



MULTIPLICATIVE FUNCTIONS ADDITIVE ON PARTITIONS OF $2k$ NONZERO SQUARES

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Abstract

For a fixed integer $k \geq 3$, we study the multiplicative functions $f: \mathbb{N} \rightarrow \mathbb{C}$ satisfying

$$f\left(\sum_{i=1}^{2k} x_i^2\right) = \sum_{j=1}^k f(x_{2j-1}^2 + x_{2j}^2)$$

for all nonzero integers $x_1, \dots, x_{2k}$. This extends a theorem of Park on sums of two nonzero squares, which established the $k = 2$ case. For $k = 3$ and $k = 4$, we prove that every such f with $f(2) \neq 0$ is the identity function on $\mathbb{N}$. For $k \geq 5$, we show that such a function f is either the identity function on $\mathbb{N}$, or else vanishes on the set $\{a^2 + b^2 : a, b \in \mathbb{N}\} \cup \{2k + t : t = 0, 3, 6, 8, 9, 11, 12\} \cup \{n \in \mathbb{N} : n \geq 2k + 14\}$.

1. Introduction

An arithmetic function $f: \mathbb{N} \rightarrow \mathbb{C}$ is said to be *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all coprime $m, n \in \mathbb{N}$. Additionally, f is *completely multiplicative* if $f(mn) = f(m)f(n)$ holds for all $m, n \in \mathbb{N}$. Multiplicative functions are determined by their values at the prime powers. We call a function f *k-additive on a set* $S \subseteq \mathbb{N}$ if

$$f(a_1 + \dots + a_k) = f(a_1) + \dots + f(a_k) \quad \text{for all } a_1, \dots, a_k \in S.$$

A multiplicative function that is also k -additive for some $k \geq 2$ is forced to be the identity function in most cases. An early result in this direction is due to Spiro [13], who showed that if a multiplicative function f is 2-additive on the set of primes and $f(p_0) \neq 0$ for some prime p_0 , then f is the identity function (i.e., $f(n) = n$ for all $n \in \mathbb{N}$). This result was later extended to functions that are k -additive on the

primes for $k = 3$ by Fang [5], and for all $k \geq 3$ by Dubickas and Šarka [3]. Following Spiro [13], we define a set $S \subseteq \mathbb{N}$ to be a *k-additive uniqueness set* for a class $\mathcal{F}$ of arithmetic functions if $\mathcal{F}$ has exactly one element f satisfying the k -additivity on S . The case $k = 2$ corresponds to the standard definition of an *additive uniqueness set*. The characterization of additive uniqueness sets, along with the exceptional functions that arise in the absence of uniqueness, has become an area of active research.

Figurate numbers often serve as additive uniqueness sets for the class of multiplicative functions. Chung [1] demonstrated that uniqueness fails for the 2-additive case on the set of squares by characterising all such multiplicative and completely multiplicative functions. In contrast, Park [8] later showed that any multiplicative function that is k -additive on the set of squares for $k \geq 3$ must be the identity function. In other words, the set of squares is a k -additive uniqueness set for multiplicative functions whenever $k \geq 3$. Besides the set of squares, the sets of triangular and tetrahedral numbers were established as 2-additive uniqueness sets for the class of multiplicative functions in the earlier work of Chung and Phong [2]. Park [9] extended this result for $k \geq 3$, by showing that the set of triangular numbers is a k -additive uniqueness set for the collection of multiplicative functions. Kim et al. [7] showed that the set of all nonzero generalized pentagonal numbers is a 2-additive uniqueness set for the collection of multiplicative functions. This result was later extended to the k -additive case by Hasanalizade [6] for $k \geq 3$, where the proof is conditional on the Generalized Riemann Hypothesis for $k = 3$ and unconditional for $k \geq 4$. The unconditional proof for $k = 3$ was later given by Park [11].

A related variant of these problems groups the summands into pairs, requiring additivity on sums of elements of $\mathcal{E}_2$ rather than on individual squares. Here,

$$\mathcal{E}_2 = \{a^2 + b^2 : a, b \in \mathbb{N}\} = \{2, 5, 8, 10, 13, 17, 18, 20, 25, 26, \dots\}$$

is the set of sums of two nonzero (or positive) squares. Park [10] determined all multiplicative functions satisfying

$$f(a^2 + b^2 + c^2 + d^2) = f(a^2 + b^2) + f(c^2 + d^2), \quad a, b, c, d \in \mathbb{N}.$$

More precisely, Park showed that any such function f is the identity function provided that $f(3)f(11) \neq 0$; otherwise, $f(n) = 0$ for all $n \geq 2$, except when $n \in \{3, 9, 11\}$. In other words, Park determined all functions that are 2-additive on $\mathcal{E}_2$. The companion problem of classifying functions that are 2-additive on T_2 , the set of sums of two positive triangular numbers, was settled in [12]. In each of these two-summand problems, the function is essentially forced to be the identity, apart from a small family of exceptional functions.

In this article, we deal with the higher analogue of Park's result [10] obtained by

summing k pairs. We study multiplicative functions f satisfying

$$f\left(\sum_{i=1}^{2k} x_i^2\right) = \sum_{j=1}^k f(x_{2j-1}^2 + x_{2j}^2), \quad x_1, \dots, x_{2k} \in \mathbb{N}. \quad (1)$$

Because every element of $\mathcal{E}_2$ is of the form $x^2 + y^2$ with $x, y \in \mathbb{N}$, Equation (1) is equivalent to

$$f\left(\sum_{i=1}^k A_i\right) = \sum_{i=1}^k f(A_i) \quad \text{for all } A_1, \dots, A_k \in \mathcal{E}_2. \quad (2)$$

In other words, we classify the multiplicative functions that are k -additive on $\mathcal{E}_2$ for $k \geq 3$. The case $k = 2$ is precisely [10, Theorem 1], and we treat the remaining cases $k \geq 3$ in this article.

The change in analysis from $k = 2$ to $k \geq 3$ comes from a structural change in the representability of integers as sums of $2k$ nonzero squares. For $k = 2$, the set of sums of $2k$ nonzero squares excludes an infinite set, namely $\{2 \cdot 4^r, 6 \cdot 4^r, 14 \cdot 4^r : r \geq 0\}$, which complicates the analysis. By contrast, for $k = 3$ (respectively, $k = 4$), the set of sums of $2k$ nonzero squares contains all but finitely many integers, as every integer greater than or equal to 20 (respectively, 22) is representable as a sum of six (respectively, eight) nonzero squares, on which our approach relies. We use this property, which simplifies the classification of multiplicative functions beyond the case settled in [10].

Throughout, the hypothesis $f(2) \neq 0$ serves as the nondegeneracy condition, similar to the role played by $f(p_0) \neq 0$ in Spiro's theorem. It cannot be omitted because the function f with $f(1) = 1$ and $f(n) = 0$ for $n > 1$ is multiplicative and satisfies Equation (1) for every k . Our first result shows that, for $k = 3$ and $k = 4$, this is the only obstruction.

Theorem 1. *Let f be a multiplicative function that satisfies Equation (1) with $k \in \{3, 4\}$. If $f(2) \neq 0$, then $f(n) = n$ for all positive integers n .*

For $k \geq 5$, we establish that f must either be the identity function or eventually behave as the zero function, regardless of the value of $f(2)$.

Theorem 2. *Let $k \geq 5$ and let f be multiplicative and satisfy Equation (1). Then either $f(n) = n$ for all positive integers n , or $f(n) = 0$ for all $n \in \{a^2 + b^2 : a, b \in \mathbb{N}\} \cup \{2k + t : t = 0, 3, 6, 8, 9, 11, 12\} \cup \{m \in \mathbb{N} : m \geq 2k + 14\}$.*

In particular, if f is a multiplicative function that satisfies Equation (1) with $k \geq 5$ and $f(\beta) \neq 0$ for some $\beta \in \{a^2 + b^2 : a, b \in \mathbb{N}\} \cup \{2k + t : t = 0, 3, 6, 8, 9, 11, 12\} \cup \{m : m \geq 2k + 14\}$, then f is the identity function (see Corollary 2).

The proofs follow the same route as in [10]. For $k \in \{3, 4\}$, one starts with the equivalent form Equation (2) and uses small decompositions in $\mathcal{E}_2^{(k)}$, combined

with multiplicativity and the assumption that $f(2) \neq 0$, to evaluate f on an initial segment of integers. The proof then continues via strong induction: since every integer beyond this segment is a sum of $2k$ nonzero squares, it can be expressed as a sum of k elements of $\mathcal{E}_2$, all smaller than the integer itself. For $k \geq 5$, the base case is replaced by a simpler argument: by comparing the two decompositions $N = 2 + 10 + \sum C_i$ and $N + 1 = 5 + 8 + \sum C_i$ (where $C_i \in \mathcal{E}_2$), we show that $f(N + 1) - f(N)$ is eventually constant for all $N > 2k + 21$, forcing f to satisfy one of the forms in Theorem 2. In all cases, we repeatedly use Corollary 1.

The remainder of the article is organized as follows. Section 2 collects the representability results we need, including Corollary 1. Sections 3 and 4 prove Theorem 1 by handling the cases $k = 3$ and $k = 4$ separately, and Section 5 proves Theorem 2 and a subsequent corollary.

2. Preliminaries

Throughout, we write $x = f(2)$, $y = f(3)$, and $z = f(5)$. We will use the following result on the integers representable as a sum of a fixed number of nonzero squares.

Lemma 1 ([4]). *Let m be an integer with $m \geq 4$. Every positive integer n can be represented as a sum of m nonzero squares, except when n lies in*

$$\begin{cases} \{1, 3, 5, 9, 11, 17, 29, 41\} \cup \{2 \cdot 4^r, 6 \cdot 4^r, 14 \cdot 4^r : r \geq 0\}, & \text{when } m = 4, \\ \{1, 2, 3, 4, 6, 7, 9, 10, 12, 15, 18, 33\}, & \text{when } m = 5, \\ \{1, 2, \dots, m - 1\} \\ \quad \cup \{m + 1, m + 2, m + 4, m + 5, m + 7, m + 10, m + 13\}, & \text{when } m \geq 6. \end{cases}$$

In particular, for an integer $m \geq 6$, every integer $n \geq m + 14$ is a sum of m nonzero squares.

Corollary 1. *Every integer $n \geq 20$ is a sum of 6 nonzero squares, and every integer $n \geq 22$ is a sum of 8 nonzero squares.*

Let $\mathcal{E}_2^{(k)} := \mathcal{E}_2 + \dots + \mathcal{E}_2$ (k summands). Since each element of $\mathcal{E}_2$ is a sum of two nonzero squares, the set of integers expressible as a sum of $2k$ nonzero squares is exactly $\mathcal{E}_2^{(k)}$. Hence, for $k \geq 3$, the set $\mathcal{E}_2^{(k)}$ contains all integers $n \geq 2k + 14$, by Lemma 1.

Remark 1. The consecutive integers $12 = 2 + 10$ and $13 = 5 + 8$ both lie in $\mathcal{E}_2 + \mathcal{E}_2$, and this is one of the key observations used in Section 5.

3. The Case $k = 3$

We begin by evaluating the function f at the first few prime values, establishing a foundation for our later analysis of larger integers.

Lemma 2. *If f is multiplicative, satisfies Equation (2) with $k = 3$, and has $f(2) \neq 0$, then $f(2) = 2$, $f(3) = 3$, and $f(5) = 5$.*

Proof. From the identity $3 \cdot 2 = 2 + 2 + 2$, we have $y = 3$. From $2 + 2 + 8 = 2 + 5 + 5$, we get $2x + f(8) = x + 2z$, that is, $f(8) = 2z - x$. From $18 = 2 + 8 + 8$, we have $f(18) = x + 2f(8) = 4z - x$. On the other hand, $f(18) = xf(9)$ and $9 = 2 + 2 + 5$ gives $f(9) = 2x + z$. Therefore, we have

$$x(2x + z) = 4z - x. \quad (3)$$

Thus, the equality $2 + 10 + 10 = 2 + 2 + 18$ gives $x + 2xz = 2x + (4z - x)$, so $z(x - 2) = 0$.

If $z = 0$, then Equation (3) yields $2x^2 = -x$. Since $x \neq 0$, this implies $x = -\frac{1}{2}$, $f(8) = 2z - x = \frac{1}{2}$, and $f(10) = xz = 0$. From $20 = 5 + 5 + 10$, we have $f(20) = 0$, while $20 = 2 + 5 + 13$ yields $f(13) = \frac{1}{2}$. Therefore, $f(26) = xf(13) = -\frac{1}{4}$, while $26 = 8 + 8 + 10$ gives $f(26) = 1$, a contradiction. Hence, $z \neq 0$, which gives $x = 2$. By Equation (3), we then have $z = 5$. $\square$

Theorem 3. *Let f be multiplicative and satisfy Equation (1) with $k = 3$. If $f(2) \neq 0$, then $f(n) = n$ for all positive integers n .*

Proof. For $k = 3$, since f satisfies Equation (1), it also satisfies Equation (2). By Lemma 2, we have $f(2) = 2$, $f(3) = 3$, and $f(5) = 5$. Hence $f(8) = 2z - x = 8$ and $f(9) = 2x + z = 9$. By multiplicativity, we have $f(6) = 6$, $f(10) = 10$, and $f(15) = 15$.

Using the decompositions $12 = 2 + 5 + 5$, $14 = 2 + 2 + 10$, $17 = 2 + 5 + 10$, $18 = 2 + 8 + 8$, $20 = 5 + 5 + 10$, and $22 = 2 + 10 + 10$ in $\mathcal{E}_2^{(3)}$, we have $f(n) = n$ for all $n \in \{12, 14, 17, 18, 20, 22\}$. Since $f(17) = 17$, the decomposition $17 = 2 + 2 + 13$ yields $f(13) = 13$. By multiplicativity, $f(7) = f(14)/f(2) = 7$, $f(11) = f(22)/f(2) = 11$, and $f(4) = f(12)/f(3) = 4$.

Also, $32 = 2 + 10 + 20$ gives $f(32) = 32$. The identities $38 = 2 + 18 + 18$ and $48 = 8 + 8 + 32$ give $f(n) = n$ for $n \in \{38, 48\}$. Therefore, $f(16) = f(48)/f(3) = 16$ and $f(19) = f(38)/f(2) = 19$. Combining everything with $f(1) = 1$, we have $f(n) = n$ for all $1 \leq n \leq 20$.

To conclude the proof, we now use the second principle of mathematical induction. Let $N \geq 20$ be an integer and suppose that $f(n) = n$ for all $n < N$. By Corollary 1, N is a sum of 6 nonzero squares, so $N = A_1 + A_2 + A_3$ with $A_i \in \mathcal{E}_2$. Since each $A_i \geq 2$, we have $A_i \leq N - 4 < N$, and Equation (2) gives $f(N) = \sum_{i=1}^3 f(A_i) = \sum_{i=1}^3 A_i = N$. $\square$

4. The Case $k = 4$

Lemma 3. *If f is multiplicative, satisfies Equation (2) with $k = 4$, and has $f(2) \neq 0$, then $f(n) = n$ for all $n \in \{2, 3, 4, 5, 6, 7, 8, 10\}$.*

Proof. By assumption, $x \neq 0$. From $8 = 2 + 2 + 2 + 2$, we have $f(8) = 4x$. From $14 = 2 + 2 + 2 + 8 = 2 + 2 + 5 + 5$, we have $7x = 2x + 2z$, so that $z = \frac{5}{2}x$. Therefore, $7x = f(14) = xf(7)$, which gives $f(7) = 7$. From $4 \cdot 5 = 5 + 5 + 5 + 5$, we have $f(4)z = 4z$, which yields $f(4) = 4$ (since $z = \frac{5}{2}x \neq 0$).

From $8 + 8 + 8 + 8 = 2 + 10 + 10 + 10$, we have $16x = 4f(8) = x + 3xz$, so that $15x = 3xz$, yielding $z = 5$. Combining $z = 5$ with $z = \frac{5}{2}x$, we get $x = 2$, hence $f(8) = 4x = 8$. Finally, $3 \cdot 8 = 2 + 2 + 10 + 10$ gives $f(24) = yf(8) = 8y = 2x + 2xz$. From $x = 2$ and $z = 5$, we have $y = 3$. Furthermore, the multiplicativity of f implies that $f(6) = 6$ and $f(10) = 10$. $\square$

Theorem 4. *Let f be multiplicative and satisfy Equation (1) with $k = 4$. If $f(2) \neq 0$, then $f(n) = n$ for all positive integers n .*

Proof. For $k = 4$, since f satisfies Equation (1), it also satisfies Equation (2). By Lemma 3, $f(n) = n$ for all $n \in \{2, 3, 4, 5, 6, 7, 8, 10\}$.

Using the decompositions $11 = 2 + 2 + 2 + 5$, $14 = 2 + 2 + 2 + 8$, $16 = 2 + 2 + 2 + 10$, $17 = 2 + 2 + 5 + 8$, $19 = 2 + 2 + 5 + 10$, $20 = 5 + 5 + 5 + 5$, $22 = 2 + 2 + 8 + 10$, and $26 = 2 + 8 + 8 + 8$ in $\mathcal{E}_2^{(4)}$, we have $f(n) = n$ for all $n \in \{11, 14, 16, 17, 19, 20, 22, 26\}$.

By multiplicativity of f , $f(12) = 12$, $f(15) = 15$, $f(21) = 21$, and $f(13) = f(26)/f(2) = 13$. The identity $4 \cdot 9 = 8 + 8 + 10 + 10$ yields $f(9) = 9$. Consequently, $f(18) = 18$, by multiplicativity. Combining everything with $f(1) = 1$, we have $f(n) = n$ for all $1 \leq n \leq 22$.

Let $K \geq 22$ be an integer and suppose that $f(n) = n$ for all $n < K$. By Corollary 1, K is a sum of 8 nonzero squares, so $K = A_1 + \cdots + A_8$ with $A_i \in \mathcal{E}_2$. Since each $A_i \geq 2$, we have $A_i \leq K - 6 < K$, and Equation (2) gives $f(K) = \sum_{i=1}^8 f(A_i) = \sum_{i=1}^8 A_i = K$. The proof is now immediate from the second principle of mathematical induction. $\square$

Proof of Theorem 1. The proof follows immediately from Theorems 3 and 4. $\square$

5. The Case $k \geq 5$

In this section, we provide a proof of Theorem 2.

Proof of Theorem 2. Since f satisfies Equation (1), f also satisfies Equation (2). Let $N_0(k) := 2k + 21$. Fix $N > N_0(k)$ and set $M = N - (2k + 2)$. Since $N > 2k + 21$, we have $M \geq 20$. Therefore, by Corollary 1, M is a sum of 6 nonzero squares, that

is, $M = C_3 + C_4 + C_5$ with $C_3, C_4, C_5 \in \mathcal{E}_2$. Setting $C_6 = \dots = C_k = 2 \in \mathcal{E}_2$ (we assign nothing when $k = 5$), we obtain

$$\sum_{i=3}^k C_i = M + 2(k-5) = (N - 2k - 2) + (2k - 10) = N - 12, \quad C_3, \dots, C_k \in \mathcal{E}_2.$$

Using $12 = 2 + 10$ and $13 = 5 + 8$, we have

$$N = 2 + 10 + \sum_{i=3}^k C_i, \quad N + 1 = 5 + 8 + \sum_{i=3}^k C_i.$$

Since $2, 5, 8, 10 \in \mathcal{E}_2$, applying Equation (2) to both yields

$$f(N) = f(2) + f(10) + \sum_{i=3}^k f(C_i), \quad f(N + 1) = f(5) + f(8) + \sum_{i=3}^k f(C_i).$$

Let $c = f(5) + f(8) - f(2) - f(10)$. Therefore, for all $N > N_0(k)$, we have

$$f(N + 1) - f(N) = c.$$

Hence, for all $n \geq N_0(k) + 2$, the telescoping sum gives

$$\begin{aligned} f(n) - f(N_0(k) + 1) &= \sum_{N=N_0(k)+1}^{n-1} (f(N + 1) - f(N)) \\ &= \sum_{N=N_0(k)+1}^{n-1} c \\ &= c(n - N_0(k) - 1). \end{aligned}$$

Setting $d = f(N_0(k) + 1) - c(N_0(k) + 1)$, we have $f(n) = cn + d$ for all $n \geq N_0(k) + 2$.

For distinct primes p, q with $p > q \geq N_0(k) + 2$, the multiplicativity of f gives

$$cpq + d = f(pq) = f(p)f(q) = (cp + d)(cq + d) = c^2pq + cd(p + q) + d^2.$$

Rearranging the above equation, we have

$$(c - c^2)pq - cd(p + q) + (d - d^2) = 0.$$

Fix a prime $q_0 \geq N_0(k) + 2$. Then, for all primes $p > q_0$, we have

$$((c - c^2)q_0 - cd)p + (d - d^2) - cdq_0 = 0.$$

Since the above equality holds for infinitely many primes p , both coefficients must vanish, yielding

$$(c - c^2)q_0 - cd = 0, \quad -cdq_0 + (d - d^2) = 0.$$

Since $q_0 \geq N_0(k) + 2$ was arbitrary, we have

$$(c - c^2)q - cd = 0, \quad -cdq + (d - d^2) = 0,$$

for infinitely many primes $q \geq N_0(k) + 2$. These further yield

$$c = c^2, \quad cd = 0, \quad d = d^2,$$

giving $(c, d) \in \{(0, 0), (0, 1), (1, 0)\}$.

When $(c, d) = (0, 1)$, we have $f(n) = 1$ for all $n \geq N_0(k) + 2$. We now choose $A_1, \dots, A_k \in \mathcal{E}_2$ such that each exceeds $N_0(k) + 2$. Then Equation (2) gives

$$1 = f\left(\sum_{i=1}^k A_i\right) = \sum_{i=1}^k f(A_i) = k \geq 5,$$

a contradiction. Hence, $(c, d) \in \{(0, 0), (1, 0)\}$; that is, either $f(N) = N$ for all $N \geq N_0(k) + 2$ or $f(N) = 0$ for all $N \geq N_0(k) + 2$.

We now prove that if $f(N) = N$ for all $N \geq N_0(k) + 2$, then f is indeed the identity function on $\mathbb{N}$. Let $n \in \mathbb{N}$ be arbitrary. Pick a prime p such that $p > \max(n, N_0(k) + 1)$. In particular, $pn \geq N_0(k) + 2$ and $\gcd(p, n) = 1$. By the multiplicativity of f , $pn = f(pn) = f(p)f(n) = pf(n)$, which yields $f(n) = n$. Since $n \in \mathbb{N}$ was arbitrary, this proves f is the identity function on $\mathbb{N}$.

We now assume that $f(N) = 0$ for all $N \geq N_0(k) + 2 = 2k + 23$. Fix $s \in \mathcal{E}_2$ and pick an integer a with $a^2 + 1 \geq 2k + 23$. Setting $A_1 = s$ and $A_2 = \dots = A_k = a^2 + 1$ in Equation (2), we get

$$0 = f(s + (k - 1)(a^2 + 1)) = f(s) + (k - 1)f(a^2 + 1),$$

which implies that $f(s) = 0$. Since $s \in \mathcal{E}_2$ was arbitrary, f vanishes at every sum of two nonzero squares. Consequently, f vanishes at every sum of $2k$ nonzero squares. Therefore, f vanishes on $\mathcal{E}_2 \cup \mathcal{E}_2^{(k)}$. By Lemma 1, for $k \geq 3$, we have

$$\begin{aligned} \mathcal{E}_2^{(k)} = \mathbb{N} \setminus (\{1, 2, \dots, 2k - 1\} \\ \cup \{2k + 1, 2k + 2, 2k + 4, 2k + 5, 2k + 7, 2k + 10, 2k + 13\}). \end{aligned}$$

In particular, $f(n) = 0$ for all integers $n \geq 2k + 14$, and additionally for $n \in \{2k + t : t = 0, 3, 6, 8, 9, 11, 12\} \cup \mathcal{E}_2$. $\square$

Corollary 2. *Let $k \geq 5$ and let f be multiplicative and satisfy Equation (1). If $f(\beta) \neq 0$ for some $\beta \in \{a^2 + b^2 : a, b \in \mathbb{N}\} \cup \{2k + t : t = 0, 3, 6, 8, 9, 11, 12\} \cup \{n \in \mathbb{N} : n \geq 2k + 14\}$, then $f(n) = n$ for all positive integers n .*

Proof. The proof is immediate from Theorem 2. $\square$

Remark 2. The threshold $k = 5$ is sharp for this method and cannot be lowered, as we require the sumset of $2(k - 2)$ nonzero squares to contain all but finitely many integers. By Lemma 1, this holds for $2(k - 2) \geq 5$ but fails at $k = 4$, where the sumset misses the infinite progressions $\{2 \cdot 4^r, 6 \cdot 4^r, 14 \cdot 4^r : r \geq 0\}$.

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