



NEW CONGRUENCES FOR $[j, k]$ -OVERPARTITION PAIRS WITH EVEN PARTS DISTINCT

Adam Paksok

*Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh,
Arunachal Pradesh, India
adam.paksok@rgu.ac.in*

Nipen Saikia¹

*Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh,
Arunachal Pradesh, India
nipennak@yahoo.com*

Received: 5/13/25, Revised: 3/26/26, Accepted: 8/3/26, Published: 10/7/26

Abstract

Let $\overline{bped}_{j,k}(n)$ denote the number of $[j, k]$ -overpartition pairs of the positive integer n with even parts distinct in which the first occurrence of each distinct part congruent to j modulo k may be overlined. Mahadeva Naika et al. (2023) proved many infinite families of congruences modulo small powers of 2 for the particular cases $\overline{bped}_{3,3}(n)$ and $\overline{bped}_{3,6}(n)$. As an extension to the above work, in this paper we prove some new infinite families of congruences for $\overline{bped}_{3,3}(n)$ and $\overline{bped}_{3,6}(n)$, which also generalize some of the known results.

1. Introduction

A *partition* of a positive integer n is a sequence of positive integers $x_1 \geq x_2 \geq \cdots \geq x_k$ with $n = \sum_{i=1}^k x_i$. The positive integers x_i are called *parts* of the partition. For example, $n = 3$ has three partitions, namely, 3 , $2 + 1$, and $1 + 1 + 1$. If $p(n)$ denotes the number of partitions of a positive integer n , then the generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}},$$

where for any complex numbers a and q ,

$$(a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| < 1. \quad (1)$$

DOI: 10.5281/zenodo.23217125

¹Corresponding author.

For any positive integer h , we will use the notation $f_h := (q^h; q^h)_\infty$ for brevity. Ramanujan [1, 11, 12] established the following beautiful congruences for the partition function $p(n)$:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7} \end{aligned}$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

An *overpartition* of a positive integer n is a partition of n wherein the first occurrence of a part may be overlined. For example, $n = 3$ has eight overpartitions, namely, $3, \bar{3}, 2 + 1, \bar{2} + 1, 2 + \bar{1}, \bar{2} + \bar{1}, 1 + 1 + 1$ and $\bar{1} + 1 + 1$. The overpartition function is usually denoted by $\bar{p}(n)$, which was first studied by Corteel and Lovejoy [2]. The generating function for $\bar{p}(n)$ is given by

$$\sum_{n \geq 0} \bar{p}(n) q^n = \frac{(q^2; q^2)_\infty}{(q; q)_\infty^2}.$$

The arithmetic properties of $\bar{p}(n)$ have been studied by many authors, for example see [3, 4, 6, 10, 13, 14].

For positive integers j and k such that $k \geq j \geq 1$, a $[j, k]$ -overpartition of n is a partition in which the first occurrence of each distinct part that is congruent to j modulo k may be overlined. If $\bar{p}_{j,k}(n)$ counts such overpartitions of n with $\bar{p}_{j,k}(0) = 1$, then its generating function is given by

$$\sum_{n=0}^{\infty} \bar{p}_{j,k}(n) q^n = \frac{(-q^j; q^k)_\infty}{(q; q)_\infty}.$$

Mahadeva Naika et al. [7] obtained many infinite families of congruences modulo powers of 2 for $\bar{p}_{j,k}(n)$.

In [8], the authors studied another partition function, denoted by $\overline{ped}_{j,k}(n)$, which counts the number of $[j, k]$ -overpartitions of n such that the even parts are distinct and the first occurrence of each part that is congruent to j modulo k may be overlined. The generating function of $\overline{ped}_{j,k}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{ped}_{j,k}(n) q^n = \frac{(q^4; q^4)_\infty (-q^j; q^k)_\infty}{(q; q)_\infty}.$$

The authors in [9] studied the partition function $\overline{bped}_{j,k}(n)$, which denotes the number of $[j, k]$ -overpartition pairs of a positive integer n with distinct even parts,

in which the first occurrence of each distinct part that is congruent to j modulo k may be overlined. The generating function of $\overline{bped}_{j,k}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{bped}_{j,k}(n) q^n = \frac{(q^4; q^4)_{\infty}^2 (-q^j; q^k)_{\infty}^2}{(q; q)_{\infty}^2}. \quad (2)$$

They also proved some infinite families of congruences modulo small powers of 2 for the particular cases $\overline{bped}_{3,3}(n)$ and $\overline{bped}_{3,6}(n)$. As an extension to the work of [9], in this paper we prove some new infinite families of congruences for $\overline{bped}_{3,3}(n)$ and $\overline{bped}_{3,6}(n)$, which generalize some of the known results. To prove our results, we will use the method of elementary q -series identities. The following q -identities are crucial in our proofs:

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}, \quad (3)$$

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} \quad (4)$$

and

$$f_{pm}^{p^{r-1}} \equiv f_m^{p^r} \pmod{p^r}, \quad (5)$$

where r and m are positive integers and p is a prime. For Identities (3) and (4), see Equations (1.61) and (1.7.1) of [5], respectively. Congruence (5) is an easy consequence of the binomial theorem and Equation (1).

2. Congruences for $\overline{bped}_{j,k}(n)$

This section is devoted to proving congruences for $\overline{bped}_{j,k}(n)$. First, we define the *Legendre symbol* $\left(\frac{a}{p}\right)$. If p is any odd prime and a is any integer relatively prime to p , then

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue of } p, \\ -1, & \text{if } a \text{ is a quadratic non-residue of } p. \end{cases}$$

Theorem 1. *Let $p > 5$ be a prime with $\left(\frac{-6}{p}\right) = -1$. Then for all integers $k, m, \alpha, \beta \geq 0$ with $p \nmid m$, we have*

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} \cdot p^{2k+1} m + \frac{7 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{16} \quad (6)$$

and

$$\overline{bped}_{3,3} \left(12 \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{7 \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{8}. \quad (7)$$

Proof. From [9, Equation (3.27)], we note that

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}n + \frac{7 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} \right) q^n \equiv 8f_1^7 \pmod{16}. \quad (8)$$

Using Equations (3) and (4), we see that the right hand side of Equation (8) becomes

$$8f_1^7 = 8f_1f_1^6 \equiv 8f_1f_2^3 \equiv 8 \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} q^{\frac{k(3k-1)}{2} + l(l+1)} \pmod{16}.$$

We need to know whether n can be written as $\frac{k(3k-1)}{2} + l(l+1)$ for some integers k and l . This is equivalent to $24n + 7 = (6k-1)^2 + 6(2l+1)^2$. Thus, if $24n + 7$ is not of form $x^2 + 6y^2$, then

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}n + \frac{7 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} \right) \equiv 0 \pmod{16}. \quad (9)$$

Also, $\left(\frac{-6}{p}\right) = -1$ implies that $\nu_p(N)$ is even if N is of the form $x^2 + 6y^2$, where $\nu_p(N)$ equals the exponent of the highest power of p dividing N . Taking $n = p^{2k+1}m + \frac{7(p^{2k+2} - 1)}{24}$, it follows that $24n + 7 = p^{2k+1}(24m + 7p)$, which cannot be written in the form $x^2 + 6y^2$. For such n , we have

$$\begin{aligned} 4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}n + \frac{7 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} \\ = 4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{7 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2}. \end{aligned}$$

Thus, Equation (9) implies

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{7 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{16}.$$

This completes the proof of Equation (6). The proof of Equation (7) is similar to the proof of Equation (6), which can be proved by using the following identity [9, Equation (3.42)], namely

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(12 \cdot 5^{2\beta}n + \frac{7 \cdot 5^{2\beta} - 1}{2} \right) q^n \equiv 4f_1^7 \pmod{8}.$$

□

Theorem 2. Let $p \geq 5$ be a prime with $\left(\frac{-2}{p}\right) = -1$. Then for all integers $k, m, \alpha, \beta \geq 0$ with $p \nmid m$, we have

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{11 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{16}, \quad (10)$$

$$\overline{bped}_{3,3} \left(12 \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{11 \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{4}, \quad (11)$$

$$\overline{bped}_{3,6} (3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2k+1}(24m + 11p)) \equiv 0 \pmod{32}, \quad (12)$$

and

$$\overline{bped}_{3,6} (3 \cdot 5^{2\beta} \cdot p^{2k+1}(24m + 11p)) \equiv 0 \pmod{32}. \quad (13)$$

Proof. From [9, Equation (3.25)], we have

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}n + \frac{11 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} \right) q^n \equiv 8f_2f_3^3 \pmod{16}. \quad (14)$$

Using Equations (3) and (4) in Equation (14), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}n + \frac{11 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} \right) q^n \\ \equiv 8 \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} q^{k(3k-1)+3l(l+1)/2} \pmod{16}. \end{aligned}$$

We note that $n = k(3k-1) + \frac{3l(l+1)}{2}$ is equivalent to $24n+11 = 2(6k-1)^2 + (6l+3)^2$. Thus, if $24n+11$ is not of the form $2x^2 + y^2$, then

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}n + \frac{11 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} \right) \equiv 0 \pmod{16}. \quad (15)$$

Also, $\left(\frac{-2}{p}\right) = -1$ implies that $\nu_p(N)$ is even if N is of the form $2x^2 + y^2$. Taking $n = p^{2k+1}m + \frac{11(p^{2k+2} - 1)}{24}$, it follows that $24n+11 = p^{2k+1}(24m+11p)$ cannot be written in the form $2x^2 + y^2$. For such n , we have

$$\begin{aligned} 4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}n + \frac{11 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} &= 4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} \cdot p^{2k+1}m \\ &\quad + \frac{11 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2}. \end{aligned}$$

Therefore, Equation (15) gives

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{11 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{16}.$$

This completes the proof of Equation (10). The proofs of Equations (11), (12), and (13) are similar to the proof of Equation (10) and can be proved by using the following identities [9, Equations (3.56), (4.28), and (4.42)], respectively:

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(12 \cdot 5^{2\beta}n + \frac{11 \cdot 5^{2\beta} - 1}{2} \right) q^n &\equiv 2f_2f_3^3 \pmod{4}, \\ \sum_{n=0}^{\infty} \overline{bped}_{3,6} (8 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}n + 11 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}) q^n &\equiv 16f_2f_3^3 \pmod{32}, \end{aligned}$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6} (72 \cdot 5^{2\beta}n + 33 \cdot 5^{2\beta}) q^n \equiv 16f_2f_3^3 \pmod{32}.$$

□

Theorem 3. Let $p \geq 5$ be a prime with $\left(\frac{-2}{p}\right) = -1$. Then for all integers $k, m, \alpha, \beta \geq 0$ with $p \nmid m$, we have

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{19 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{16}, \quad (16)$$

$$\overline{bped}_{3,3} \left(12 \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{19 \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{8}, \quad (17)$$

$$\overline{bped}_{3,6} (3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2k+1}(24m + 19p)) \equiv 0 \pmod{32}, \quad (18)$$

and

$$\overline{bped}_{3,6} (3 \cdot 5^{2\beta} \cdot p^{2k+1}(24m + 19p)) \equiv 0 \pmod{32}. \quad (19)$$

Proof. From [9, Equation (3.29)], we note that

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}n + \frac{19 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} \right) q^n \equiv 8f_1f_6^3 \pmod{16}, \quad (20)$$

Using Equations (3) and (4) in Equation (20), we obtain

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}n + \frac{19 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} \right) q^n$$

$$\equiv 8 \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} q^{k(3k-1)/2+6l(l+1)/2} \pmod{16}.$$

We see that $n = \frac{k(3k-1)}{2} + \frac{6l(l+1)}{2}$ for some integers k and l . This is equivalent to $24n + 19 = (6k-1)^2 + 2(6l+3)^2$. Thus, if $24n + 19$ is not of the form $x^2 + 2y^2$, then

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} n + \frac{19 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} \right) \equiv 0 \pmod{16}. \quad (21)$$

Also, $\left(\frac{-2}{p}\right) = -1$ implies that $\nu_p(N)$ is even if N is of the form $x^2 + 2y^2$. Taking $n = p^{2k+1}m + \frac{19(p^{2k+2} - 1)}{24}$, it follows that $24n + 19 = p^{2k+1}(24m + 19p)$ cannot be written in the form $x^2 + 2y^2$. For such n , we have

$$\begin{aligned} 4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} n + \frac{19 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} - 1}{2} \\ = 4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{19 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2}. \end{aligned}$$

So by Equation (21), we obtain

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{19 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{16},$$

which completes the proof of Equation (16). The proofs of Equations (17), (18), and (19) are identical to the proof of Equation (16) and can be proved by using the following identities [9, Equations (3.44), (4.25), and (4.40)], respectively:

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(12 \cdot 5^{2\beta} n + \frac{19 \cdot 5^{2\beta} - 1}{2} \right) q^n &\equiv 4f_1 f_6^3 \pmod{8}, \\ \sum_{n=0}^{\infty} \overline{bped}_{3,6} (8 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} n + 19 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}) q^n &\equiv 16f_1 f_6^3 \pmod{32}, \end{aligned}$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6} (72 \cdot 5^{2\beta} n + 57 \cdot 5^{2\beta}) q^n \equiv 16f_1 f_6^3 \pmod{32}.$$

□

Theorem 4. Let $p \geq 3$ be a prime with $\left(\frac{-1}{p}\right) = -1$. Then for all $k, m, \alpha, \gamma \geq 0$, such that $p \nmid m$, we have

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+2} \cdot 7^{2\gamma} \cdot p^{2k+1}m + \frac{5 \cdot 3^{4\alpha+1} \cdot 7^{2\gamma} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{8}, \quad (22)$$

$$\overline{bped}_{3,6} (3^{4\alpha+2} \cdot 7^{2\gamma} \cdot p^{2k+1}(24m + 5p)) \equiv 0 \pmod{16}, \quad (23)$$

and

$$\overline{bped}_{3,6} (3 \cdot 7^{2\gamma} \cdot p^{2k+1} (24m + 5p)) \equiv 0 \pmod{16}. \quad (24)$$

Proof. From [9, Equation (3.38)], we have

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+2} \cdot 7^{2\gamma} n + \frac{5 \cdot 3^{4\alpha+1} \cdot 7^{2\gamma} - 1}{2} \right) q^n \equiv 4f_1^5 \pmod{8}. \quad (25)$$

Employing Equation (5) in Equation (25) and then using Equations (3) and (4), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+2} \cdot 7^{2\gamma} n + \frac{5 \cdot 3^{4\alpha+1} \cdot 7^{2\gamma} - 1}{2} \right) q^n \\ \equiv 4 \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} q^{k(3k-1)/2 + 4l(3l-1)/2} \pmod{8}. \end{aligned}$$

We note that $n = \frac{k(3k-1)}{2} + \frac{4l(3l-1)}{2}$ is equivalent to $24n + 5 = (6k-1)^2 + (12l-2)^2$. Thus, if $24n + 5$ is not of the form $x^2 + y^2$, then

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+2} \cdot 7^{2\gamma} n + \frac{5 \cdot 3^{4\alpha+1} \cdot 7^{2\gamma} - 1}{2} \right) \equiv 0 \pmod{8}. \quad (26)$$

Also, $\left(\frac{-1}{p}\right) = -1$ implies that $\nu_p(N)$ is even if N is of the form $x^2 + y^2$. Taking $n = p^{2k+1}m + \frac{5(p^{2k+2} - 1)}{24}$, it follows that $24n + 5 = p^{2k+1}(24m + 5p)$, which cannot be written in the form $x^2 + y^2$. For such n , we have

$$\begin{aligned} 4 \cdot 3^{4\alpha+2} \cdot 7^{2\gamma} n + \frac{5 \cdot 3^{4\alpha+1} \cdot 7^{2\gamma} - 1}{2} \\ = 4 \cdot 3^{4\alpha+2} \cdot 7^{2\gamma} \cdot p^{2k+1}m + \frac{5 \cdot 3^{4\alpha+1} \cdot 7^{2\gamma} \cdot p^{2\alpha+2} - 1}{2}. \end{aligned}$$

So, Equation (22) follows easily from Equation (26). The proofs of Equations (23) and (24) are similar to that of Equation (22) and can be proved by using the following identities [9, Equations (4.52) and (4.56)], respectively:

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6} (8 \cdot 3^{4\alpha+3} \cdot 7^{2\gamma} n + 5 \cdot 3^{4\alpha+2} \cdot 7^{2\gamma}) q^n \equiv 8f_1^5 \pmod{16}$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6} (72 \cdot 7^{2\gamma} n + 15 \cdot 7^{2\gamma}) q^n \equiv 8f_1^5 \pmod{16}.$$

□

Theorem 5. *Let $p \geq 5$ be a prime with $\left(\frac{-3}{p}\right) = -1$. Then for all integers $k, m, \alpha, \beta \geq 0$ with $p \nmid m$, we have*

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{13 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{8}, \quad (27)$$

$$\overline{bped}_{3,6} (3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2k+1}(24m + 13p)) \equiv 0 \pmod{16}, \quad (28)$$

and

$$\overline{bped}_{3,6} (3 \cdot 5^{2\beta} \cdot p^{2k+1}(24m + 13p)) \equiv 0 \pmod{16}. \quad (29)$$

Proof. From [9, Equation (3.40)], we have

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}n + \frac{13 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} - 1}{2} \right) q^n \equiv 4f_1^{13} \pmod{8}. \quad (30)$$

Employing Equation (5) in Equation (30) and then using Equations (3) and (4), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}n + \frac{13 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} - 1}{2} \right) q^n \\ \equiv 4 \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} q^{k(3k-1)/2+2l(l+1)} \pmod{8}. \end{aligned}$$

We see that $n = \frac{k(3k-1)}{2} + 2l(l+1)$ is equivalent to $24n+13 = (6k-1)^2 + 3(4l+2)^2$. Thus, if $24n+13$ is not of the form $x^2 + 3y^2$, then

$$\overline{bped}_{3,3} \left(4 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}n + \frac{13 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} - 1}{2} \right) \equiv 0 \pmod{8}. \quad (31)$$

Also, $\left(\frac{-3}{p}\right) = -1$ implies that $\nu_p(N)$ is even if N is of the form $x^2 + 3y^2$. Taking $n = p^{2k+1}m + \frac{13(p^{2k+2} - 1)}{24}$, it follows that $24n+13 = p^{2k+1}(24m+13p)$, which cannot be written in the form $x^2 + 3y^2$. For such n , we have

$$\begin{aligned} 4 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}n + \frac{13 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} - 1}{2} \\ = 4 \cdot 3^{4\alpha+2} \cdot 5^{2\beta} \cdot p^{2k+1}m + \frac{13 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} \cdot p^{2\alpha+2} - 1}{2}. \end{aligned}$$

Thus, Equation (27) follows immediately from Equation (31). The proofs of Equations (28) and (29) are similar to that of Equation (27) and can be proved using the following identities [9, Equations (4.50) and (4.54)], respectively:

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6} (8 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} n + 13 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}) q^n \equiv 8f_1^{13} \pmod{16}$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6} (72 \cdot 5^{2\beta} n + 39 \cdot 5^{2\beta}) q^n \equiv 8f_1^{13} \pmod{16}.$$

□

Theorem 6. *Let $p \geq 3$ be a prime with $\left(\frac{-1}{p}\right) = -1$. Then for all integers $k, m, \alpha, \beta \geq 0$ with $p \nmid m$, we have*

$$\overline{bped}_{3,6} (3^{4\alpha+2} \cdot p^{2k+1}(12m + 5p)) \equiv 0 \pmod{16} \quad (32)$$

and

$$\overline{bped}_{3,6} (3 \cdot p^{2k+1}(12m + 5p)) \equiv 0 \pmod{16}. \quad (33)$$

Proof. From [9, Equation(4.61)], we note that

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6} (4 \cdot 3^{4\alpha+3} n + 5 \cdot 3^{4\alpha+2}) q^n \equiv 8 \frac{f_2 f_3^3}{f_1} \pmod{16}. \quad (34)$$

Employing Equation (5) in Equation (34) and then using Equations (3) and (4), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{bped}_{3,6} (4 \cdot 3^{4\alpha+3} n + 5 \cdot 3^{4\alpha+2}) q^n \\ \equiv 8 \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} q^{k(3k-1)/2+3l(l+1)/2} \pmod{16}. \end{aligned}$$

We see that $n = \frac{k(3k-1)}{2} + \frac{3l(l+1)}{2}$ can be equivalently expressed as $24n + 10 = (6k-1)^2 + (6l+3)^2$. Thus, if $24n + 10$ is not of the form $x^2 + y^2$, then

$$\overline{bped}_{3,6} (4 \cdot 3^{4\alpha+3} n + 5 \cdot 3^{4\alpha+2}) \equiv 0 \pmod{16}. \quad (35)$$

Also, $\left(\frac{-1}{p}\right) = -1$ implies that $\nu_p(N)$ is even if N is of the form $x^2 + y^2$. Taking $n = p^{2k+1}m + \frac{5(p^{2k+2} - 1)}{12}$, it follows that $24n + 10 = p^{2k+1}(24m + 10p)$, which cannot be written in the form $x^2 + y^2$. For such n , we have

$$4 \cdot 3^{4\alpha+3} n + 5 \cdot 3^{4\alpha+2} = 3^{4\alpha+2} \cdot p^{2k+1}(12m + 5p).$$

So Equation (32) follows from Equation (35). Similarly, we can prove Equation (33) by using the following identity from [9, Equation (4.66)], namely

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6} (36n + 15) q^n \equiv 8 \frac{f_2 f_3^3}{f_1} \pmod{16}.$$

□

Acknowledgement. The authors gratefully acknowledge the Editor and the anonymous reviewers for carefully reading our manuscript and providing valuable suggestions, which greatly improved this manuscript. The first author also thanks University Grants Commission (UGC) of India for supporting her research work through CSIR-UGC Junior Research Fellowship (JRF) vide NTA-Ref. No.: 221610074648, dated 29/11/2022.

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