



A NEW EXPRESSION FOR THE BERNOULLI NUMBERS AND ITS APPLICATIONS

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Abstract

This paper shows that a finite discrete convolution involving Stirling numbers of both kinds and harmonic numbers can be expressed in terms of the Bernoulli numbers. As applications of this expression, the linear recurrence relation for the Bernoulli numbers given by Agoh is reproved, and a new recurrence relation for the Bernoulli numbers is obtained. Furthermore, it is shown that a cumulative sum of the Bernoulli numbers can be written in terms of the Bernoulli and di-Bernoulli numbers. Finally, congruences for the sums of the Bernoulli and Euler numbers are established.

1. Introduction

The Bernoulli numbers, denoted by B_n , first arose in the context of computing sums of powers of positive integers in closed form, a topic that has been studied since antiquity. They may be defined by means of the generating function ([4])

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

It is easy to find the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, and $B_{2n+1} = 0$ for all $n \geq 1$. They also appear in the Taylor series expansions of the tangent and hyperbolic tangent functions, the Euler–Maclaurin summation formula, and certain representations of the Riemann zeta function (see, e.g., [4]). The

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Bernoulli numbers have consequently become one of the cornerstones of mathematical science, with numerous approaches available for their introduction, including generating functions, recurrence relations, integral representations, and explicit formulas. Among the numerous known explicit formulas, one of the most famous is Worpitzky's identity

$$\sum_{k=1}^n (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{k!}{k+1} = B_n,$$

which was later generalized to

$$\sum_{k=j}^n \frac{(-1)^{k-j}}{k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k \\ j \end{matrix} \right] = \binom{n-1}{j} \frac{B_{n-j}}{n-j}, \quad (1)$$

where $1 \leq j \leq n$ (cf. [15, p. 217]). Here $\left[\begin{matrix} n \\ k \end{matrix} \right]$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ denote the Stirling numbers of the first and second kind defined by

$$(x)_n = x(x-1) \cdots (x-n+1) = \sum_{k=0}^n (-1)^{n-k} \left[\begin{matrix} n \\ k \end{matrix} \right] x^k, \quad (2)$$

and

$$x^n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (x)_k,$$

respectively (see, for example, [15, Chapters 9 and 12]), and $\binom{n}{j}$ is the *binomial coefficient*.

In 2010, Mezö [11] extended Equation (1) and derived a formula for the Bernoulli polynomials by using the r -Whitney numbers. Motivated by this approach, Merca further showed that a finite discrete convolution involving different generalizations of the Stirling numbers of both kinds can be expressed in terms of the Bernoulli polynomials [12, 13]. In a recent paper [3], new explicit expressions for the Bernoulli numbers, namely,

$$\sum_{k=1}^n (-1)^{k-1} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (k-1)! H_k = B_{n-1}, \quad (3)$$

and

$$\sum_{k=2}^n (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (k-1)! H_{k-1} H_k = \frac{n+1}{2} B_{n-2}, \quad (4)$$

were introduced. Here

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is the n th harmonic number. The identities

$$(k-1)! = \begin{bmatrix} k \\ 1 \end{bmatrix} \text{ and } (k-1)!H_{k-1} = \begin{bmatrix} k \\ 2 \end{bmatrix}$$

motivate us to examine the closed form of the sum

$$\sum_{k=j}^n (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \begin{bmatrix} k \\ j \end{bmatrix} H_k.$$

As stated in Theorem 1, this sum gives rise to a new formula for the Bernoulli numbers. Several consequences of the main theorem are then explored. In particular, we provide an alternative proof of the linear recurrence for the Bernoulli numbers obtained by Agoh [1] and establish a new recurrence relation. We also derive a closed-form expression for the cumulative sum of the Bernoulli numbers, which is subsequently used to obtain congruences involving sums of Bernoulli and Euler numbers modulo an odd prime and its square. These applications are discussed in Section 3.

2. Main Result

In this section, we present our main result together with its proof.

Theorem 1 (Main Theorem). *For integers n and j with $0 \leq j \leq n$, we have*

$$\sum_{k=j}^n (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \begin{bmatrix} k \\ j \end{bmatrix} H_k = \left(\binom{n}{j} - 1 \right) \frac{B_{n-j}}{n-j}. \quad (5)$$

We note that when $n = j$, the right-hand side is understood in the limiting sense. Indeed,

$$\lim_{n \rightarrow j} \left(\binom{n}{j} - 1 \right) \frac{B_{n-j}}{n-j} = H_j,$$

which agrees with the left-hand side. For $j = 1$ and $j = 2$, Equation (5) reduces to Equations (3) and (4), respectively. Moreover, using the known relation

$$\begin{bmatrix} k \\ 3 \end{bmatrix} = \frac{1}{2!} (k-1)! \left[(H_{k-1})^2 - H_{k-1}^{(2)} \right],$$

where

$$H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$$

is the *generalized harmonic number*, we obtain the following special case of Equation (5).

$$\sum_{k=3}^n (-1)^{k-1} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (k-1)! \left[(H_{k-1})^2 - H_{k-1}^{(2)} \right] H_k = \frac{n^2 + 2}{3} B_{n-3}.$$

Proof of Theorem 1. Let

$$\sum_{k=j}^n (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k \\ j \end{matrix} \right] H_k = \mathcal{B}_{n,j},$$

where the symbol $\mathcal{B}_{n,j}$ is used solely as a formal notation for the closed-form expression of the above sum; it does not represent a new sequence or family of numbers. Using the well-known recurrence relation for the Stirling numbers of the second kind [15, Equation (9.1)]

$$\left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k + \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\},$$

we obtain that

$$\mathcal{B}_{n+1,j} = \sum_{k=j}^{n+1} (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k \\ j \end{matrix} \right] k H_k + \sum_{k=j}^{n+1} (-1)^{k-j} \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\} \left[\begin{matrix} k \\ j \end{matrix} \right] H_k. \quad (6)$$

Applying the well-known recurrence relation of the Stirling numbers of the first kind [15, Equation (12.3)]

$$\left[\begin{matrix} k+1 \\ j \end{matrix} \right] = \left[\begin{matrix} k \\ j \end{matrix} \right] k + \left[\begin{matrix} k \\ j-1 \end{matrix} \right],$$

together with the identity

$$H_{k+1} = H_k + \frac{1}{k+1},$$

and the fact that $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = 0$ for $k > n$, Equation (6) becomes

$$\begin{aligned} \mathcal{B}_{n+1,j} &= \sum_{k=j}^n (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k+1 \\ j \end{matrix} \right] H_k - \sum_{k=j}^n (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k \\ j-1 \end{matrix} \right] H_k \\ &\quad - \sum_{k=j-1}^n (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k+1 \\ j \end{matrix} \right] H_k - \sum_{k=j-1}^n (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k+1 \\ j \end{matrix} \right] \frac{1}{k+1}. \end{aligned}$$

Employing Equation (1) in the form

$$\sum_{k=j-1}^{n-1} \frac{(-1)^{k+1-j}}{k+1} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} \left[\begin{matrix} k+1 \\ j \end{matrix} \right] = \binom{n-1}{j} \frac{B_{n-j}}{n-j} \quad (1 \leq j \leq n),$$

we find that

$$\mathcal{B}_{n+1,j} = \mathcal{B}_{n,j-1} + \binom{n}{j} \frac{B_{n+1-j}}{n+1-j}.$$

From this reduction formula, we see that

$$\mathcal{B}_{n+1,j} = \mathcal{B}_{n-1,j-2} + \left(\binom{n}{j} + \binom{n-1}{j-1} \right) \frac{B_{n+1-j}}{n+1-j}.$$

Consecutive $j - 2$ applications yield

$$\mathcal{B}_{n+1,j} = \mathcal{B}_{n-j,0} + \frac{B_{n+1-j}}{n+1-j} \sum_{k=0}^{j-1} \binom{n-k}{j-k}. \quad (7)$$

Here, we have

$$\sum_{k=0}^{j-1} \binom{n-k}{j-k} = \binom{n+1}{j} - 1, \quad (8)$$

upon the use of [9, p. 7, Equation (1.49)]

$$\sum_{k=0}^n \binom{x+k}{k} = \binom{x+n+1}{n}.$$

Substituting Equation (8) into Equation (7), with the use of $\mathcal{B}_{n-j,0} = 0$, we arrive at

$$\mathcal{B}_{n+1,j} = \left(\binom{n+1}{j} - 1 \right) \frac{B_{n+1-j}}{n+1-j},$$

which is the desired result. $\square$

3. Applications of the Main Result

Multiplying both sides of Equation (5) by x^j and summing over j from 1 to $n - 1$, and then interchanging the order of summation, we obtain

$$\begin{aligned} \sum_{j=1}^n \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} x^{n-j} &= \sum_{j=1}^{n-1} \sum_{k=j}^n (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k \\ j \end{matrix} \right] H_k x^j \\ &= \sum_{j=1}^n \sum_{k=j}^n (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k \\ j \end{matrix} \right] H_k x^j - H_n x^n \\ &= \sum_{k=1}^n \sum_{j=1}^k (-1)^{k-j} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k \\ j \end{matrix} \right] H_k x^j - H_n x^n. \end{aligned}$$

Finally, applying the falling factorial expansion (2), we obtain

$$\sum_{j=1}^n \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} x^{n-j} = {}_H w_n^{(x)} - H_n x^n, \quad (9)$$

where n is a positive integer, x is a non-zero complex variable and

$${}_H w_n^{(x)} = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \binom{x}{k} k! H_k. \quad (10)$$

In this section, we list some special cases of Equation (9). First, we have the following linear identity which was also proved by a different method in [1].

Corollary 1. *For any positive integers n and m , we have*

$$\sum_{j=1}^n \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} m^{n-j} = m^n (H_m - H_n) - \sum_{j=1}^m \frac{(m-j)^n}{j}. \quad (11)$$

Proof. Setting $x = m \geq 1$ in Equation (9), we obtain

$$\sum_{j=1}^n \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} m^{n-j} = {}_H w_n^{(m)} - H_n m^n. \quad (12)$$

To examine ${}_H w_n^{(m)}$, we first use the generating function of the Stirling numbers of the second kind

$$\sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}, \quad (13)$$

and obtain

$$\sum_{n=1}^{\infty} {}_H w_n^{(m)} \frac{t^n}{n!} = \sum_{k=1}^m \binom{m}{k} H_k (e^t - 1)^k.$$

The equality

$$\sum_{k=1}^m \binom{m}{k} H_k (z - 1)^k = H_m z^m - \sum_{k=0}^{m-1} \frac{z^k}{m - k}$$

([2, Equation (2.4, iii)]) gives

$$\sum_{n=1}^{\infty} {}_H w_n^{(m)} \frac{t^n}{n!} = H_m \sum_{n=0}^{\infty} m^n \frac{t^n}{n!} - \sum_{n=0}^{\infty} \left(\sum_{k=0}^{m-1} \frac{k^n}{m - k} \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $t^n/n!$ yields

$${}_H w_n^{(m)} = H_m m^n - \sum_{j=1}^m \frac{(m-j)^n}{j}.$$

Inserting this equation in Equation (12) completes the proof of Equation (11). $\square$

Note that Equation (11) can also be written as

$$\sum_{j=1}^n (-1)^j \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} m^{n-j} = m^n \left(H_m - H_n + \frac{(n-1)}{m} \right) - \sum_{j=1}^m \frac{(m-j)^n}{j},$$

since $(-1)^n B_n = B_n$ for $n \geq 2$. In particular, we have

$$\sum_{j=1}^n (-1)^j \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} = n - H_n.$$

Moreover, if we set $m = 1$ and $m = 2$ in Equation (11) and combine the resulting formulas, we arrive at

$$\sum_{j=1}^n \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} (1 - 2^{-j}) = \frac{1 - 2^{n-1}}{2^n}.$$

A slightly different form of the above equation can serve as a recurrence relation for the Bernoulli numbers.

Corollary 2. *For a positive integer n , we have*

$$\sum_{j=1}^n \left(\binom{n}{j} + 1 \right) \frac{B_j}{j} (1 - 2^j) = 1. \quad (14)$$

Proof. We first set $x = -1/2$ in Equation (9) to obtain

$$\sum_{j=1}^n (-1)^j \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} 2^j = (-1)^n 2^n {}_Hw_n^{(-1/2)} - H_n.$$

Since

$$\sum_{k=1}^{\infty} \binom{2k}{k} H_k t^k = \frac{2}{\sqrt{1-4t}} \ln \left(\frac{1 + \sqrt{1-4t}}{2\sqrt{1-4t}} \right)$$

(see [6, Equation (4)]) and

$$\binom{-1/2}{k} = \frac{(-1)^k}{2^{2k}} \binom{2k}{k},$$

we obtain that

$$\sum_{n=0}^{\infty} {}_Hw_n^{(-1/2)} \frac{(-2t)^n}{n!} = 2e^t \ln \left(\frac{e^t + 1}{2} \right).$$

From the generating function of Euler polynomials [15, Section 15.3],

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{xt},$$

we deduce that

$$\sum_{n=1}^{\infty} E_{n-1}(1) \frac{t^n}{n!} = 2 \ln \left(\frac{e^t + 1}{2} \right).$$

Hence, we obtain

$$(-1)^n 2^n {}_Hw_n^{(-1/2)} = \sum_{k=0}^{n-1} \binom{n}{k+1} E_k(1).$$

We then use [15, Section 15.3]

$$E_k(1) = -E_k(0) = 2(2^{k+1} - 1) \frac{B_{k+1}}{k+1} \quad (15)$$

to obtain

$$\sum_{j=1}^n (-1)^j \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} 2^j - 2 \sum_{j=1}^n \binom{n}{j} \frac{B_j}{j} (2^j - 1) = 2n - H_n.$$

Since $(-1)^n B_n = B_n$ for $n \geq 2$, we arrive at

$$\sum_{j=1}^n \binom{n}{j} \frac{B_j}{j} (2^{j-1} - 1) + \sum_{j=1}^n \frac{B_j}{j} 2^{j-1} = \frac{1}{2} H_n - 1.$$

Equation (14) now follows after some elementary manipulations. $\square$

Before presenting the next result, we recall the poly-Bernoulli polynomials. The poly-Bernoulli polynomials are defined [5] via the generating function

$$\sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!} = \frac{\text{Li}_p(1 - e^{-t})}{1 - e^{-t}} e^{xt}, \quad (16)$$

where $\text{Li}_p(z) = \sum_{n=1}^{\infty} z^n/n^p$ is the polylogarithm function. In particular, $B_n^{(p)}(0) = B_n^{(p)}$ is the n th poly-Bernoulli number and when $p = 2$, $B_n^{(2)}$ is called a di-Bernoulli number [10].

We now state a cumulative sum of the Bernoulli numbers in terms of the Bernoulli numbers, di-Bernoulli numbers, and the values of the di-Bernoulli polynomials at $x = 1$.

Corollary 3. *For an integer $n \geq 2$, we have*

$$\sum_{j=0}^n B_j = B_n^{(2)}(1) + B_n - B_n^{(2)} - 1.$$

Proof. The derivative of Equation (9) with respect to x is

$$\begin{aligned} n \sum_{j=1}^{n-1} \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} x^{n-j-1} - \sum_{j=1}^{n-1} \left(\binom{n}{j} - 1 \right) B_j x^{n-j-1} \\ = \frac{d}{dx} {}_H w_n^{(x)} - n H_n x^{n-1}. \end{aligned} \quad (17)$$

By Equation (9), we have

$$n \sum_{j=1}^n \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} x^{n-j-1} = \frac{n}{x} \left({}_H w_n^{(x)} - H_n x^n \right). \quad (18)$$

The Bernoulli polynomials $B_n(x)$ are defined by

$$\sum_{j=0}^n \binom{n}{j} B_j x^{n-j} = B_n(x).$$

Thus, we have

$$\sum_{j=1}^{n-1} \left(\binom{n}{j} - 1 \right) B_j x^{n-j-1} = \frac{1}{x} B_n(x) - \frac{1}{x} \sum_{j=1}^{n-1} B_j x^{n-j}. \quad (19)$$

Now, using the obvious relation

$$\frac{d}{dx} \binom{x}{k} = \binom{x}{k} \sum_{j=0}^{k-1} \frac{1}{x-j}$$

in Equation (10), we find that

$$\frac{d}{dx} {}_H w_n^{(x)} = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \binom{x}{k} k! H_k \sum_{j=0}^{k-1} \frac{1}{x-j}. \quad (20)$$

Inserting Equations (18), (19), and (20) into Equation (17) gives

$$\begin{aligned} \sum_{j=0}^n B_j x^{n-j} &= x \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \binom{x}{k} k! H_k \sum_{j=0}^{k-1} \frac{1}{x-j} - n H_n x^n \\ &\quad - n \sum_{j=1}^{n-1} \left(\binom{n}{j} - 1 \right) \frac{B_j}{j} x^{n-j} + B_n(x). \end{aligned}$$

Setting $x = -1$ and using the relation $(-1)^n B_n(-x) = B_n(x) + nx^{n-1}$ in the above equation yields

$$\sum_{j=0}^n B_j = \sum_{k=1}^n (-1)^{n-k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! (H_k)^2 + B_n(1) + n - n^2 - 1. \quad (21)$$

To evaluate the first sum on the right-hand side of Equation (21), we use Equation (13) and find that

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{k=1}^n (-1)^{n-k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! (H_k)^2 = \sum_{k=1}^{\infty} (H_k)^2 (1 - e^{-t})^k. \quad (22)$$

With the use of the generating function of the harmonic numbers

$$\sum_{k=1}^{\infty} H_k t^k = -\frac{\ln(1-x)}{1-x},$$

one can obtain that

$$\sum_{k=1}^{\infty} (H_k)^2 t^k = \frac{\text{Li}_2(x)}{1-x} + \frac{\ln^2(1-x)}{1-x}.$$

Using this relation with Equation (16) in Equation (22), we arrive at

$$\sum_{k=1}^n (-1)^{n-k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! (H_k)^2 = B_n^{(2)}(1) - B_n^{(2)} + n(n-1).$$

Combining these results in Equation (21) gives the desired formula. $\square$

We conclude this section by presenting some congruences for cumulative sums of the Bernoulli and Euler numbers. Here we call $E_n(0) = E_n$ the n th Euler number.

Corollary 4. *For an odd prime p , we have*

$$\sum_{j=0}^p p B_j \equiv -1 \pmod{p}, \quad (23)$$

$$\sum_{j=0}^{p-3} B_j \equiv -1 \pmod{p}, \quad (p \geq 5), \quad (24)$$

$$\sum_{j=0}^p E_j \equiv \frac{3}{2} \pmod{p}, \quad (25)$$

$$p \sum_{j=0}^p \frac{B_j}{(p-j+1)} \equiv -1 \pmod{p}. \quad (26)$$

Proof. Let $n = p$ be an odd prime in Equation (21). Since $p \geq 3$, we have $B_p(1) = B_p = 0$. Thus,

$$\begin{aligned} \sum_{j=0}^p B_j &= \sum_{k=1}^p (-1)^{p-k} \left\{ \begin{matrix} p \\ k \end{matrix} \right\} k! (H_k)^2 + B_p(1) + p - p^2 - 1 \\ &= p - p^2 - 1 + \left\{ \begin{matrix} p \\ 1 \end{matrix} \right\} (H_1)^2 + \left\{ \begin{matrix} p \\ p \end{matrix} \right\} p! (H_p)^2 + \sum_{k=2}^{p-1} (-1)^{p-k} \left\{ \begin{matrix} p \\ k \end{matrix} \right\} k! (H_k)^2. \end{aligned}$$

Now, $\left\{ \begin{matrix} p \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} p \\ p \end{matrix} \right\} = 1$, $H_1 = 1$, and

$$p! (H_p)^2 = p! (H_{p-1})^2 + 2(p-1)! H_{p-1} + \frac{(p-1)!}{p},$$

so

$$\sum_{j=0}^p B_j = p - p^2 + p! (H_{p-1})^2 + 2(p-1)! H_{p-1} + \frac{(p-1)!}{p} + \sum_{k=2}^{p-1} (-1)^{p-k} \left\{ \begin{matrix} p \\ k \end{matrix} \right\} k! (H_k)^2,$$

or equivalently

$$\begin{aligned} \sum_{j=0}^p pB_j &= p^2 - p^3 + p(p!) (H_{p-1})^2 + 2p! H_{p-1} + (p-1)! \\ &\quad + p \sum_{k=2}^{p-1} (-1)^{p-k} \left\{ \begin{matrix} p \\ k \end{matrix} \right\} k! (H_k)^2. \end{aligned} \quad (27)$$

Since $\left\{ \begin{matrix} p \\ k \end{matrix} \right\} \equiv 0 \pmod{p}$ for $k = 2, 3, \dots, p-1$, we find that

$$\sum_{j=0}^p pB_j \equiv (p-1)! \equiv -1 \pmod{p},$$

which is the congruence given in Equation (23).

Now, Babbage's theorem states that $H_{p-1} \equiv 0 \pmod{p}$, so Equation (27) can be written as

$$\sum_{j=0}^p pB_j \equiv (p-1)! \pmod{p^2}.$$

A famous congruence by Glaisher [8] is

$$(p-1)! \equiv -p + pB_{p-1} \pmod{p^2}.$$

Thus,

$$\begin{aligned} -p + pB_{p-1} &\equiv \sum_{j=0}^p pB_j = pB_0 + pB_1 + pB_p + pB_{p-1} + pB_{p-2} + \sum_{j=2}^{p-3} pB_j \\ &= p - \frac{p}{2} + pB_{p-1} + \sum_{j=2}^{p-3} pB_j \pmod{p^2}, \end{aligned}$$

which implies that

$$\sum_{j=2}^{p-3} pB_j \equiv -\frac{3p}{2} \pmod{p^2}$$

for $p \geq 3$, or equivalently

$$\sum_{j=2}^{p-3} B_j \equiv -\frac{3}{2} \pmod{p}$$

if $p \geq 5$. Equation (24) now follows by using the values B_0 and B_1 .

Inserting Equation (15) into Equation (14) yields

$$\sum_{j=1}^n \left(\binom{n}{j} + 1 \right) \frac{E_{j-1}}{2} = 1,$$

or

$$\sum_{j=0}^{n-1} E_j = 2 - nE_n - E_{n-1} - \sum_{j=2}^{n-1} \binom{n}{j} E_{j-1}.$$

Let $n = p$ be an odd prime. Then, we have

$$\sum_{j=0}^p E_j = 2 + E_p - p - \sum_{j=2}^{p-1} \binom{p}{j} E_{j-1},$$

since $E_0 = 1$ and $E_{2m} = 0$ for $m \geq 1$. Moreover, $2^m E_m \in \mathbb{Z}$, which means that E_m does not contain primes p in the denominator when $p > 2$. Thus,

$$\binom{p}{j} E_{j-1} \equiv 0 \pmod{p}$$

for $j = 1, 2, \dots, p-1$. Therefore, we have

$$\sum_{j=0}^p E_j \equiv 2 - \frac{1}{2} = \frac{3}{2} \pmod{p} \quad (28)$$

for $p \geq 5$. Note that

$$\sum_{j=0}^3 E_j = E_0 + E_1 + E_2 + E_3 = 1 - \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Hence, Equation (28) is also true when $p = 3$, that is, we have Equation (25).

To prove Equation (26), we first integrate both sides of Equation (9) with respect to x from 0 to 1 and use Equation (11) and the well-known formula

$$\sum_{j=0}^n \binom{n}{j} \frac{B_j}{n-j+1} = \int_0^1 B_n(x) dx = 0, \quad n \geq 2.$$

We then obtain

$$\sum_{j=0}^n \frac{B_j}{(n-j+1)} = 1 - (n+1) \int_0^1 Hw_n^{(x)} dx.$$

From the definition of the Cauchy numbers of the first kind c_n (cf. [14])

$$\frac{c_n}{n!} = \int_0^1 \binom{x}{n} dx,$$

we arrive at

$$\sum_{j=0}^n \frac{B_j}{(n-j+1)} = 1 - (n+1) \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} c_k H_k.$$

We write $n = p$, where p is an odd prime, in the equation above to obtain

$$\sum_{j=0}^p \frac{B_j}{(p-j+1)} = 1 - (p+1) \sum_{k=1}^p \left\{ \begin{matrix} p \\ k \end{matrix} \right\} c_k H_k. \quad (29)$$

Employing $\left\{ \begin{matrix} p \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} p \\ p \end{matrix} \right\} = 1$ and $\left\{ \begin{matrix} p \\ p-1 \end{matrix} \right\} = \binom{p}{2}$, as well as the explicit expression of Cauchy numbers of the first kind [14]

$$c_k = \sum_{j=1}^k \left[\begin{matrix} k \\ j \end{matrix} \right] \frac{(-1)^{k-j}}{j+1},$$

we have

$$\begin{aligned} \sum_{k=1}^p \left\{ \begin{matrix} p \\ k \end{matrix} \right\} c_k H_k &= \frac{1}{2} + c_p H_{p-1} + \frac{c_p}{p} + \frac{p(p-1)}{2} c_{p-1} H_{p-1} \\ &\quad + \sum_{k=2}^{p-2} \sum_{j=1}^k \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \left[\begin{matrix} k \\ j \end{matrix} \right] \frac{(-1)^{k-j}}{j+1} H_k, \end{aligned}$$

or equivalently

$$\begin{aligned} p \sum_{k=1}^p \left\{ \begin{matrix} p \\ k \end{matrix} \right\} c_k H_k &= \frac{p}{2} + p c_p H_{p-1} + c_p + \frac{p(p-1)}{2} p c_{p-1} H_{p-1} \\ &\quad + p \sum_{k=2}^{p-2} \sum_{j=1}^k \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \left[\begin{matrix} k \\ j \end{matrix} \right] \frac{(-1)^{k-j}}{j+1} H_k. \end{aligned} \quad (30)$$

Let

$$A(p) = \sum_{k=2}^{p-2} \sum_{j=1}^k \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \left[\begin{matrix} k \\ j \end{matrix} \right] \frac{(-1)^{k-j}}{j+1} H_k.$$

The product $\frac{1}{j+1} H_k$ does not involve p in the denominator, so $A(p) \equiv 0 \pmod{p}$ since $\left\{ \begin{matrix} p \\ k \end{matrix} \right\} \equiv 0 \pmod{p}$ for $k = 2, 3, \dots, p-2$. Therefore, from Equation (29) and

Equation (30), we conclude that

$$p \sum_{j=0}^p \frac{B_j}{(p-j+1)} = p - \frac{p^2}{2} - \frac{p}{2} - p^2 c_p H_{p-1} - p c_p H_{p-1} - p c_p - c_p \\ - \frac{p^2(p-1)}{2} p c_{p-1} H_{p-1} - \frac{p(p-1)}{2} p c_{p-1} H_{p-1} - p^2 A(p) - p A(p). \quad (31)$$

Now, by [7, Corollary 5.2], we know that if $\lambda \equiv 0 \pmod{p}$, then

$$c_p(\lambda) \equiv 1 - \lambda + (1 - \lambda)(-\lambda) \cdots (2 - \lambda - p) \pmod{p}$$

and

$$p c_{p-1}(\lambda) \equiv 1 \pmod{p},$$

where $c_k(\lambda)$ is the degenerate Cauchy number. Since $\lim_{\lambda \rightarrow 0} c_k(\lambda) = c_k$, we conclude that

$$c_p \equiv 1 \pmod{p} \quad \text{and} \quad p c_{p-1} = \lim_{\lambda \rightarrow 0} p c_{p-1}(\lambda) \equiv 1 \pmod{p}$$

Therefore, Equation (31) implies the congruence (26). $\square$

Remark 1. Equation (23) also follows from the von Staudt-Clausen theorem for the Bernoulli numbers.

$$p B_{2j} \equiv \begin{cases} 0 \pmod{p}, & \text{if } (p-1) \nmid 2j, \\ -1 \pmod{p}, & \text{if } (p-1) \mid 2j. \end{cases}$$

Indeed, we first write

$$\sum_{j=0}^p p B_j = p B_0 + p B_1 + p B_p + p B_{p-1} + p B_{p-2} + \sum_{j=1}^{\frac{p-3}{2}} p B_{2j} \\ = p - \frac{p}{2} + p B_{p-1} + \sum_{j=1}^{\frac{p-3}{2}} p B_{2j}.$$

As j ranges from 1 to $\frac{p-3}{2}$, the integer $p-1$ does not divide $2j$. So $p B_{2j} \equiv 0 \pmod{p}$ for $j = 1, 2, \dots, \frac{p-3}{2}$. Moreover, $p B_{p-1} \equiv -1 \pmod{p}$ and hence we have

$$\sum_{j=0}^p p B_j \equiv -1 \pmod{p}$$

for an odd prime p .

We also note that, from Equation (31), the congruence (26) is equivalent to

$$p \sum_{j=0}^p \frac{B_j}{(p-j+1)} = -\frac{p}{2} - c_p \pmod{p^2}.$$

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