



FINITE DIVISIBILITY RUNS BETWEEN GENERALIZED FIBONACCI AND TRIBONACCI SEQUENCES

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Abstract

We study term-by-term divisibility of a generalized Tribonacci sequence (Q_n) by a generalized Fibonacci sequence (P_n) . We first prove that if (P_n) is everywhere nonzero and $P_n \mid Q_n$ for all $n \geq 1$, then (Q_n) must be identically zero. Thus, every nontrivial instance of such divisibility is necessarily finite. We then ask how long a divisibility run can persist and how many initial configurations produce a run of length at least r . For initial values in $[-B, B]$, let $E(B)$ denote the maximal divisibility run length and let $A_B(r)$ denote the number of initial configurations producing a run of length at least r . We prove $E(B) \leq C \log B$ and obtain $A_B(1) = 32B^4 \log B + O(B^4)$ and $A_B(2) = 32B^3(\log B)^2 + O(B^3 \log B)$. For $r = 3$ and $r = 4$, we obtain exact formulas; in the latter case, a coupling between the variables prevents the independent factorization available for smaller values of r .

1. Introduction

Linear recurrence sequences have been studied for a long time in number theory, and Fibonacci-type sequences are among the most familiar examples. Early work of Lucas and Ward laid the foundations of the arithmetic theory of linear recurrence sequences [7, 8, 9, 12]. Following these classical studies, many authors have investigated the arithmetic properties of Fibonacci, Lucas, and related recurrence sequences, with particular emphasis on divisibility phenomena [11, 2, 3, 6].

Generalized Fibonacci sequences and higher-order analogues such as the Tribonacci sequence have also been studied from several directions. For example, Dresden and Du gave a simplified Binet formula for k -generalized Fibonacci numbers, and Spickerman gave a Binet formula for the Tribonacci sequence [4, 10]. On the divisibility side, Aoki and Sakai studied generalized Fibonacci numbers, while Flórez, Higuera, and Mukherjee studied strong divisibility for generalized Fibonacci polynomials [1, 5].

Most of this work studies divisibility within a single recurrence family. In this paper, we instead study divisibility between two different recurrence families. More precisely, if (P_n) is a generalized Fibonacci sequence and (Q_n) is a generalized Tribonacci sequence, we ask when it is possible to have

$$P_n \mid Q_n \quad \text{for all } n \geq 1,$$

or at least for a long initial run. As far as we know, this question has not been studied in this form.

Throughout the paper, for integers x, y, a, b, c , let (P_n) and (Q_n) be defined by

$$P_1 = x, \quad P_2 = y, \quad P_{n+2} = P_{n+1} + P_n,$$

and

$$Q_1 = a, \quad Q_2 = b, \quad Q_3 = c, \quad Q_{n+3} = Q_{n+2} + Q_{n+1} + Q_n.$$

We define the *divisibility run length* by

$$L(x, y; a, b, c) := \sup \left\{ \ell \in \mathbb{Z}_{\geq 0} : P_n \neq 0 \text{ and } P_n \mid Q_n \text{ for every } 1 \leq n \leq \ell \right\}, \quad (1)$$

with $L = \infty$ if the stated conditions hold for every $n \geq 1$. Thus, a zero denominator term ends the run immediately. For $B \geq 1$, let

$$\mathcal{C}_B := \left\{ (x, y, a, b, c) \in ([-B, B] \cap \mathbb{Z})^5 : (x, y) \neq (0, 0), (a, b, c) \neq (0, 0, 0) \right\},$$

and define

$$\begin{aligned} E(B) &:= \max_{(x, y, a, b, c) \in \mathcal{C}_B} L(x, y; a, b, c), \\ A_B(r) &:= \#\{(x, y, a, b, c) \in \mathcal{C}_B : L(x, y; a, b, c) \geq r\}. \end{aligned} \quad (2)$$

Our first main result shows that this kind of divisibility is much more rigid than it may first appear. If (P_n) has no zero terms and $P_n \mid Q_n$ for every $n \geq 1$, then (Q_n) must be identically zero. Thus, nontrivial global divisibility cannot occur. This reduces the problem to a finite one: how long can divisibility persist, and how many choices of initial values produce a run of a given length?

We then study these finite runs quantitatively through the functions $E(B)$ and $A_B(r)$ defined in Equation (2). We prove that

$$E(B) \leq C \log B$$

for some absolute constant $C > 0$, and we also obtain exact formulas in the first few cases, including

$$A_B(1) = 32B^4 \log B + O(B^4), \quad A_B(2) = 32B^3(\log B)^2 + O(B^3 \log B).$$

The paper is organized as follows. In Section 2 we collect the basic facts we need about generalized Fibonacci and Tribonacci sequences. In Section 3 we prove that

nontrivial global divisibility cannot occur. In Section 4 we study finite divisibility runs and the associated counting problems, while Section 5 proves the logarithmic upper bound for the maximal run length. Computational data are presented in Section 6.

2. Preliminaries

The generalized Fibonacci sequence (P_n) and the generalized Tribonacci sequence (Q_n) are both linear, so their terms can be written as linear combinations of the initial values. We record the standard formulas that will be used repeatedly.

Proposition 1. *For all $n \geq 2$, $P_n = F_{n-2}x + F_{n-1}y$, where (F_n) is the standard Fibonacci sequence with $F_1 = F_2 = 1$.*

Proof. Setting $F_0 = 0$, the base cases are $P_2 = F_0x + F_1y = y$ and $P_3 = F_1x + F_2y = x + y$, both correct. For the inductive step, the recurrence gives

$$P_{n+2} = P_{n+1} + P_n = (F_{n-1}x + F_ny) + (F_{n-2}x + F_{n-1}y) = F_nx + F_{n+1}y. \quad \square$$

The Tribonacci sequence has an analogous linear representation.

Proposition 2. *There are integer sequences u_n, v_n, w_n each satisfying the Tribonacci recurrence, with initial conditions $(u_1, u_2, u_3) = (1, 0, 0)$, $(v_1, v_2, v_3) = (0, 1, 0)$, $(w_1, w_2, w_3) = (0, 0, 1)$, such that $Q_n = u_na + v_nb + w_nc$ for all n .*

Proof. The linear combination $R_n = u_na + v_nb + w_nc$ matches Q_n at $n = 1, 2, 3$ by construction, and satisfies the same recurrence. Two Tribonacci sequences that agree on three consecutive terms are equal everywhere. $\square$

The first few terms, written out explicitly, are

$$\begin{aligned} P_1 &= x, & P_2 &= y, & P_3 &= x + y, & P_4 &= x + 2y, & P_5 &= 2x + 3y, \\ Q_1 &= a, & Q_2 &= b, & Q_3 &= c, & Q_4 &= a + b + c, & Q_5 &= a + 2b + 2c. \end{aligned}$$

These identities will be used repeatedly below.

2.1. Growth

Setting $\psi = (1 - \sqrt{5})/2 = -\varphi^{-1}$, the Binet formulas are

$$\begin{aligned} P_n &= A\varphi^{n-1} + B_1\psi^{n-1}, & A &= \frac{y - \psi x}{\sqrt{5}}, & B_1 &= \frac{\varphi x - y}{\sqrt{5}}, \\ Q_n &= C\tau^{n-1} + D\sigma^{n-1} + \overline{D}\overline{\sigma}^{n-1}, \end{aligned}$$

where σ and $\overline{\sigma}$ are the two non-real roots of $x^3 - x^2 - x - 1 = 0$, D and $\overline{D}$ are conjugate coefficients, and $|\sigma| = |\overline{\sigma}| < 1$.

Lemma 1. *Let (P_n) be as above with $(x, y) \neq (0, 0)$. The following bounds hold:*
 (i) $|P_n| \leq (|x| + |y|)\varphi^{n-1}$ for all $n \geq 1$; (ii) for n large enough depending on $|x| + |y|$, we have $|P_n| \geq c_0 \varphi^n / (|x| + |y|)$ for some absolute constant $c_0 > 0$; and
 (iii) $|Q_n| \leq (|a| + |b| + |c|)\tau^{n-1}$ for all $n \geq 1$.

Proof. For (i), we argue by induction on n . The cases $n = 1, 2$ are immediate:

$$|P_1| = |x| \leq |x| + |y|, \quad |P_2| = |y| \leq (|x| + |y|)\varphi.$$

Now suppose for some $n \geq 1$ that

$$|P_n| \leq (|x| + |y|)\varphi^{n-1} \quad \text{and} \quad |P_{n+1}| \leq (|x| + |y|)\varphi^n.$$

Then by the recurrence,

$$|P_{n+2}| \leq |P_{n+1}| + |P_n| \leq (|x| + |y|)(\varphi^n + \varphi^{n-1}) = (|x| + |y|)\varphi^{n+1},$$

since $\varphi^2 = \varphi + 1$. This proves (i).

For (ii), the key observation is that

$$-5AB_1 = y^2 - xy - x^2.$$

This is a nonzero integer: if $x \neq 0$, then y/x would satisfy $t^2 - t - 1 = 0$, which has no rational solutions; if $x = 0$, then $y \neq 0$ so $y^2 \geq 1$. Hence

$$|A||B_1| \geq \frac{1}{5}.$$

Also, from the explicit formula for B_1 ,

$$|B_1| \leq \frac{\varphi|x| + |y|}{\sqrt{5}} \leq |x| + |y|,$$

so

$$|A| \geq \frac{1}{5|B_1|} \geq \frac{1}{5(|x| + |y|)}.$$

Now from the Binet formula

$$P_n = A\varphi^{n-1} + B_1\psi^{n-1},$$

we get

$$|P_n| \geq |A|\varphi^{n-1} - |B_1|\varphi^{-(n-1)}.$$

For n large enough depending on $|x| + |y|$, the second term is at most half the first, so

$$|P_n| \geq \frac{1}{2}|A|\varphi^{n-1} \geq \frac{1}{10(|x| + |y|)}\varphi^{n-1}.$$

Absorbing the factor φ^{-1} into the constant gives (ii).

For (iii), let $M = |a| + |b| + |c|$. The claim is clear for $n = 1, 2, 3$. If it holds for $n, n + 1, n + 2$, then

$$|Q_{n+3}| \leq |Q_{n+2}| + |Q_{n+1}| + |Q_n| \leq M\tau^{n+1} + M\tau^n + M\tau^{n-1} = M\tau^{n+2},$$

since $\tau^3 = \tau^2 + \tau + 1$. This proves (iii). $\square$

We will also use the following two elementary lemmas. The first identifies the relevant greatest common divisor of separated Fibonacci-type terms.

Lemma 2. *With $d = \gcd(x, y)$, we have $\gcd(P_n, P_{n-2}) = d$ for all $n \geq 3$.*

Proof. Using the recurrence, we have $\gcd(P_n, P_{n-2}) = \gcd(P_{n-1} + P_{n-2}, P_{n-2}) = \gcd(P_{n-1}, P_{n-2})$, and iterating this gives $\gcd(P_2, P_1) = \gcd(y, x) = d$. $\square$

We also record two elementary recurrence identities.

Lemma 3. *For all $n \geq 4$, we have $P_{n-1} = P_n - P_{n-2}$ and $P_{n-3} = P_n - 2P_{n-2}$.*

Proof. The first identity is the recurrence rearranged. Apply it twice for the second: $P_{n-3} = P_{n-1} - P_{n-2} = (P_n - P_{n-2}) - P_{n-2}$. $\square$

3. Global Non-Divisibility

We first rule out nontrivial global divisibility.

Theorem 1. *Let (P_n) be a generalized Fibonacci sequence with $P_n \neq 0$ for all n , and let (Q_n) be any generalized Tribonacci sequence. If $P_n \mid Q_n$ for all $n \geq 1$, then $Q_n = 0$ for all n .*

The proof forms the integer sequence $R_n = Q_n/P_n$ and derives two linear relations involving an integer error term. The growth comparison $\tau < \varphi^2$ forces that error term to vanish for all sufficiently large n ; the resulting homogeneous relations then force several consecutive values of R_n to be zero, and hence (Q_n) to be identically zero.

Proof of Theorem 1. Let $d = \gcd(x, y)$. Since $P_n \mid Q_n$ implies $(P_n/d) \mid Q_n$, we may divide through by d and assume $\gcd(x, y) = 1$. Define $R_n = Q_n/P_n \in \mathbb{Z}$. Substituting $Q_n = R_n P_n$ into the Tribonacci recurrence and using Lemma 3, we get

$$(R_n - R_{n-1} - R_{n-3})P_n = (R_{n-2} - R_{n-1} - 2R_{n-3})P_{n-2}.$$

By Lemma 2, $\gcd(P_n, P_{n-2}) = 1$, so there exists $t_n \in \mathbb{Z}$ such that

$$R_n - R_{n-1} - R_{n-3} = t_n P_{n-2}, \quad (3)$$

$$R_{n-2} - R_{n-1} - 2R_{n-3} = t_n P_n. \quad (4)$$

By Lemma 1(ii) and (iii), for all sufficiently large m we have $|R_m| = |Q_m|/|P_m| \ll (\tau/\varphi)^m$, so the left-hand side of Equation (4) is $O((\tau/\varphi)^n)$, whereas $|P_n| \gg \varphi^n$. Hence, $|t_n| \ll (\tau/\varphi^2)^n$. Since $\tau < \varphi^2$, it follows that $t_n = 0$ for all sufficiently large n .

Now fix n so large that $t_n = t_{n+1} = t_{n+2} = 0$, and write $r_0 = R_{n-3}$, $r_1 = R_{n-2}$, $r_2 = R_{n-1}$, $r_3 = R_n$, and $r_4 = R_{n+1}$. Then the relations at indices $n, n+1, n+2$ become

$$r_3 = r_2 + r_0, \quad (5)$$

$$r_1 = r_2 + 2r_0, \quad (6)$$

$$r_4 = r_3 + r_1, \quad (7)$$

$$r_2 = r_3 + 2r_1, \quad (8)$$

$$r_3 = r_4 + 2r_2. \quad (9)$$

From Equations (7) and (9), we get $r_3 = (r_3 + r_1) + 2r_2$, hence $r_1 = -2r_2$. Substituting into Equation (8) gives $r_2 = r_3 - 4r_2$, so $r_3 = 5r_2$. Then Equation (5) yields $r_0 = r_3 - r_2 = 4r_2$, while Equation (6) gives $r_1 = r_2 + 2r_0 = r_2 + 8r_2 = 9r_2$. Thus, $9r_2 = -2r_2$, so $r_2 = 0$, and therefore $r_0 = r_1 = r_3 = r_4 = 0$ as well. Hence, $R_{n-3} = R_{n-2} = R_{n-1} = R_n = R_{n+1} = 0$, so the corresponding five consecutive terms of (Q_n) are zero. In particular, (Q_n) has three consecutive zero terms, and the Tribonacci recurrence then forces $Q_n = 0$ for all n . Thus, (Q_n) is identically zero. $\square$

Remark 1. The hypothesis $P_n \neq 0$ for all n is necessary in Theorem 1, since the proof defines $R_n = Q_n/P_n$. This is consistent with Equation (1): if $P_n = 0$ at some index, the finite divisibility run terminates before that index.

4. Counting Finite Runs

Theorem 1 rules out nontrivial global divisibility. The next question is how to count configurations that produce a long finite run. For small r , the conditions on (x, y, a, b, c) reduce to divisibility constraints on the individual coordinates, which makes the counting manageable.

Proposition 3. *Let $L = L(x, y; a, b, c)$ be as in Equation (1). Then (i) $L \geq 1$ if and only if $x \neq 0$ and $x \mid a$; (ii) $L \geq 2$ if and only if, in addition, $y \neq 0$ and $y \mid b$; and (iii) $L \geq 3$ if and only if, in addition, $x + y \neq 0$ and $x + y \mid c$.*

Proof. This is immediate from the definitions. For $n = 1, 2, 3$, we have

$$P_1 = x, \quad P_2 = y, \quad P_3 = x + y, \quad Q_1 = a, \quad Q_2 = b, \quad Q_3 = c.$$

The stated equivalences therefore follow directly from the condition $P_n \mid Q_n$. $\square$

Write $d_B(m) = 2 \lfloor B/m \rfloor + 1$ for the count of multiples of m in $[-B, B]$.

Proposition 4. *We have*

$$A_B(1) = 4(2B + 1)^3 D(B) + 2B(2B + 1)((2B + 1)^2 - 1),$$

where $D(B) = \sum_{m=1}^B \lfloor B/m \rfloor$. In particular, $A_B(1) = 32B^4 \log B + O(B^4)$.

Proof. For fixed nonzero x , the number of valid choices for a is $d_B(|x|) = 2 \lfloor B/|x| \rfloor + 1$. The values b , c , and y are unconstrained, giving $(2B + 1)^3$ combinations, minus 1 for $(a, b, c) = (0, 0, 0)$. Summing over nonzero x and using the symmetry $x \leftrightarrow -x$ gives

$$\begin{aligned} A_B(1) &= 2 \sum_{m=1}^B [d_B(m)(2B + 1)^2 - 1] (2B + 1) \\ &= 4(2B + 1)^3 D(B) + 2B(2B + 1)((2B + 1)^2 - 1). \end{aligned}$$

The asymptotic uses Dirichlet's estimate $D(B) = B \log B + (2\gamma - 1)B + O(\sqrt{B})$ and $(2B + 1)^3 = 8B^3 + O(B^2)$. $\square$

Proposition 5. *Setting $S(B) = \sum_{m=1}^B d_B(m) = 2D(B) + B$,*

$$A_B(2) = 4(2B + 1)S(B)^2 - 4B^2.$$

In particular, $A_B(2) = 32B^3(\log B)^2 + O(B^3 \log B)$.

Proof. For fixed nonzero x and y , the conditions $x \mid a$ and $y \mid b$ are independent of each other and of c , so the count is $(2B + 1) d_B(|x|) d_B(|y|)$, minus 1 for the zero triple. Summing over nonzero x and y and using symmetry to replace the sum by $4 \sum_{m,n \geq 1}$ gives

$$A_B(2) = 4 \left[(2B + 1) \left(\sum_{m=1}^B d_B(m) \right)^2 - B^2 \right] = 4(2B + 1)S(B)^2 - 4B^2.$$

Since $S(B) = 2B \log B + O(B)$, the asymptotic follows. $\square$

Proposition 6. *We have*

$$A_B(3) = \sum_{\substack{x \neq 0 \\ y \neq 0 \\ x+y \neq 0}} \left(d_B(|x|) d_B(|y|) d_B(|x+y|) - 1 \right).$$

Grouping by $m = |x|$, $n = |y|$, and the signs of x and y gives

$$\begin{aligned} A_B(3) &= 2 \sum_{m,n=1}^B d_B(m) d_B(n) d_B(m+n) \\ &\quad + 4 \sum_{1 \leq n < m \leq B} d_B(m) d_B(n) d_B(m-n) \\ &\quad - (4B^2 - 2B). \end{aligned}$$

Proof. For fixed (x, y) with x, y , and $x+y$ nonzero, the conditions $x \mid a$, $y \mid b$, $x+y \mid c$ each constrain a separate variable, so the corresponding counts are independent and multiply. The sum in the second formula comes from sorting the sign combinations: same-sign pairs give $|x+y| = m+n$ (two combinations, contributing factor 2); opposite-sign pairs with $m \neq n$ give $|x+y| = m-n$ (four combinations for each unordered pair $\{m, n\}$ with $m > n$). The correction $4B^2 - 2B$ is the number of pairs with $x \neq 0$, $y \neq 0$ minus those with $x+y = 0$. $\square$

Remark 2. These formulas were checked by exhaustive computation for $1 \leq B \leq 7$, and in each case they agree exactly with the values reported in Table 2.

Proposition 7. *We have*

$$A_B(4) = \sum_{\substack{x \neq 0, y \neq 0 \\ x+y \neq 0, x+2y \neq 0}} (T_B(x, y) - 1),$$

where

$$T_B(x, y) := \# \left\{ (a, b, c) \in [-B, B]^3 : x \mid a, y \mid b, x+y \mid c, x+2y \mid a+b+c \right\}.$$

Writing $a = x\alpha$, $b = y\beta$, $c = (x+y)\gamma$, define

$$\chi_{x,y}(\alpha, \beta, \gamma) := \begin{cases} 1, & \text{if } x+2y \mid x\alpha + y\beta + (x+y)\gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$A_B(4) = \sum_{\substack{x \neq 0, y \neq 0 \\ x+y \neq 0, x+2y \neq 0}} \left[\sum_{\substack{|\alpha| \leq \lfloor B/|x| \rfloor \\ |\beta| \leq \lfloor B/|y| \rfloor \\ |\gamma| \leq \lfloor B/|x+y| \rfloor}} \chi_{x,y}(\alpha, \beta, \gamma) - 1 \right].$$

Proof. Since $P_4 = x + 2y$ and $Q_4 = a + b + c$, the condition $L \geq 4$ adds $x + 2y \neq 0$ and $x + 2y \mid a + b + c$ to the $L \geq 3$ conditions. The substitution $a = x\alpha$, $b = y\beta$, and $c = (x + y)\gamma$ absorbs the first three divisibility conditions into the ranges of α, β, γ , leaving the fourth condition as a single congruence. Unlike the cases $r \leq 3$, this congruence couples all three free parameters, so $T_B(x, y)$ does not factor into a product of independent counts. $\square$

Example 1. The quintuple $(x, y, a, b, c) = (1, 1, 1, 2, 0)$ achieves $E(2) = 6$. The sequences are

$$(P_n) : 1, 1, 2, 3, 5, 8, 13, \dots \quad (Q_n) : 1, 2, 0, 3, 5, 8, 16, \dots$$

One checks $1 \mid 1$, $1 \mid 2$, $2 \mid 0$, $3 \mid 3$, $5 \mid 5$, $8 \mid 8$, and then $13 \nmid 16$. The exhaustive search at $B = 2$ finds exactly four extremizers: $(\varepsilon, \varepsilon, \delta, 2\delta, 0)$ with $\varepsilon, \delta \in \{\pm 1\}$.

5. The Logarithmic Upper Bound

We now establish the logarithmic upper bound for the maximal run length.

Theorem 2. *There is an absolute constant $C > 0$ with $E(B) \leq C \log B$ for all $B \geq 2$.*

Proof. Fix a nontrivial quintuple and let $L = L(x, y; a, b, c)$. As before, normalize to $\gcd(x, y) = 1$ and let $R_n = Q_n/P_n \in \mathbb{Z}$ for $n \leq L$. By the growth bounds, for $n \geq c_1 \log B$ (where c_1 absorbs the $\log(|x| + |y|) \leq \log(2B)$ from Lemma 1(ii)):

$$|P_n| \geq c_2 B^{-1} \varphi^n, \quad |Q_n| \leq 3B \tau^{n-1}.$$

From the same derivation as Theorem 1, there are integers t_n satisfying Equations (3)–(4), and

$$|t_n| \leq c_4 B^3 \left(\frac{\tau}{\varphi^2} \right)^n.$$

Since $\tau/\varphi^2 < 1$, there exists an absolute constant $c_5 > 0$ such that $|t_n| < 1$ for all $n \geq c_5 \log B$. Because $t_n \in \mathbb{Z}$, it follows that $t_n = 0$ throughout this range.

Now suppose $L \geq N + 2$ where $N = \max\{4, \lceil c_1 \log B \rceil, \lceil c_5 \log B \rceil\}$. Then there are five consecutive values $R_{n-3}, \dots, R_{n+1}$ with $t_n = t_{n+1} = t_{n+2} = 0$. Writing $r_k = R_{n-3+k}$, Equations (3)–(4) at indices $n, n+1, n+2$ give:

$$r_3 = r_2 + r_0, \tag{10}$$

$$r_1 = r_2 + 2r_0, \tag{11}$$

$$r_4 = r_3 + r_1, \tag{12}$$

$$r_2 = r_3 + 2r_1, \tag{13}$$

$$r_3 = r_4 + 2r_2. \tag{14}$$

From Equations (12) and (14) we get $r_1 = -2r_2$. Then Equation (13) gives $r_3 = 5r_2$, and Equation (10) gives $r_0 = 4r_2$. Substitution into Equation (11) yields $r_1 = r_2 + 8r_2 = 9r_2$. Since also $r_1 = -2r_2$, we obtain $11r_2 = 0$, and hence $r_2 = 0$. The preceding relations then imply $r_0 = r_1 = r_3 = r_4 = 0$.

Therefore, $Q_k = 0$ for $k = n - 3, \dots, n + 1$. In particular, (Q_n) contains three consecutive zero terms, which forces the entire Tribonacci sequence to be identically zero. This contradicts $(a, b, c) \neq (0, 0, 0)$. Consequently, $L \leq N + 1 = O(\log B)$. $\square$

6. Computational Results

We searched all nontrivial quintuples $(x, y, a, b, c) \in [-B, B]^5$ for $2 \leq B \leq 13$. For each quintuple, we computed L by checking divisibility up to $n = 30$. In every case, the first failure occurred before this cutoff, so the reported values are unaffected by the truncation. Table 1 records the extremal run lengths and the number of extremizers, while Table 2 records the tail counts.

B	Quintuples searched	$E(B)$	Extremizers
2	2,976	6	4
3	16,416	7	4
4	58,240	7	12
5	159,600	7	12
6	368,928	8	4
7	755,776	8	8
8	1,414,656	8	16
9	2,468,880	8	16
10	4,074,400	8	16
11	6,423,648	8	24
12	9,749,376	9	8
13	14,328,496	9	8

Table 1: Extremal run lengths $E(B)$ and extremizer counts.

The formulas in Propositions 4–6 reproduce the first three rows exactly. The tail counts decrease rapidly; by $r = 8$, only a small number of configurations remain even for $B = 7$.

7. Conclusion

The principal conclusion is that global term-by-term divisibility of a nontrivial generalized Tribonacci sequence by a zero-free generalized Fibonacci sequence is

r	$B = 2$	$B = 3$	$B = 4$	$B = 5$	$B = 6$	$B = 7$
1	1,980	8,876	29,088	66,440	149,240	263,040
2	1,264	4,696	14,336	27,400	59,968	91,064
3	516	1,764	5,716	9,300	21,588	28,588
4	112	572	1,700	2,664	6,572	8,216
5	28	100	372	612	1,540	1,828
6	4	16	72	88	232	284
7	0	4	12	12	44	72
8	0	0	0	0	4	8

Table 2: Tail counts $A_B(r) = \#\{(x, y, a, b, c) \in [-B, B]^5 : (x, y) \neq (0, 0), (a, b, c) \neq (0, 0, 0), L \geq r\}$.

impossible. This is Theorem 1. The finite version is more interesting: runs can be surprisingly long given how fast these sequences grow. The asymptotic formulas

$$A_B(1) = 32B^4 \log B + O(B^4), \quad A_B(2) = 32B^3(\log B)^2 + O(B^3 \log B)$$

give explicit leading asymptotics, while the formula for $r = 3$ remains exact but is more involved. At $r = 4$, the counting problem becomes more complicated: the condition $x + 2y \mid a + b + c$ couples all three free parameters, so the count no longer factors into independent terms. Obtaining a sharp asymptotic would require understanding the average frequency with which this congruence is satisfied over the resulting family of moduli.

A remaining question is whether $E(B) \rightarrow \infty$. The data ($E(B) \in \{6, 7, 8, 9\}$ for $B \leq 13$) are consistent with logarithmic growth, but do not establish unboundedness. Resolving this question would likely require either substantially larger computations or an additional structural argument.

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A. Appendix

The following remark records the role of the code used for the computations.

Remark 3. The following Python code reproduces all the data in Section 6. Listing 1 computes the two recurrence sequences and their divisibility run length, while Listing 2 performs the brute-force counts used for the reported values of $A_B(r)$.

run_length.py

The following function computes the two sequences and returns the run length, the sequences, and the index of first failure.

```

1 def gen_fibonacci(x: int, y: int, n: int):
2     P = [0] * (n + 1)
3     if n >= 1:
4         P[1] = x
5     if n >= 2:
6         P[2] = y
7     for k in range(3, n + 1):
8         P[k] = P[k - 1] + P[k - 2]
9     return P
10
11 def gen_tribonacci(a: int, b: int, c: int, n: int):
12     Q = [0] * (n + 1)
13     if n >= 1:
14         Q[1] = a
15     if n >= 2:
16         Q[2] = b
17     if n >= 3:
18         Q[3] = c
19     for k in range(4, n + 1):
20         Q[k] = Q[k - 1] + Q[k - 2] + Q[k - 3]
21     return Q

```

```

22
23 def run_length(x: int, y: int, a: int, b: int, c: int, N: int):
24     P = gen_fibonacci(x, y, N)
25     Q = gen_tribonacci(a, b, c, N)
26
27     L = 0
28     first_fail = None
29     for k in range(1, N + 1):
30         if P[k] == 0:
31             first_fail = k
32             break
33         if Q[k] % P[k] != 0:
34             first_fail = k
35             break
36     L = k
37
38     if first_fail is None:
39         first_fail = N + 1
40
41     return L, P, Q, first_fail

```

Listing 1: The file `run_length.py`.**count_AB.py**

The following script imports the preceding function and performs the brute-force count over $[-B, B]^5$.

```

1 from run_length import run_length
2
3 def count_AB(B: int, r: int, N: int = 30):
4     total = 0
5
6     for x in range(-B, B + 1):
7         for y in range(-B, B + 1):
8             if (x, y) == (0, 0):
9                 continue
10
11         for a in range(-B, B + 1):
12             for b in range(-B, B + 1):
13                 for c in range(-B, B + 1):
14                     if (a, b, c) == (0, 0, 0):
15                         continue
16
17                     L, _, _, _ = run_length(x, y, a, b, c, N)
18                     if L >= r:
19                         total += 1
20
21     return total
22
23 if __name__ == "__main__":
24     for B in range(2, 8):
25         for r in range(1, 9):
26             value = count_AB(B, r, N=30)
27             print(f"B={B}, r={r}, A_B(r)={value}")

```

Listing 2: The file `count_AB.py`.