(n)$ divisors of n . We study the function

$$D_n(X, Y) := |\{d \in \mathcal{D}_n : X \leq d \leq X + Y\}|$$

for $Y \leq X$. We establish a general upper bound for $D_n(n^\theta, n^\eta)$ when $0 < \eta < \theta < 1$. We also show that there exist infinitely many integers n having several divisors in intervals of the form $[n^\theta, n^\theta + n^{\theta-\epsilon}]$, for any fixed $0 < \theta < 1$ and $0 < \epsilon < \theta$.

– *Dedicated to Carl Pomerance on the occasion of his 80th birthday.*

1. Introduction and Notation

Let $\mathcal{D}_n \subset \mathbb{N}$ denote the set of the $\tau(n)$ divisors of n . For each integer $n \geq 1$, we consider the function

$$D_n(X, Y) := |\{d \in \mathcal{D}_n : X \leq d \leq X + Y\}|.$$

We first observe that $D_n(X, Y) = D_n(\frac{n}{X+Y}, \frac{Yn}{X(X+Y)})$. Some nontrivial estimates are known for $D_n(X, Y)$. For example, Lemma 6 from [8] implies that

$$D_n(X, X) \ll \frac{\tau(n)}{\sqrt{\Omega_2(n)}} \quad (n \geq 2),$$

where $\Omega_2(n) := \sum_{p^\beta \parallel n} \beta^2$. In general, this inequality is known to be best possible, but the simpler inequality

$$\max_{1 \leq X \leq \frac{n}{2}} D_n(X, X) \gg 1 + \frac{\tau(n)}{\log n} \quad (n \geq 2)$$

always holds. The following conjecture is suggested in the literature; see [5], [1], and [2].

Conjecture 1. Let $\epsilon > 0$ be fixed. There exists a constant k_ϵ such that, for each integer $n \geq 1$, we have

$$D_n(n^{1/2}, n^{1/2-\epsilon}) \leq k_\epsilon.$$

In this paper, we address the possibility that a strong general upper bound holds when $Y \ll X^{1-\epsilon}$ for each fixed $\epsilon > 0$. Precisely, we propose the following conjecture.

Conjecture 2. Let $0 < \theta < 1$ and $0 < \epsilon < \theta$ be fixed. There exists a constant $k_\epsilon(\theta)$ such that, for each integer $n \geq 1$, we have

$$D_n(n^\theta, n^{\theta-\epsilon}) \leq k_\epsilon(\theta).$$

Throughout the paper, we use the functions

$$\omega(n) := \sum_{p|n} 1 \quad \text{and} \quad V(n) := \max_{p^\beta \| n} \beta.$$

2. Main Results

The first theorem encapsulates our best result that holds in full generality when $Y \leq X^{1-\epsilon}$.

Theorem 1. Let η and θ be fixed real numbers such that $0 < \eta < \theta < 1$. For each integer $n \geq 2$, we have

$$D_n(n^\theta, n^\eta) \ll \tau(n)^{1-\xi(\theta, \eta)} \frac{V(n) \log \tau(n)}{\theta(1-\theta)}, \quad (1)$$

where

$$\xi(\theta, \eta) := \begin{cases} 1 & \text{if } \eta \leq \theta^2; \\ \frac{\theta^2}{\eta} & \text{if } \eta > \theta^2, \theta \leq \frac{1}{2}, \text{ and } 2\eta \leq \theta; \\ 4(\theta - \eta) & \text{if } \eta > \theta^2, \theta \leq \frac{1}{2}, \text{ and } 2\eta > \theta; \\ \frac{(1-\theta)^2}{(1-\theta)^2 + \eta - \theta^2} & \text{if } \eta > \theta^2, \theta > \frac{1}{2}, \text{ and } 2\eta \leq 3\theta - 1; \\ 4(\theta - \eta) & \text{if } \eta > \theta^2, \theta > \frac{1}{2}, \text{ and } 2\eta > 3\theta - 1. \end{cases} \quad (2)$$

A relaxed version of Conjecture 2 would ask for a larger region where $\xi(\theta, \eta) = 1$; see Proposition 1 below.

Observe that if p is a prime divisor of n and d is any divisor of n , then for every $X \geq 1$ there is at most one $\beta \in \mathbb{Z}$ such that $X \leq p^\beta d < 2X$. It follows that, for all $0 < \eta < \theta < 1$,

$$D_n(n^\theta, n^\eta) \leq \frac{\tau(n)}{V(n) + 1} \quad (n \geq 1).$$

Comparing this estimate with Inequality (1), we obtain the following saving in the exponent of $\tau(n)$.

Corollary 1. *Let η and θ be fixed real numbers such that $0 < \eta < \theta < 1$. For each integer $n \geq 2$, we have*

$$D_n(n^\theta, n^\eta) \ll \tau(n)^{1 - \frac{\xi(\theta, \eta)}{2}} \sqrt{\frac{\log \tau(n)}{\theta(1 - \theta)}}.$$

In the opposite direction, the second theorem is inspired by a simple idea that can be found on the first page of [4].

Theorem 2. *Let $0 < \theta < 1$ and $0 < \epsilon < \theta$ be fixed real numbers. If Conjecture 2 holds, then we have*

$$k_\epsilon(\theta) \gg \sqrt{\epsilon}(\theta(1 - \theta))^{3/2} \left(\frac{1}{\theta^\theta(1 - \theta)^{1 - \theta}} \right)^{\frac{1}{\epsilon}}.$$

3. Preliminary Lemmas

The proof of the following lemma is given in Corollary 1.4 of [3]; see also [7].

Lemma 1. *Let $d_1, \dots, d_k$ be positive integers. Then, for all $t \in \mathbb{Z}$, we have*

$$[d_1, \dots, d_k]^{\frac{t(t+1)}{2}} \prod_{1 \leq i < j \leq k} (d_i, d_j) \geq \prod_{1 \leq i \leq k} d_i^t,$$

where $[d_1, \dots, d_k]$ denotes the least common multiple of $d_1, \dots, d_k$ and where (d_i, d_j) denotes the greatest common divisor of d_i and d_j .

Lemma 2 ([8], Lemma 7). *Let (x_i, y_i) for $i = 1, \dots, k$ be pairs of positive real numbers that satisfy $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$. There exists a permutation $\sigma = (\sigma_1, \dots, \sigma_k)$ such that for each $1 \leq s \leq k$,*

$$\sum_{i=1}^s x_{\sigma_i} \leq \sum_{i=1}^s y_{\sigma_i}.$$

Lemma 3. *Suppose $\mathcal{M}$ is a set of integers contained in an interval with $N \geq 1$ integers, and assume that the elements of $\mathcal{M}$ avoid a set $\Omega_{\mathfrak{p}} \neq \mathbb{Z}/\mathfrak{p}\mathbb{Z}$ of congruence classes for every $\mathfrak{p} := p^{\beta_p} \in \mathcal{P}$, a set of powers of distinct primes. Then we have*

$$|\mathcal{M}| \leq \frac{N + Q^2}{H}$$

for any $Q \geq 1$, where

$$H = \sum_{q \leq Q} h(q)$$

and where $h(q)$ is the multiplicative function defined for $\beta \geq 1$ as

$$h(p^\beta) := \begin{cases} \frac{|\Omega_{\mathfrak{p}}|}{\mathfrak{p} - |\Omega_{\mathfrak{p}}|} & \text{if } p^\beta = \mathfrak{p} \in \mathcal{P}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is essentially the last part of Theorem 7.14 from [6], with a slight generalization that allows us to consider powers of primes instead of only primes. The proof of this generalization is included in Lemma 7.15, assuming that $q = \mathfrak{p}_1 \cdots \mathfrak{p}_k$, and that the $\star$ is now interpreted as $\mathfrak{p}_i \nmid a$ for $i = 1, \dots, k$. One can sum inequality (7.39) over $q \leq Q$ and use Theorem 7.7 to conclude. $\square$

In the next lemma, we use the Legendre symbol $\left(\frac{\cdot}{p}\right)$ for $p \geq 3$.

Lemma 4. *Let $p \geq 3$ be a prime number and $u \in \mathbb{Z}$ an integer not divisible by p . Let S_v be the number of x modulo p^{v+1} for which the equation*

$$x^2 + p^v u \equiv y^2 \pmod{p^{v+1}}$$

is unsolvable. Then $S_0 = \frac{p - \left(\frac{-u}{p}\right)}{2}$, $S_1 = p$, and $S_v = pS_{v-2}$ for $v \geq 2$.

Proof. For $v = 0$, we consider the function

$$g(k) := \sum_{x=1}^p \left(\frac{x^2 + k}{p} \right).$$

From here, using changes of variable of the form $x = cz$ with $c \not\equiv 0 \pmod{p}$, we deduce that $g(k)$ only takes two values when $p \nmid k$ depending on the value of $\left(\frac{k}{p}\right)$. Thus, by summing the functions $g(k)$ and $\left(\frac{k}{p}\right)g(k)$ over k , we arrive at a linear system of two equations with two variables that allows us to evaluate $g(k)$ exactly. We find that $g(k) = -1$ for each $k \not\equiv 0 \pmod{p}$ and we conclude that $S_0 = \frac{p - \left(\frac{-u}{p}\right)}{2}$.

Now, for a fixed $v \geq 1$, we consider the change of variables defined by $x = ip^v + j$ and $y = rp^v + s$, where $i, r = 1, \dots, p$ and $j, s = 1, \dots, p^v$. We then treat x as fixed and not divisible by p , that is, with i, j fixed and $p \nmid j$. We verify that the y defined by $s = j$ and $r \equiv i + (2j)^{-1}u \pmod{p}$ satisfies the equation, and thus these x do not contribute to S_v .

Finally, again for a fixed $v \geq 1$, we consider the change of variables $x = pX$ and $y = pY$ (since clearly $x \equiv \pm y \pmod{p}$). We then obtain the equation

$$p^2 X^2 + p^v u \equiv p^2 Y^2 \pmod{p^{v+1}}.$$

We deduce that $S_1 = p$, and for $v \geq 2$, we can divide the equation by p^2 . After doing so, we consider the change of variables defined by $X = ip^{v-1} + j$ and $Y = rp^{v-1} + s$, where $i, r = 1, \dots, p$ and $j, s = 1, \dots, p^{v-1}$. We conclude that $S_v = pS_{v-2}$, and the result is proven. $\square$

4. Proof of Theorem 1

We first need to establish three very specific statements.

Proposition 1. *Let $n \geq 1$ be a fixed integer, and let $0 < \theta < 1$ and $0 < \epsilon < \theta^2$ be fixed real numbers. Then*

$$D_n(n^\theta, n^{\theta^2-\epsilon}) \ll \frac{\theta(1-\theta)}{\epsilon} + \frac{1}{\theta(1-\theta)}.$$

Proof. We can assume that we have $k \geq 2$ such divisors. From Lemma 1, we derive

$$\begin{aligned} n^{\frac{t(t+1)}{2}} n^{(\theta^2-\epsilon)\frac{k(k-1)}{2}} &\geq [d_1, \dots, d_k]^{\frac{t(t+1)}{2}} \prod_{1 \leq i < j \leq k} (d_i, d_j) \\ &\geq \prod_{1 \leq i \leq k} d_i^t \\ &\geq n^{\theta tk}. \end{aligned}$$

From here, we set $t = \theta k + \zeta$ for some $0 \leq \zeta < 1$. By comparing the exponents, we get the inequality

$$\epsilon k(k-1) \leq (\theta - \theta^2)k + \zeta(\zeta + 1).$$

Thus, if $k \leq \frac{4}{\theta(1-\theta)}$ does not hold, we find $k \leq 3\frac{\theta(1-\theta)}{\epsilon}$, and the result follows. $\square$

Proposition 2. *Let η and θ be fixed real numbers such that $0 < \theta^2 \leq \eta < \theta < 1$. The largest $\alpha(\theta, \eta)$ for which*

$$\left(\frac{\theta-t}{\alpha}\right)^2 - \left(\frac{\eta-t}{\alpha}\right) \geq 0 \quad \text{for all } 0 \leq t \leq 1-\alpha \quad (3)$$

holds for $0 < \alpha \leq \alpha(\theta, \eta)$ is given by

$$\alpha(\theta, \eta) := \begin{cases} \frac{\theta^2}{\eta} & \text{if } \theta \leq \frac{1}{2} \text{ and } 2\eta \leq \theta; \\ 4(\theta - \eta) & \text{if } \theta \leq \frac{1}{2} \text{ and } 2\eta > \theta; \\ \frac{(1-\theta)^2}{(1-\theta)^2 + \eta - \theta^2} & \text{if } \theta > \frac{1}{2} \text{ and } 2\eta \leq 3\theta - 1; \\ 4(\theta - \eta) & \text{if } \theta > \frac{1}{2} \text{ and } 2\eta > 3\theta - 1. \end{cases} \quad (4)$$

Proof. Given an $\alpha > 0$, we define $F(t) := \left(\frac{\theta-t}{\alpha}\right)^2 - \left(\frac{\eta-t}{\alpha}\right)$ and $I := [0, 1-\alpha]$. The function F is a polynomial of degree two with zeros

$$\theta - \frac{\alpha}{2} \pm \frac{\sqrt{\alpha(\alpha - \Delta)}}{2}, \quad \text{where } \Delta = 4(\theta - \eta).$$

If this polynomial is positive on the interval I , it means that its two zeros are either to the left or to the right of I , or that $\alpha \leq \Delta$. Also, we observe that $\alpha(\theta, \eta) \geq \Delta$ and in particular, $\alpha > 0$ is possible in (3).

Now, assume that F has a real zero. There are two possible situations. First,

$$\theta - \frac{\alpha}{2} + \frac{\sqrt{\alpha(\alpha - \Delta)}}{2} \leq 0,$$

which gives

$$\alpha \leq \frac{\theta^2}{\eta} \quad \text{if } \alpha \geq 2\theta \text{ and } \alpha \geq \Delta, \quad (5)$$

and second,

$$\theta - \frac{\alpha}{2} - \frac{\sqrt{\alpha(\alpha - \Delta)}}{2} \geq 1 - \alpha,$$

which gives

$$\alpha \leq \frac{(1 - \theta)^2}{(1 - \theta)^2 + \eta - \theta^2} \quad \text{if } \alpha \geq 2(1 - \theta) \text{ and } \alpha \geq \Delta. \quad (6)$$

We remark that in all possible scenarios, $\alpha \leq \Delta$, $\alpha \leq \frac{\theta^2}{\eta}$, and $\alpha \leq \frac{(1-\theta)^2}{(1-\theta)^2 + \eta - \theta^2}$, we have $\alpha \leq 1$.

The first case (5) is possible only when $\theta \leq \frac{1}{2}$. We verify that $\frac{\theta^2}{\eta} - \Delta = \frac{(\theta - 2\eta)^2}{\eta}$. The condition $\alpha \geq 2\theta$ allows us to complete the first two lines of (4). Similarly, the second case (6) is possible only when $\theta \geq \frac{1}{2}$ and we then verify that $\frac{(1-\theta)^2}{(1-\theta)^2 + \eta - \theta^2} - \Delta = \frac{(2\eta - 3\theta + 1)^2}{(1-\theta)^2 + \eta - \theta^2}$. Again, the condition $\alpha \geq 2(1 - \theta)$ allows us to complete the two last lines of (4). The proof is complete. $\square$

Proposition 3. *Let η and θ be fixed real numbers such that*

$$0 < \theta^2 - (\theta(1 - \theta))^2 \delta \leq \eta < \theta < 1$$

for some fixed $0 \leq \delta \leq 1$. We can choose $\alpha(\theta, \eta, \delta)$ such that the property

$$\left(\frac{\theta - t}{\alpha}\right)^2 - \left(\frac{\eta - t}{\alpha}\right) \geq (\theta(1 - \theta))^2 \delta \quad \text{for all } 0 \leq t \leq 1 - \alpha \quad (7)$$

holds for $0 < \alpha \leq \alpha(\theta, \eta, \delta)$ with

$$\xi(\theta, \eta) - \delta \leq \alpha(\theta, \eta, \delta) \leq \xi(\theta, \eta). \quad (8)$$

Proof. For simplicity, we define $\varepsilon := (\theta(1 - \theta))^2 \delta$. From Proposition 2, we can assume that $\varepsilon > 0$. Also, the setup being comparable to that of Proposition 2, the initial treatment is the same. We define the function $F_\varepsilon(t) := F(t) - \varepsilon$ and the same interval $I := [0, 1 - \alpha]$. We then consider the zeros of $F_\varepsilon(t)$ given by

$$\theta - \frac{\alpha}{2} \pm \frac{\sqrt{\alpha((1 + 4\varepsilon)\alpha - \Delta)}}{2}, \quad \text{where } \Delta = 4(\theta - \eta).$$

We distinguish three scenarios. In the first case, we consider the situation where $F_\varepsilon(t)$ has no roots or just one double root. We find $0 < \alpha \leq \frac{\Delta}{1 + 4\varepsilon}$. The second case is that of two roots less than or equal to 0. This situation is described by

$$\alpha \leq \frac{\theta^2}{\eta + \varepsilon\alpha} \quad \text{if } \alpha \geq 2\theta \text{ and } \alpha \geq \frac{\Delta}{1 + 4\varepsilon}. \quad (9)$$

We can express the solution to the first inequality as $0 < \alpha \leq \frac{-\eta + \sqrt{\eta^2 + 4\varepsilon\theta^2}}{2\varepsilon} =: \alpha_0(\theta, \eta, \delta)$. The third case is that of two roots greater than or equal to $1 - \alpha$, a situation that is described by

$$\alpha \leq \frac{(1 - \theta)^2}{(1 - \theta)^2 + \eta - \theta^2 + \varepsilon\alpha} \quad \text{if} \quad \alpha \geq 2(1 - \theta) \quad \text{and} \quad \alpha \geq \frac{\Delta}{1 + 4\varepsilon}. \quad (10)$$

The explicit solution is written as $0 < \alpha \leq \frac{-(1 - 2\theta + \eta) + \sqrt{(1 - 2\theta + \eta)^2 + 4\varepsilon(1 - \theta)^2}}{2\varepsilon} =: \alpha_1(\theta, \eta, \delta)$, we also note that $1 - 2\theta + \eta > 0$. With these descriptions in hand, we can address Inequality (8). First, let us verify that each valid α is less than or equal to 1. For $\eta \geq \theta^2$, this is obvious, but for $\theta^2 - (\theta(1 - \theta))^2\delta \leq \eta < \theta^2$, we must check the three possibilities separately. The case $\frac{\Delta}{1 + 4\varepsilon}$ is straightforward. For case (9), if this were not true, we would have $\varepsilon + \eta < \varepsilon\alpha^2 + \eta\alpha \leq \theta^2$, which is a contradiction. The case (10) follows in the same way.

The condition $\alpha \geq 2\theta$ is necessary for the first inequality in (9) to hold, but this is not the case for $\alpha \geq \frac{\Delta}{1 + 4\varepsilon}$. The same is true in (10). Also, we note that $2\theta \leq \frac{\Delta}{1 + 4\varepsilon}$ for $2\eta \leq \theta - 4\theta\varepsilon$ and that $2(1 - \theta) \leq \frac{\Delta}{1 + 4\varepsilon}$ for $2\eta \leq 3\theta - 1 - 4(1 - \theta)\varepsilon$. We deduce that if $2\eta \leq \theta - 4\theta\varepsilon$ and $\alpha_0(\theta, \eta, \delta) \geq \frac{\Delta}{1 + 4\varepsilon}$, then (7) holds for $0 < \alpha \leq \alpha_0(\theta, \eta, \delta)$. Similarly, if $2\eta \leq 3\theta - 1 - 4(1 - \theta)\varepsilon$ and $\alpha_1(\theta, \eta, \delta) \geq \frac{\Delta}{1 + 4\varepsilon}$, then (7) holds for $0 < \alpha \leq \alpha_1(\theta, \eta, \delta)$.

Now, we need to address the five cases of (2) separately. Let us begin with Cases 3 and 5, that is, when $\xi(\theta, \eta) = \Delta$. Since the property (7) holds for every $0 < \alpha \leq \alpha(\theta, \eta, \delta) := \frac{\Delta}{1 + 4\varepsilon}$, we have $\xi(\theta, \eta) - \alpha(\theta, \eta, \delta) \leq 16(\theta(1 - \theta))^3\delta \leq \frac{\delta}{4}$.

Suppose now we are in Case 2 with $\xi(\theta, \eta) = \frac{\theta^2}{\eta}$. We distinguish two situations. The first one is when $2\eta \leq \theta - 4\theta\varepsilon$. We then choose $\alpha(\theta, \eta, \delta) := \alpha_0(\theta, \eta, \delta)$. We deduce from above that property (7) holds for every $0 < \alpha \leq \alpha(\theta, \eta, \delta)$ and we have

$$\begin{aligned} \xi(\theta, \eta) - \alpha(\theta, \eta, \delta) &= \frac{\theta^2}{\eta} - \frac{\theta^2}{\eta + \varepsilon\alpha_0(\theta, \eta, \delta)} \\ &\leq \frac{\varepsilon\theta^2}{\eta^2} \leq (1 - \theta)^2\delta \leq \delta, \end{aligned}$$

since $\alpha_0(\theta, \eta, \delta) \leq 1$.

The second possibility is when $\theta - 4\theta\varepsilon < 2\eta \leq \theta$, where we choose $\alpha(\theta, \eta, \delta) := \frac{\Delta}{1 + 4\varepsilon}$. In this situation, property (7) holds for every $0 < \alpha \leq \alpha(\theta, \eta, \delta)$, and also

$$\begin{aligned} \xi(\theta, \eta) - \alpha(\theta, \eta, \delta) &= \xi(\theta, \eta) - \Delta + \Delta - \frac{\Delta}{1 + 4\varepsilon} \\ &= \frac{(\theta - 2\eta)^2}{\eta} + \frac{4\varepsilon\Delta}{1 + 4\varepsilon} \\ &\leq (16(\theta(1 - \theta))^4 + 16(\theta(1 - \theta))^3)\delta \leq \frac{5\delta}{16}, \end{aligned}$$

as required. Case 4 is treated similarly.

It only remains to handle Case 1. We will start by considering the case where $\theta \leq \frac{1}{2}$, which corresponds to either $\alpha_0(\theta, \eta, \delta)$ or $\frac{\Delta}{1+4\varepsilon}$. We proceed as previously and we first consider when $2\eta \leq \theta - 4\theta\varepsilon$, where we set $\alpha(\theta, \eta, \delta) := \alpha_0(\theta, \eta, \delta)$. Thus, property (7) holds for every $0 < \alpha \leq \alpha(\theta, \eta, \delta)$. Also, we already know that $\alpha_0(\theta, \eta, \delta) \leq 1$ and we have

$$\begin{aligned} \xi(\theta, \eta) - \alpha(\theta, \eta, \delta) &= 1 - \frac{\theta^2}{\eta + \varepsilon\alpha_0(\theta, \eta, \delta)} \\ &= \frac{\eta + \varepsilon\alpha_0(\theta, \eta, \delta) - \theta^2}{\eta + \varepsilon\alpha_0(\theta, \eta, \delta)} \\ &\leq \frac{\varepsilon}{\theta^2} = (1 - \theta)^2\delta \leq \delta, \end{aligned}$$

as desired. Now, in the situation where $2\eta > \theta - 4\theta\varepsilon$, we choose $\alpha(\theta, \eta, \delta) := \frac{\Delta}{1+4\varepsilon}$. From $\eta \leq \theta^2$, we deduce that $\theta \geq \frac{1-4\varepsilon}{2}$, so that

$$\begin{aligned} \alpha(\theta, \eta, \delta) &\geq \frac{4(\theta - \theta^2)}{1 + 4\varepsilon} \\ &\geq 4\left(\frac{1-4\varepsilon}{2} - \left(\frac{1-4\varepsilon}{2}\right)^2\right)/(1 + 4\varepsilon) \\ &= 1 - 4\varepsilon \geq 1 - \frac{\delta}{4}, \end{aligned}$$

which gives the result. The case $\theta > \frac{1}{2}$ is treated in a similar way. The proof is complete. $\square$

Proof of Theorem 1. We can assume that $\tau(n) \geq 3$ since otherwise the result is trivial. We will distinguish two cases. The first case is when $\eta < \theta^2 - \frac{(\theta(1-\theta))^2}{\log \tau(n)}$. By Proposition 1, we have

$$\begin{aligned} D_n(n^\theta, n^\eta) &\ll \frac{\log \tau(n)}{\theta(1-\theta)} \\ &\ll \tau(n)^{1-\xi(\theta, \eta)} \frac{V(n) \log \tau(n)}{\theta(1-\theta)}, \end{aligned}$$

which is the desired result. The second case is when $\eta \geq \theta^2 - \frac{(\theta(1-\theta))^2}{\log \tau(n)}$.

Let $n = p_1^{\beta_1} \cdots p_{\omega(n)}^{\beta_{\omega(n)}}$ be the factorization of n into distinct primes and suppose that $n = ab$ where $(a, b) = 1$. We notice the identity

$$D_n(X, Y) = \sum_{e|b} D_a\left(\frac{X}{e}, \frac{Y}{e}\right). \quad (11)$$

To choose the factorization, we begin by setting $\delta := \frac{1}{\log \tau(n)}$. Let us remark right away that we can assume that $\delta < \xi(\theta, \eta)$, since otherwise the result is trivial. Then,

we consider the pairs (ρ_i, θ_i) , with $i = 1, \dots, \omega(n)$, defined by $(\beta_i + 1) = \tau(n)^{\rho_i}$ and $p_i^{\beta_i} = n^{\theta_i}$. From Lemma 2, there exists a permutation σ such that

$$\sum_{i=1}^s \rho_{\sigma_i} \leq \sum_{i=1}^s \theta_{\sigma_i}$$

for every $1 \leq s \leq \omega(n)$. We consider σ fixed and choose the minimal s such that

$$\sum_{i=1}^s \theta_{\sigma_i} \geq 1 - \alpha(\theta, \eta, \delta),$$

where $\alpha(\theta, \eta, \delta)$ is as in Proposition 3. Thus,

$$\sum_{i=1}^{s-1} \theta_{\sigma_i} \leq 1 - \alpha(\theta, \eta, \delta).$$

We define

$$b := \prod_{i=1}^s p_{\sigma_i}^{\beta_{\sigma_i}}.$$

With this factorization $n = ab$ established, we observe that

$$\begin{aligned} \tau(b) &= (\beta_{\sigma_s} + 1) \cdot \prod_{i=1}^{s-1} (\beta_{\sigma_i} + 1) \\ &\leq 2V(n)\tau(n)^{1-\alpha(\theta, \eta, \delta)} \\ &\ll V(n)\tau(n)^{1-\xi(\theta, \eta)} \end{aligned} \tag{12}$$

from Proposition 3. Also, since $a = n/b$, we have $1 \leq a = n^\alpha \leq n^{\alpha(\theta, \eta, \delta)}$. If $a = 1$, the result follows from (11), (12), and the fact that $D_1(\cdot, \cdot) \leq 1$. We can now assume that $a > 1$. We seek to bound $D_a\left(\frac{X}{e}, \frac{Y}{e}\right)$ for a given divisor e of b , which we write as $e = n^t$. Let us remark that $D_a\left(\frac{X}{e}, \frac{Y}{e}\right) \leq 2$ if either $e \geq Y$ or $\frac{X}{e} \geq \frac{a}{2}$. Also, by Proposition 3, we know that

$$\begin{aligned} \left(\frac{\log(X/e)}{\log a}\right)^2 - \left(\frac{\log(Y/e)}{\log a}\right) &= \left(\frac{\theta - t}{\alpha}\right)^2 - \left(\frac{\eta - t}{\alpha}\right) \\ &\geq \frac{(\theta(1 - \theta))^2}{\log \tau(n)} \end{aligned}$$

for $t \in [0, 1 - \alpha]$. In any case, from Proposition 1 and Equation (11), we deduce that

$$\begin{aligned} D_n(X, Y) &\ll V(n)\tau(n)^{1-\xi(\theta, \eta)} \left(\frac{\log \tau(n)}{\theta(1 - \theta)} + \frac{1}{\theta(1 - \theta)} \right) \\ &\ll \tau(n)^{1-\xi(\theta, \eta)} \frac{V(n) \log \tau(n)}{\theta(1 - \theta)}. \end{aligned}$$

The proof is complete. □

5. Proof of Theorem 2

We need the following proposition to deal with certain cases.

Proposition 4. *Let $0 < \theta < 1$ and $0 < \epsilon < \theta$ be fixed real numbers. If Conjecture 2 holds, then*

$$k_\epsilon(\theta) \geq 1.$$

Proof. Let us begin with the case $\theta \leq \frac{1}{2}$. Fix a sufficiently large integer m , and the smallest integer v such that $m < v^\theta$. There is a multiple of m in the interval $[v - m, v - 1]$, say n_0 . We then have

$$\begin{aligned} n_0^\theta &\leq m \leq (n_0 + m)^\theta \\ &\leq n_0^\theta \left(1 + \frac{m}{n_0}\right)^\theta \\ &\leq n_0^\theta \left(1 + \theta \frac{m}{n_0}\right) \\ &\leq n_0^\theta + \frac{1.1\theta}{n_0^{1-2\theta}}, \end{aligned}$$

which is more than sufficient to show that $k_\epsilon(\theta) \geq 1$ for $0 < \theta \leq \frac{1}{2}$ and $0 < \epsilon < \theta$. Now, consider the case where $\theta > \frac{1}{2}$. A similar argument allows us to find an integer n_1 having a divisor m such that

$$n_1^{1-\theta} - \frac{1.1(1-\theta)}{n_1^{2\theta-1}} \leq m \leq n_1^{1-\theta},$$

so

$$n_1^\theta \leq \frac{n_1}{m} \leq \frac{n_1^\theta}{1 - \frac{1.1(1-\theta)}{n_1^\theta}} \leq n_1^\theta + 1.2(1-\theta)$$

from which the result follows. $\square$

Proof of Theorem 2. The parameters θ and ϵ are fixed throughout the argument. Let M be a sufficiently large number for these parameters, and let $L = (\log M)^2$. By the Prime Number Theorem, we can assume that

$$\pi(M + L) - \pi(M) \geq s,$$

where $s \geq 1$ is an integer to be determined later. Define the integer

$$N_0 := p_1 \cdots p_s,$$

where p_i are the s smallest primes greater than M . Note that for any group of $1 \leq \ell \leq s$ integers n_j in the interval $[M, M + L]$, we have

$$M^\ell \leq \prod_{j=1}^{\ell} n_j \leq (M + L)^\ell,$$

so that

$$\prod_{j=1}^{\ell} n_j = M^{\ell} + \lambda \ell L(M+L)^{\ell-1} \quad (13)$$

for some $0 \leq \lambda \leq 1$.

Let us begin with the case $\theta \leq \frac{1}{2}$. Denoting the ceiling function by $\lceil \cdot \rceil$ and the fractional part by $\{\cdot\}$, we will choose an integer $r \leq s$ defined by

$$r := \begin{cases} \lceil \theta s \rceil & \text{if } 0 < \{\theta s\} \leq \frac{1}{2}; \\ \lceil \theta s \rceil + 1 & \text{otherwise} \end{cases}$$

and also

$$N := \max_{v \geq 1} v N_0, \\ M^{r \geq v^{\theta} M^{\theta s} \left(1 + \theta s L \frac{(M+L)^{s-1}}{M^s}\right)}$$

which in turn defines an integer m such that $N = m N_0$. Now, let $\mathcal{Q}$ be the set of the $\binom{s}{r}$ divisors of N_0 formed by r primes. We will now fix a $q \in \mathcal{Q}$ and show that $N^{\theta} \leq q \leq N^{\theta} + N^{\theta-\epsilon}$ if $\epsilon \leq \frac{1}{s+\frac{3}{2\theta}}$. On the one hand, we have

$$\begin{aligned} N^{\theta} &\leq m^{\theta} (M^s + sL(M+L)^{s-1})^{\theta} \\ &= m^{\theta} M^{\theta s} \left(1 + \frac{sL(M+L)^{s-1}}{M^s}\right)^{\theta} \\ &\leq m^{\theta} M^{\theta s} \left(1 + \frac{\theta sL(M+L)^{s-1}}{M^s}\right), \end{aligned} \quad (14)$$

from which we obtain $q \geq N^{\theta}$ by the choice of N and (13). On the other hand, since

$$\left(1 + \frac{1}{m}\right)^{\theta} \leq 1 + \frac{\theta}{m} \leq 1 + \frac{1}{M}$$

by the choice of m , we find

$$\begin{aligned} q &\leq M^r \left(1 + \frac{rL(M+L)^{r-1}}{M^r}\right) \\ &\leq (m+1)^{\theta} M^{\theta s} \left(1 + \theta sL \frac{(M+L)^{s-1}}{M^s}\right) \left(1 + \frac{rL(M+L)^{r-1}}{M^r}\right) \\ &\leq \frac{(m+1)^{\theta}}{m^{\theta}} N^{\theta} \left(1 + \theta sL \frac{(M+L)^{s-1}}{M^s}\right) \left(1 + \frac{rL(M+L)^{r-1}}{M^r}\right) \\ &\leq N^{\theta} \left(1 + \frac{1}{M}\right) \left(1 + \theta sL \frac{(M+L)^{s-1}}{M^s}\right) \left(1 + \frac{rL(M+L)^{r-1}}{M^r}\right) \\ &\leq N^{\theta} \left(1 + O\left(\frac{sL}{M}\right)\right) \\ &\leq N^{\theta} + N^{\theta-\epsilon}, \end{aligned} \quad (15)$$

where the last line follows from $\frac{r}{\theta} < s + \frac{3}{2\theta}$.

It remains to show that $\binom{s}{r}$ satisfies the announced inequality. We begin by choosing $s := \lfloor \frac{1}{\epsilon} - \frac{3}{2\theta} \rfloor$, in accordance with the previous steps, and we will temporarily assume that $\epsilon \leq \frac{\theta}{2}$. Note that this assumption implies that $1 \leq \frac{1}{2\theta} \leq \frac{1}{\epsilon} - \frac{3}{2\theta}$, and therefore $\theta s \geq \frac{1}{4}$ (since $\lfloor x \rfloor \geq \frac{x}{2}$ for $x \geq 1$). We then deduce that $r \asymp \theta s$, and similarly $s - r + 1 \asymp s$. We also deduce that $\frac{1}{\epsilon} \geq \frac{1}{\epsilon} - \frac{3}{2\theta} \geq \frac{1}{4\epsilon} \geq 1$, and hence $s \asymp \frac{1}{\epsilon}$.

Now, using the Stirling formula, we have

$$\begin{aligned} \binom{s}{r} &\gg \sqrt{\frac{s}{r(s-r+1)}} \frac{s^s}{r^r (s-r)^{s-r}} \\ &= \sqrt{\frac{s}{r(s-r+1)}} \left(\frac{1}{\frac{r}{s} (1 - \frac{r}{s})^{1-\frac{r}{s}}} \right)^s \\ &\gg \sqrt{\frac{s}{r(s-r+1)}} \left(\frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \right)^s \\ &\gg \sqrt{\frac{\epsilon}{\theta}} \left(\frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \right)^{\frac{1}{\epsilon} - \frac{3}{2\theta}} \\ &\gg \sqrt{\epsilon \theta^3 (1-\theta)^3} \left(\frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \right)^{\frac{1}{\epsilon}}. \end{aligned} \tag{16}$$

The first two lines of (16) are automatic. To proceed, we first study the function $t(x) = \log \frac{1}{x^x (1-x)^{1-x}}$. We show that $t(x)$ is increasing for $x \in [0, 1/2]$ and that its values lie within the interval $[0, \log 2]$. In the third line of (16), for $r \leq s/2$, we use the monotonicity of $t(x)$, whereas for $r > s/2$ and s large enough, we write

$$\begin{aligned} \exp(t(r/s)) &= \exp(t(\theta)) + \exp(t(\theta))(\exp(t(r/s) - t(\theta)) - 1) \\ &= \exp(t(\theta)) + O\left(\frac{1}{s}\right), \end{aligned}$$

which follows from $|\frac{r}{s} - \theta| \ll \frac{1}{s}$ and the fact that $1 \leq \exp(t(x)) \leq 2$ for $x \in [0, \frac{1}{2}]$. In the fourth line of (16), we introduce an approximation of s in the exponent, along with estimates for r , $s - r + 1$, and s . For the last line of (16), we simplify the fourth line.

If the inequality $\epsilon \leq \frac{\theta}{2}$ does not hold, we observe that

$$\sqrt{\epsilon \theta^3 (1-\theta)^3} \left(\frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \right)^{\frac{1}{\epsilon}} \ll 1$$

and thus the announced result holds from Proposition 4. This completes the argument for the case $\theta \leq \frac{1}{2}$.

Only the case $\theta > \frac{1}{2}$ remains. Let us begin with the case where $\epsilon < 1 - \theta$. The previous construction can be adapted to find arbitrarily large integers n for which

$$|\{d \in \mathcal{D}_n \mid n^{1-\theta} - 0.5n^{1-\theta-\epsilon} \leq d \leq n^{1-\theta}\}| \gg \sqrt{\epsilon} (\theta(1-\theta))^{3/2} \left(\frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \right)^{\frac{1}{\epsilon}},$$

so that, as we did in the proof of Proposition 4, we conclude that

$$D_n(n^\theta, n^{\theta-\epsilon}) \gg \sqrt{\epsilon}(\theta(1-\theta))^{3/2} \left(\frac{1}{\theta^\theta(1-\theta)^{1-\theta}} \right)^{\frac{1}{\epsilon}}.$$

The details of this construction are mostly the same. We choose the primes in the interval $[M-L, M]$ and use the approximation $\prod_{j=1}^\ell n_j = M^\ell - \lambda \ell L M^{\ell-1}$ as the analogue of (13). We let $1-\theta$ replace θ everywhere in the argument, including the definition of r . We define N_1 , the analogue of N_0 , and we note

$$N' := \min_{\substack{v \geq 1 \\ M^r \leq v^{1-\theta} M^{(1-\theta)s} \left(1 - \frac{(1-\theta)sL}{M(1-sL/M)^\theta} \right)}} v N_1.$$

The analogue of (14) gives $q \leq N'^{1-\theta}$ and the analogue of (15) gives $q \geq N'^{1-\theta} - 0.5N'^{1-\theta-\epsilon}$. We leave the verification of the details to the reader. Finally, in the case where $\epsilon \in [1-\theta, \theta)$, we verify that the announced result is simply $k_\epsilon(\theta) \gg 1$, which follows from Proposition 4. $\square$

6. A Discussion around Conjecture 1

It is well known that the large divisors of an integer n are isolated from each other. Indeed, let d and $d+h$ be two consecutive divisors of n in increasing order. We observe that

$$1 \leq \frac{n}{d} - \frac{n}{d+h} = \frac{hn}{d(d+h)}, \quad \text{which gives} \quad h > \frac{d^2}{n}.$$

This bound is non-trivial when $d \geq \sqrt{n}$. Proposition 4.1 of [5] suggests a completely different method that is most efficient when the divisors are close to $\sqrt{n}$.

Let a, b be two fixed coprime positive integers and consider the function

$$f(x) := a \frac{n}{x} + bx \quad (x > 0).$$

This simple function has the following desired properties. First of all, its derivative vanishes at $x_0 := \sqrt{\frac{an}{b}}$, where $f(x)$ attains its minimum. For $x \geq x_0$, the function $f(x)$ is strictly increasing. Moreover, from its Taylor series we have

$$\begin{aligned} f(x_0 + h) &= f(x_0) + f''(\zeta) \frac{h^2}{2} \\ &= f(x_0) + \frac{an}{\zeta^3} h^2, \end{aligned}$$

for some $\zeta \in [x_0, x_0 + h]$. Secondly, $f(d)$ is an integer for every divisor d of n . Also, as we restrict our attention to the divisors $d \geq x_0$, the value of $f(d)$ determines d uniquely.

We deduce that $|f(x_0 + h) - f(x_0)| \geq i$ requires $h \geq \sqrt{\frac{ix_0}{b}}$ and therefore

$$D_n(x_0, \sqrt{\frac{ix_0}{b}}) \leq i + 1.$$

Furthermore, we note that

$$f(d)^2 - f^*(d)^2 = 4abn, \quad (17)$$

where

$$f^*(x) := a \frac{n}{x} - bx.$$

This associated function f^* also takes integer values at divisors d of n . This implies that Equation (17) is, in fact, a nontrivial algebraic relation satisfied by $(f(d), f^*(d))$. Interestingly enough, this algebraic identity allows us to get constraints modulo well-chosen prime powers on the values taken by $f(d)$. This is made explicit in Lemma 4, and everything is in place to apply the large sieve from Lemma 3. For each $p \geq 3$, we choose the set $\mathcal{P}$ to contain $p^{v_p(4abn)+1}$, where $v_p(\cdot)$ is the usual p -adic valuation. We can verify that the value of the function $h(p^\beta)$ in Lemma 3 is

$$h(p) = 1 + O\left(\frac{1}{p}\right), \quad h(p^2) = \frac{1}{p-1}, \quad h(p^3) = \frac{1}{2p} + O\left(\frac{1}{p^2}\right),$$

and that it satisfies $h(p^\beta) \ll \frac{1}{p^2}$ for $\beta \geq 4$. For all n , we obtain

$$D_n(x_0, \sqrt{\frac{ix_0}{b}}) \leq \frac{i + 1 + Q^2}{H}, \quad (18)$$

where $H = \sum_{q \leq Q} h(q)$. Choosing the parameter Q so that $\pi(Q) \approx 2\omega(4abn)$ forces $H \gg \omega(4abn)$, and therefore gives the following result.

Proposition 5. *Let a, b be two fixed coprime positive integers and $n \geq 1$ a fixed integer. Let us define $x_0 := \sqrt{\frac{an}{b}}$ and $w := \omega(4abn)$. For $i \geq 1$, we have*

$$D_n(x_0, \sqrt{\frac{ix_0}{b}}) \ll \min\left(i, \frac{i}{w} + w(\log w)^2\right).$$

Remark 1. It is well known that $\omega(n) \ll \frac{\log n}{\log \log n}$. Hence, taking Q such that $\pi(Q) \approx \omega(2abn) + \frac{\log n}{\log \log n}$ implies that

$$D_n(x_0, \sqrt{\frac{ix_0}{b}}) = o_{a,b}(i) \quad \text{whenever} \quad \frac{i}{\log n \log \log n} \rightarrow \infty.$$

By choosing $Q = \sqrt{i}$, one can also deduce that, when n squarefree or cubefree, the right-hand side of (18) is respectively $\ll_{a,b} \frac{i}{\log i}$ and $\ll_{a,b} \frac{i}{\sqrt{\log i}}$ ($i \geq 2$), respectively. Finally, again with $Q = \sqrt{i}$, if $\pi(Q) \geq 2\omega(4abn)$ then the right-hand side of (18) is $\ll \sqrt{i} \log i$.

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