



A BOMBIERI-VINOGRADOV RESULT TO ADDRESS THE PARITY PROBLEM

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Abstract

We propose a weighted sieve that, combined with a sufficiently good distribution of the sifting function in arithmetic progressions, allows us to detect the existence of primes in the sequence under study. In the general case, a level of distribution equal to $1/2$ is required; however, for those sequences that exhibit a certain regularity, we show that a Bombieri–Vinogradov hypothesis is sufficient to overcome the parity problem. Moreover, applying these methods to classical sequences, we obtain further results of interest.

1. Introduction

Classical sieve methods, while powerful for detecting almost-primes, fail to isolate primes themselves due to the parity problem. These methods cannot distinguish integers with an odd number of prime factors from those with an even number of prime factors. This obstruction highlights the limitations of unconditional sieve-based approaches to problems in the style of the Twin Primes and Goldbach conjectures.

However, in certain situations, it has been possible to detect primes by combining the sieve with additional information. In the works of Bombieri, Fouvry, Friedlander, and Iwaniec ([1],[3],[5]), this information consists of estimates for certain bilinear forms involving the Möbius function. With a similar aim, Murty and Vatwani [7] address this barrier by formulating an equidistribution conjecture for the Möbius function evaluated on shifted primes in arithmetic progressions. Their work demonstrates that, when combined with the Elliott–Halberstam conjecture for primes, such Möbius equidistribution is sufficient to establish the expected asymptotic for $\sum \Lambda(n)\Lambda(n+a)$, thereby overcoming the parity obstacle. With current knowledge regarding the Elliott–Halberstam conjecture (moduli up to $q \leq x^{1/2-\varepsilon}$), it would be necessary to prove the shifted conjecture for moduli beyond the exponent $1/2$, which poses a significant challenge. Nevertheless, this approach highlights

that, if sieve methods are combined with the good distribution of an appropriate arithmetic function, this framework can potentially overcome the parity barrier.

We will see how applying a weighted sieve to a generic sequence leads to the evaluation of such arithmetic progressions, but, in our case, the function to be evaluated is $f_z(\cdot)$, the indicator function of the integers whose prime factors are greater than z . The framework imposed by the applied sieve introduces two conditions that allow us to simplify the treatment. On the one hand, the required level for the equidistribution of $\mathbf{1}_A(n)f_z(n)$ is $1/2$, and on the other hand, it suffices to restrict to prime moduli p . Moreover, if the sequence $\{A_p\}$ contains, in order of magnitude, the expected number of integers, then the required level for the equidistribution of $\mathbf{1}_A(n)f_z(n)$ is $1/2 - \epsilon$, which implies that a Bombieri–Vinogradov type result would suffice to detect primes in the sequence $\{A\}$.

We define a generic sequence of positive integers $\{A\}$, whose characteristic function we call $\mathbf{1}_A$. We apply to this sequence a weighted sieve similar to the one used by Chen in the proof of his celebrated results (see, for example, [2] and [4] for a general description of sieve methods and [6] for a summary of Chen’s proof).

In the standard formulation of the sieve, P denotes a subset of the primes. We shall retain this notation when stating general results. However, when applying these results to the sequence $\{p + a\}$, we shall assume that

$$P = \mathbb{P},$$

where $\mathbb{P}$ denotes the set of all primes. For this purpose, we define

$$S_0(A, z) := S(A, P, z) - \sum_{z < p \leq x^{1/2}} S(A_p, P, z), \quad (1)$$

where $S(A, P, z)$ and $S(A_p, P, z)$ are the usual sifting functions

$$S(A, P, z) = \#\{n \in A : p \mid n \Rightarrow p \geq z, p \in P\} = \sum_{n \in A} f_z(n), \quad (2)$$

$$S(A_p, P, z) = \sum_{n \in A_p} f_z(n). \quad (3)$$

Here, the sequence A_p is defined by

$$\{A_p\} := \{n \in A : p \mid n\},$$

and $f_z(n)$ is the characteristic function of integers whose prime factors are $> z$,

$$f_z(n) = \begin{cases} 1, & \text{if } (n, P(z)) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This function is particularly relevant in the context of sieve theory because of the role played by z as the sieve level, determining up to which limit elements with

“small” prime factors are excluded when sifting a given sequence. We recall that f_z is multiplicative and can be written in the following way:

$$f_z(n) = \sum_{d|n} \mu(d) \mathbf{1}_{d|P(z)},$$

where $\mu(\cdot)$ is the Möbius function and

$$P(z) = \prod_{p < z} p.$$

Specifically, we are interested in the shifted version $f_z(n + a)$, due to this identity, which is a consequence of the inclusion–exclusion principle and is valid for any arithmetic function $h(n)$:

$$\sum_{d|P(z)} \mu(d) \sum_{\substack{n \leq x \\ n \equiv -a \pmod{d}}} h(n) = \sum_{n \leq x} h(n) f_z(n + a). \quad (4)$$

If we think of $h(\cdot)$ as the indicator function of a given sequence of integers, then Equation (4) highlights the relationship between f_z and the elements of the sequence that survive the sieve process.

We shall show that if we select z properly, then $S_0(A, z)$ allows us to obtain a lower bound for the number of primes contained in $\{A\}$. Our results strongly depend on the value of z . As we shall see later, the lower bound $z \geq x^{1/5}$ is required to obtain Theorem 1, while condition $z \leq x^{1/4}$ allows us to make use of established results for the specific sequence $\{p + a\}$, to which Theorem 2 is applied. We assume $z = x^{1/5}$ and then, expressing $S_0(A, x^{1/5})$ in terms of $f_{x^{1/5}}(\cdot)$, we formulate our main results.

Theorem 1. *Let $\{A\}$ be a sequence of positive integers, and let φ denote the Euler totient function. Then, for sufficiently large x , there exists a constant $K > 0$ such that*

$$\#\{p \in A\} \geq KS(A, P, x^{1/5}) - \sum_{x^{1/5} < p \leq x^{1/2}} \Delta(A_p), \quad (5)$$

where

$$\Delta(A_p) := S(A_p, P, x^{1/5}) - \frac{1}{\varphi(p)} S(A, P, x^{1/5}). \quad (6)$$

Moreover, if the sequence $\{A\}$ satisfies the following condition

$$S(A_p, P, x^{1/5}) \ll \frac{1}{\varphi(p)} S(A, P, x^{1/5}), \quad (7)$$

for p in the range $x^{1/5} < p \leq x^{1/2}$, then there exists a constant $K > 0$ such that

$$\#\{p \in A\} \geq KS(A, P, x^{1/5}) - \sum_{x^{1/5} < p \leq x^{1/2} (\log x)^{-B}} \Delta(A_p), \quad (8)$$

where B is an arbitrary positive constant.

The following result is an application of Theorem 1 to the sequence $\{p + a\}$.

Theorem 2. *Let a be an even integer $\neq 0$. Then the sequence*

$$\mathcal{A} = \{p + a : p \in \mathbb{P}, p \leq x\},$$

satisfies Equation (7), and the following holds

$$\#\{p \in \mathcal{A}\} \geq C \frac{x}{(\log x)^2} - \sum_{x^{1/5} < p \leq x^{1/2} (\log x)^{-B}} \bar{\Delta}(\mathcal{A}_p), \quad (9)$$

where C is a positive constant, B is an arbitrary positive constant, and

$$\bar{\Delta}(\mathcal{A}_p) = \left(\sum_{\substack{n \leq x \\ n \equiv -a \pmod{p}}} \mathbf{1}_{\mathbb{P}}(n) f_{x^{1/5}}(n + a) - \frac{1}{\varphi(p)} \sum_{\substack{n \leq x \\ (n, p) = 1}} \mathbf{1}_{\mathbb{P}}(n) f_{x^{1/5}}(n + a) \right). \quad (10)$$

Here, $\mathbf{1}_{\mathbb{P}}$ denotes the indicator function of the primes.

Theorem 2 implies that a Bombieri–Vinogradov type result for $f_z(p + a)$ allows us to detect primes in $\mathcal{A}$.

Compared with Theorem 1.1 of [7], we observe that Theorem 2 imposes an apparently more accessible condition for proving the infinitude of twin primes, since the necessary Bombieri–Vinogradov result seems to be within the reach of currently available techniques. Indeed, taking into account the extensive existing literature on the subject and the advances in understanding the properties required for an arithmetic function to satisfy a Bombieri–Vinogradov–type mean value theorem (see, for example, [1]), it seems plausible that the required equidistribution property for $f_z(p + a)$ may be accessible to existing methods.

Remark 1. An analogous result to Theorem 2 can be derived for the sequence

$$G := \{2N - p : p \in \mathbb{P}, p < 2N\}.$$

2. Previous Results

In order to deduce Theorem 1, we will require two preliminary estimates involving the sum of $\frac{1}{\varphi(p)}$ over the specific ranges that appear in our setup.

Lemma 1. *Let $B > 0$ be fixed. For all sufficiently large x , we have*

$$\sum_{x^{1/5} < p \leq x^{1/2}} \frac{1}{\varphi(p)} = 0.916 \dots + o(1), \quad (11)$$

and

$$\sum_{x^{1/2}(\log x)^{-B} < p \leq x^{1/2}} \frac{1}{\varphi(p)} \ll_B \frac{\log \log x}{\log x}. \quad (12)$$

Proof. Since p is prime, we have $\varphi(p) = p - 1$, and hence

$$\frac{1}{\varphi(p)} = \frac{1}{p-1} = \frac{1}{p} + O\left(\frac{1}{p^2}\right).$$

Therefore, uniformly for $2 \leq w < y$,

$$\sum_{w < p \leq y} \frac{1}{\varphi(p)} = \sum_{w < p \leq y} \frac{1}{p} + O\left(\sum_{p > w} \frac{1}{p^2}\right) = \sum_{w < p \leq y} \frac{1}{p} + O\left(\frac{1}{w}\right).$$

Now, we apply Mertens' theorem,

$$\sum_{p \leq t} \frac{1}{p} = \log \log t + M + O\left(\frac{1}{\log t}\right),$$

where M is the Meissel–Mertens constant. Taking $w = x^{1/5}$ and $y = x^{1/2}$, we obtain

$$\begin{aligned} \sum_{x^{1/5} < p \leq x^{1/2}} \frac{1}{\varphi(p)} &= \log \log \left(x^{1/2}\right) - \log \log \left(x^{1/5}\right) + O\left(\frac{1}{\log x}\right) \\ &= \log \left(\frac{\frac{1}{2} \log x}{\frac{1}{5} \log x}\right) + O\left(\frac{1}{\log x}\right) \\ &= \log \frac{5}{2} + O\left(\frac{1}{\log x}\right). \end{aligned}$$

Since

$$\log \frac{5}{2} = 0.916290 \dots,$$

this concludes the proof of Equation (11).

For the second estimate, put

$$Y = x^{1/2}(\log x)^{-B}.$$

Applying the same argument gives

$$\begin{aligned} \sum_{Y < p \leq x^{1/2}} \frac{1}{\varphi(p)} &= \log \log \left(x^{1/2}\right) - \log \log Y + O_B\left(\frac{1}{\log x}\right) \\ &= \log \left(\frac{\frac{1}{2} \log x}{\frac{1}{2} \log x - B \log \log x}\right) + O_B\left(\frac{1}{\log x}\right) \\ &= -\log \left(1 - \frac{2B \log \log x}{\log x}\right) + O_B\left(\frac{1}{\log x}\right). \end{aligned}$$

For sufficiently large x ,

$$\frac{2B \log \log x}{\log x} < \frac{1}{2},$$

and the estimate $-\log(1-u) \ll u$, valid for $0 \leq u \leq 1/2$, yields

$$\sum_{x^{1/2}(\log x)^{-B} < p \leq x^{1/2}} \frac{1}{\varphi(p)} \ll_B \frac{\log \log x}{\log x},$$

which completes the proof of Equation (12). $\square$

In the following lemmas, we state the classical bounds obtained from the Selberg and Rosser–Iwaniec sieves and apply them to the sequence $\mathcal{A} = \{p + a\}$. Most of the material presented here consists of standard applications of sieve methods. We mainly follow Heath-Brown’s notation and presentation [6], although these results are also covered in the usual references on sieve methods.

The following form of the Selberg sieve is Equation 3.8 of [6].

Lemma 2. *Let $\mathcal{A}$ be a finite sequence of positive integers, and let P be a set of primes. For every squarefree integer $d \mid P(z)$, suppose that*

$$\#\mathcal{A}_d = \frac{\omega(d)}{d}X + R_d, \quad \mathcal{A}_d := \{n \in \mathcal{A} : d \mid n\},$$

where $X > 0$ and ω is a non-negative multiplicative function satisfying

$$(\Omega_1) \quad 0 \leq \omega(p) < p \quad (p \in P).$$

Assume that the sieve problem has dimension $\kappa > 0$, in the sense that

$$\sum_{\substack{t \leq p < z \\ p \in P}} \frac{\omega(p) \log p}{p} = \kappa \log \frac{z}{t} + O(1)$$

uniformly for $2 \leq t \leq z$. Then, we have

$$S(\mathcal{A}, P, z) \leq X e^{\gamma \kappa} \Gamma(\kappa + 1) \prod_{\substack{p < z \\ p \in P}} \left(1 - \frac{\omega(p)}{p}\right) \left(1 + O\left(\frac{1}{\log z}\right)\right) + \sum_{\substack{d < z^2 \\ d \mid P(z)}} 3^{\nu(d)} |R_d|, \quad (13)$$

where γ is Euler’s constant, Γ denotes the gamma function, and

$$\nu(d) := \sum_{p \mid d} 1$$

is the number of distinct prime divisors of d . The implied constant may depend on κ and on the constants in the sieve-dimension hypothesis.

Lemma 3 (Euler product for a sieve of dimension k). *Let ω be the function defined in Lemma 2. Assume that the corresponding sieve problem has dimension k . Then*

$$V(z) := \prod_{p < z} \left(1 - \frac{\omega(p)}{p}\right) \asymp_k \frac{1}{(\log z)^k}.$$

Proof. Since $\omega(p) \ll_k 1$, we have, for all sufficiently large primes p ,

$$\log \left(1 - \frac{\omega(p)}{p}\right) = -\frac{\omega(p)}{p} + O_k \left(\frac{1}{p^2}\right).$$

Hence, by partial summation and the sieve-dimension hypothesis, we have

$$\begin{aligned} \log V(z) &= \sum_{p < z} \log \left(1 - \frac{\omega(p)}{p}\right) \\ &= -\sum_{p < z} \frac{\omega(p)}{p} + O_k \left(\sum_p \frac{1}{p^2}\right) \\ &= -k \log \log z + O_k(1). \end{aligned}$$

Exponentiating the last estimate yields

$$V(z) \asymp_k \frac{1}{(\log z)^k}.$$

□

The following result is an application of Equation (13) to the sequence $\mathcal{A}_p$, when $\mathcal{A}$ is the sequence of shifted primes.

Lemma 4. *Let $\mathcal{A}$ be the sequence $\{p+a\}$, defined in Theorem 2, and put $z = x^{1/5}$. Then, uniformly for $z < p \leq x^{1/2}$,*

$$S(\mathcal{A}_p, P, z) \ll \frac{x}{\varphi(p)(\log x)^2}. \quad (14)$$

Proof. Set

$$M := \left\lfloor \frac{x+a}{p} \right\rfloor$$

and consider the polynomial

$$F_p(m) := m(pm - a),$$

where m satisfies

$$m \leq M, \quad (m, P(z)) = 1, \quad pm - a \in \mathbb{P}.$$

If $pm - a \geq z$, then also

$$(pm - a, P(z)) = 1.$$

Moreover, since $p > z$ and $pm - a \in \mathbb{P}$, the exceptional condition $pm - a < z$ is equivalent to

$$0 < pm - a < z.$$

Consequently,

$$\frac{a}{p} < m < \frac{a+z}{p}.$$

Since this interval has length $z/p < 1$, it follows that

$$\#\{m : pm - a < z\} = O(1).$$

If q denotes a prime, we can write

$$\mathcal{A} = \{q + a : q \leq x, q \in \mathbb{P}\}.$$

Thus, an element $n = q + a$ belongs to $\mathcal{A}_p$ if

$$p \mid q + a.$$

In that case, there exists an integer m such that

$$q + a = pm, \quad \text{that is,} \quad q = pm - a.$$

Therefore, we have

$$S(\mathcal{A}_p, P, z) = \sum_{\substack{m \leq M \\ pm - a \in \mathbb{P}}} f_z(pm) = \sum_{\substack{m \leq M \\ pm - a \in \mathbb{P}}} f_z(m).$$

Hence, we obtain the following relation,

$$\begin{aligned} S(\mathcal{A}_p, P, z) &= \#\{m \leq M : pm - a \in \mathbb{P}, (m, P(z)) = 1\} \\ &\leq \#\{m \leq M : (F_p(m), P(z)) = 1\} + O_a(1). \end{aligned}$$

Here, the $O_a(1)$ term only appears if $pm - a < z$.

For every square-free integer $d \mid P(z)$, define

$$\rho_p(d) := \#\{m \bmod d : F_p(m) \equiv 0 \pmod{d}\}.$$

Then

$$\#\{m \leq M : d \mid F_p(m)\} = \frac{\rho_p(d)}{d} M + R_d, \quad |R_d| \leq \rho_p(d).$$

Since $p > z$, we have $p \neq \ell$ for every prime $\ell < z$. Consequently,

$$\rho_p(\ell) = \begin{cases} 1, & \ell \mid a, \\ 2, & \ell \nmid a. \end{cases}$$

The function ρ_p is multiplicative on square-free integers and satisfies

$$\rho_p(d) \leq 2^{\nu(d)}$$

and

$$\sum_{w \leq \ell < z} \frac{\rho_p(\ell) \log \ell}{\ell} = 2 \log \frac{z}{w} + O_a(1).$$

Therefore, the sieve problem has dimension $\kappa = 2$.

Since $\Gamma(3) = 2$, taking $X = M$ and $\omega(\ell) = \rho_p(\ell)$, the first term in Equation (13) is

$$2Me^{2\gamma} \prod_{\ell < z} \left(1 - \frac{\rho_p(\ell)}{\ell}\right) \left(1 + O_a\left(\frac{1}{\log z}\right)\right).$$

Thus, we have

$$\#\{m \leq M : (F_p(m), P(z)) = 1\} \ll_a M \prod_{\ell < z} \left(1 - \frac{\rho_p(\ell)}{\ell}\right) + \sum_{\substack{d < z^2 \\ d|P(z)}} 3^{\nu(d)} |R_d|.$$

By Lemma 3, we have

$$\prod_{\ell < z} \left(1 - \frac{\rho_p(\ell)}{\ell}\right) \asymp \frac{1}{(\log z)^2}.$$

Furthermore,

$$\begin{aligned} \sum_{\substack{d < z^2 \\ d|P(z)}} 3^{\nu(d)} |R_d| &\leq \sum_{d < z^2} \mu^2(d) 3^{\nu(d)} \rho_p(d) \\ &\leq \sum_{d < z^2} \mu^2(d) 6^{\nu(d)} \ll z^2 (\log z)^5. \end{aligned}$$

Here we used

$$\mu^2(d) 6^{\nu(d)} \leq \tau_6(d),$$

where $\tau_k(n)$ is the generalized divisor function of order k , defined by

$$\tau_k(n) := \sum_{d_1 d_2 \cdots d_k = n} 1.$$

We have also applied the estimate

$$\sum_{d \leq y} \tau_k(d) \ll y (\log y)^{(k-1)}.$$

Therefore,

$$S(\mathcal{A}_p, P, z) \ll_a \frac{M}{(\log z)^2} + z^2 (\log z)^5.$$

Since $p \leq x^{1/2}$, we have

$$M \gg x^{1/2}.$$

Taking $z = x^{1/5}$, we obtain

$$\frac{z^2(\log z)^5}{M/(\log z)^2} \ll \frac{pz^2(\log z)^7}{x} \ll x^{-1/10}(\log x)^7 = o(1).$$

Thus, the error term is absorbed into the main term, and

$$S(\mathcal{A}_p, P, x^{1/5}) \ll \frac{x}{p(\log x)^2}.$$

Finally, for p prime,

$$\frac{1}{p} \leq \frac{1}{\varphi(p)},$$

which concludes the proof. $\square$

We shall use this version of the Rosser–Iwaniec sieve, given in Corollary 5.1 of [6].

Lemma 5. *With the notation of Lemma 2, assume that the one-dimensional sieve condition*

$$\sum_{\substack{t \leq \ell < z \\ \ell \in P}} \frac{\omega(\ell) \log \ell}{\ell} \leq \log \frac{z}{t} + O(1)$$

holds uniformly for $2 \leq t \leq z$. Define

$$s := \frac{\log y}{\log z},$$

where $y \geq z$ is the level of the Rosser–Iwaniec sieve. Then there exists a constant $C > 0$, depending at most on the constant occurring in the preceding sieve condition, such that

$$S(\mathcal{A}, P, z) \leq XV(z) \left\{ F(s) + Ce^{-s}(\log y)^{-1/3} \right\} + \sum_{\substack{d < y \\ d|P(z)}} |R_d|,$$

and

$$S(\mathcal{A}, P, z) \geq XV(z) \left\{ f(s) - Ce^{-s}(\log y)^{-1/3} \right\} - \sum_{\substack{d < y \\ d|P(z)}} |R_d|.$$

Here F and f are the standard upper and lower linear-sieve functions, respectively, determined by

$$F(s) = \frac{2e^\gamma}{s} \quad (0 < s \leq 3),$$

$$f(s) = 0 \quad (0 < s \leq 2),$$

and by the differential–difference relations

$$(sF(s))' = f(s-1), \quad (sf(s))' = F(s-1).$$

In particular, for $2 \leq s \leq 4$,

$$f(s) = \frac{2e^\gamma}{s} \log(s-1).$$

Lemma 6. *Let $\mathcal{A}$ be the sequence $\{p+a\}$, defined in Theorem 2. There exist two positive constants $C_1 < C_2$ such that*

$$\frac{C_1 x}{(\log x)^2} < S(\mathcal{A}, P, x^{1/5}) < \frac{C_2 x}{(\log x)^2}. \quad (15)$$

Furthermore, Equation (7) is satisfied and, for p in the range $x^{1/5} < p \leq x^{1/2}$, we have

$$S(\mathcal{A}_p, P, x^{1/5}) \ll \frac{1}{\varphi(p)} S(\mathcal{A}, P, x^{1/5}). \quad (16)$$

Proof. The bound

$$S(\mathcal{A}, P, x^{1/5}) \gg \frac{x}{(\log x)^2}$$

is deduced in [6], Example 3 of Corollary 5.1, as an application of Lemma 5 to the sequence of twin primes. This result can be generalized to every even integer $|a| \geq 2$, with the implicit constant depending on a . To deduce the upper bound, we apply the corresponding bound in the same Lemma. Taking

$$X = \pi(x), \quad z = x^{1/5}, \quad y = x^{1/2}(\log x)^{-c_2},$$

and

$$\omega(p) = \begin{cases} \frac{p}{p-1}, & p \nmid a, \\ 0, & p \mid a, \end{cases}$$

we have

$$s = \frac{\log y}{\log z} = \frac{5}{2} - 5c_2 \frac{\log \log x}{\log x} = \frac{5}{2} + o(1).$$

Thus,

$$F(s) = \frac{2e^\gamma}{s} = \frac{4e^\gamma}{5} + O\left(\frac{\log \log x}{\log x}\right).$$

The sum over R_d is bounded by the Bombieri–Vinogradov theorem. We have

$$\sum_{\substack{d < y \\ d \mid P(z)}} |R_d| \ll \frac{x}{(\log x)^3}.$$

Hence, there exists a constant K_a , depending on a , such that

$$S(\mathcal{A}, P, x^{1/5}) \leq \left(K_a + O_a \left((\log x)^{-1/3} \right) \right) \frac{x}{(\log x)^2},$$

which concludes the proof of Equation (15).

Finally, Equation (16) is an immediate consequence of Equations (14) and (15). $\square$

3. Proofs of the Main Results

Proof of Theorem 1. Recalling Equation (1), we have

$$S_0(A, x^{1/5}) = S(A, P, x^{1/5}) - \sum_{x^{1/5} < p \leq x^{1/2}} S(A_p, P, x^{1/5}). \quad (17)$$

The key property of S_0 is its ability to estimate the number of primes in A that remain after sieving. Let $m \in A$ be an element surviving the sieve, that is, contributing to $S(A, P, z)$. Let $w(m)$ denote the contribution of m to S_0 . Suppose first that m is a prime p_1 . Then the following statements hold:

- If $m > x^{1/2}$, its contribution to S_0 is $w(m = p_1) = 1$;
- If $m \leq x^{1/2}$, its contribution to S_0 is $w(m = p_1) = 0$.

If $m = p_1 p_2$ has two prime factors, then at least one of them is less than $x^{1/2}$ and contributes to the second term in S_0 . Thus, $w(m = p_1 p_2) \leq 0$. If m has more than two prime factors, then at least two of them are less than $x^{1/2}$, again implying that $w(m) \leq 0$. Hence, $w(m) \leq 0$ whenever m is not prime, and therefore

$$S_0(A, x^{1/5}) \leq \#\{n \in A : n = p\}. \quad (18)$$

Recalling Equations (6) and (17), we can write

$$S_0(A, x^{1/5}) = S(A, P, x^{1/5}) - \sum_{x^{1/5} < p \leq x^{1/2}} \frac{1}{\varphi(p)} S(A, P, x^{1/5}) - \sum_{x^{1/5} < p \leq x^{1/2}} \Delta(A_p).$$

By Equation (11), we have

$$S(A, P, x^{1/5}) - \sum_{x^{1/5} < p \leq x^{1/2}} \frac{1}{\varphi(p)} S(A, P, x^{1/5}) > (1 - 0.92) S(A, P, x^{1/5}).$$

Therefore,

$$S_0(A, x^{1/5}) \geq 0.08 S(A, P, x^{1/5}) - \sum_{x^{1/5} < p \leq x^{1/2}} \Delta(A_p). \quad (19)$$

Substituting Equation (19) into Equation (18), we obtain Equation (5), with $K \approx 0.08$. Equation (11) shows that $z = x^{1/5}$ is close to the lower bound required to ensure that S_0 is positive.

To establish Equation (8), we note that Equation (7) implies

$$\Delta(A_p) = O\left(\frac{1}{\varphi(p)}\right)S(A, P, x^{1/5}).$$

As a consequence of Equation (12), we have

$$\sum_{x^{1/2}(\log x)^{-B} < p \leq x^{1/2}} \frac{1}{\varphi(p)} = o(1).$$

Hence,

$$\sum_{x^{1/5} < p \leq x^{1/2}} \Delta(A_p) = \sum_{x^{1/5} < p \leq x^{1/2}(\log x)^{-B}} \Delta(A_p) + o(1)S(A, P, x^{1/5}).$$

This shows that Equation (8) holds, and concludes the proof of Theorem 1. $\square$

Proof of Theorem 2. In order to obtain the result, we apply the bound stated in Equation (15) to Equation (8). This yields the first term of Equation (9) with $C = KC_1$.

Equation (10) for $\Delta(\mathcal{A}_p)$ follows from the specific shape of the sequence $\mathcal{A}$. From Equation (2) we obtain

$$S(\mathcal{A}, P, z) = \sum_{p \leq x} f_z(p + a). \quad (20)$$

Since $n \in \mathcal{A}_q$ implies $n \equiv -a \pmod{q}$, it follows from Equation (3) that

$$S(\mathcal{A}_q, P, z) = \sum_{\substack{p \leq x \\ p \equiv -a \pmod{q}}} f_z(p + a). \quad (21)$$

Substituting Equations (20) and (21) into Equation (6), we obtain

$$\Delta(\mathcal{A}_p) = \left(\sum_{\substack{n \leq x \\ n \equiv -a \pmod{p}}} \mathbf{1}_{\mathbb{P}}(n) f_{x^{1/5}}(n + a) - \frac{1}{\varphi(p)} \sum_{n \leq x} \mathbf{1}_{\mathbb{P}}(n) f_{x^{1/5}}(n + a) \right).$$

To derive Equation (10), we note that condition $(n, p) = 1$ introduces an admissible difference between $\Delta(\mathcal{A}_p)$ and $\tilde{\Delta}(\mathcal{A}_p)$, as a consequence of the following relation:

$$\frac{1}{\varphi(p)} \sum_{n \leq x} \mathbf{1}_{\mathbb{P}}(n) f_{x^{1/5}}(n + a) = \frac{1}{\varphi(p)} \sum_{\substack{n \leq x \\ (n, p) = 1}} \mathbf{1}_{\mathbb{P}}(n) f_{x^{1/5}}(n + a) + O\left(\frac{1}{\varphi(p)}\right).$$

Thus, summing the term $O\left(\frac{1}{\varphi(p)}\right)$ over the required range of primes and applying Equation (11), we obtain a total error of $O(1)$, which is absorbed into the main term in Equation (9). $\square$

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