



A PROBABILISTIC REPRESENTATION OF GENERALIZED HARMONIC NUMBERS AND AN IDENTITY FOR THEIR SUMS

Toshio Nakata

*Department of Mathematics, University of Teacher Education Fukuoka, Munakata,
Fukuoka, Japan*

`nakata@fukuoka-edu.ac.jp`

Received: 7/15/26, Accepted: 9/23/26, Published: 10/7/26

Abstract

We present a probabilistic representation of generalized harmonic numbers in terms of the standard gamma distribution. Using the Laplace transform and the reproductive property of the gamma distribution, we give a probabilistic proof of a summation identity for generalized harmonic numbers.

1. Introduction

Let α be a real number. For any integer $n \geq 1$, we define the *harmonic numbers* by

$$H_n = \sum_{k=1}^n \frac{1}{k},$$

and the *generalized harmonic numbers* by

$$H_n^{(\alpha)} = \sum_{k=1}^n \frac{1}{k^\alpha}.$$

In what follows, we assume that $n \geq 2$. It is well-known [3, Equation (6.67)] that

$$\sum_{j=1}^{n-1} H_j = nH_n - n.$$

More generally,

$$\sum_{j=1}^{n-1} H_j^{(\alpha)} = nH_n^{(\alpha)} - H_n^{(\alpha-1)} \tag{1}$$

holds for every real α [10, Equation (43)]. Another identity involving harmonic numbers is

$$\sum_{j=1}^{n-1} \binom{j}{m} H_j = \binom{n}{m+1} \left(H_n - \frac{1}{m+1} \right), \quad (2)$$

which holds for every integer $m \geq 0$ [3, Equation (6.70)]. Note that we use the convention that $\binom{j}{m} = 0$ for integers j and m satisfying $0 \leq j < m$.

A common generalization of Equations (1) and (2) is the following proposition.

Proposition 1 ([10, Theorem 25]). *For every integer $m \geq 0$, we have*

$$\sum_{j=1}^{n-1} \binom{j}{m} H_j^{(\alpha)} = \binom{n}{m+1} H_n^{(\alpha)} - \frac{1}{(m+1)!} \sum_{r=0}^{m+1} (-1)^{m+1-r} \left[\begin{matrix} m+1 \\ r \end{matrix} \right] H_n^{(\alpha-r)}, \quad (3)$$

where $\left[\begin{matrix} m \\ j \end{matrix} \right]$ denotes the unsigned Stirling number of the first kind, defined by

$$(x)_m = \sum_{j=0}^m (-1)^{m-j} \left[\begin{matrix} m \\ j \end{matrix} \right] x^j \quad (4)$$

and

$$(x)_m = \begin{cases} x(x-1) \cdots (x-m+1) & \text{for } m \geq 1, \\ 1 & \text{for } m = 0. \end{cases}$$

Setting $m = 0$ in Equation (3) yields Equation (1), while setting $\alpha = 1$ yields Equation (2). As a numerical illustration, we verify Equation (3) in a simple case.

Example 1. Let $n = 3$, $\alpha = 2$, and $m = 1$. We have

$$H_1^{(2)} = 1, \quad H_2^{(2)} = \frac{5}{4}, \quad \text{and} \quad H_3^{(2)} = \frac{49}{36}.$$

The left-hand side of Equation (3) is

$$\sum_{j=1}^2 \binom{j}{1} H_j^{(2)} = \binom{1}{1} H_1^{(2)} + \binom{2}{1} H_2^{(2)} = 1 + \frac{5}{2} = \frac{7}{2}.$$

For the right-hand side, we also need

$$H_3^{(1)} = \frac{11}{6}, \quad H_3^{(0)} = 3,$$

together with $\left[\begin{matrix} 2 \\ 0 \end{matrix} \right] = 0$, $\left[\begin{matrix} 2 \\ 1 \end{matrix} \right] = 1$, and $\left[\begin{matrix} 2 \\ 2 \end{matrix} \right] = 1$. Hence,

$$\binom{3}{2} H_3^{(2)} - \frac{1}{2!} \left(-H_3^{(1)} + H_3^{(0)} \right) = \frac{49}{12} - \frac{7}{12} = \frac{7}{2},$$

which agrees with the left-hand side, confirming Equation (3) in this case.

The aim of this note is to give a probabilistic interpretation of generalized harmonic numbers in terms of the gamma distribution and to provide a probabilistic proof of Equation (3).

Some important identities in combinatorics admit probabilistic interpretations. For example, Peterson [8] gave probabilistic proofs of several identities involving binomial coefficients by using the standard exponential distribution. That is, he considered a random variable X_1 having probability density function

$$f_{X_1}(x) = \begin{cases} e^{-x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

This approach has subsequently been applied to various combinatorial identities; see, for example, [1, 2, 5, 6, 7]. Among these works, Farhadian [1] gave an elegant probabilistic proof of Equation (1) using the Zipf distribution. Motivated by Farhadian's work, we establish the more general identity (3) using the gamma distribution, a natural extension of the exponential distribution.

Throughout this note, let $X_s \sim \text{Gamma}(s, 1)$ denote the gamma distribution with shape parameter $s > 0$ and scale parameter 1. Its probability density function is

$$f_{X_s}(x) = \begin{cases} \frac{e^{-x} x^{s-1}}{\Gamma(s)} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases}$$

where

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$$

is the standard gamma function (see [9, Section 5.6.1]). Our proof is based on the Laplace transform and the reproductive property of the gamma distribution. These provide a unified probabilistic approach to identities involving generalized harmonic numbers.

In what follows, α denotes the parameter of generalized harmonic numbers, whereas $s > 0$ denotes the shape parameter of the gamma distribution. The assumption $s > 0$ is required only for the probabilistic argument, since the gamma distribution is defined only for positive shape parameters. The identities themselves remain valid for arbitrary real values of α .

2. A Representation of Generalized Harmonic Numbers via the Gamma Distribution

We first establish two lemmas describing the relationship between generalized harmonic numbers and the gamma distribution. These results lead directly to the proof of the main identity (3).

Lemma 1. *Let $s > 0$ and let $X_s \sim \text{Gamma}(s, 1)$. Then, we have*

$$H_n^{(s)} = \sum_{k=0}^{n-1} \mathbb{E}(e^{-kX_s}) = \mathbb{E}\left(\frac{1 - e^{-nX_s}}{1 - e^{-X_s}}\right), \quad (5)$$

where $\mathbb{E}(X)$ denotes the expectation of a random variable X .

Proof. According to [4, Section 5.8 (8) and Appendix I.F.12], the Laplace transform of X_s is

$$\mathbb{E}(e^{-tX_s}) = \int_0^\infty e^{-tx} \frac{e^{-x} x^{s-1}}{\Gamma(s)} dx = \left(\frac{1}{1+t}\right)^s \quad \text{for } t > -1.$$

Setting $t = k$, where $k = 0, 1, \dots$, gives

$$\mathbb{E}(e^{-kX_s}) = \frac{1}{(k+1)^s}.$$

Summing both sides over $k = 0, \dots, n-1$ immediately yields Equation (5). $\square$

To explain the property of the gamma distribution that we use below, we first introduce the notion of a reproductive property.

Definition 1. A parametric family of distributions is said to possess the *reproductive property* if the sum of independent random variables from this family again belongs to the same family.

The gamma distribution is known to possess the reproductive property (see [9, Example 3b]): for any $u \in (0, s)$, $X_s \sim \text{Gamma}(s, 1)$ can be represented as

$$X_s \stackrel{d}{=} X_{s-u} + X'_u, \quad (6)$$

where $X'_u \sim \text{Gamma}(u, 1)$ is independent of $X_{s-u} \sim \text{Gamma}(s-u, 1)$. This property enables us to relate generalized harmonic numbers with different parameters.

Lemma 2. *For $0 < u < s$, we have*

$$H_n^{(s-u)} = \sum_{k=0}^{n-1} (k+1)^u \mathbb{E}(e^{-kX_s}). \quad (7)$$

Proof. Since X_{s-u} and X'_u are independent, Equation (6) implies

$$\mathbb{E}(e^{-kX_s}) = \mathbb{E}(e^{-kX_{s-u}}) \mathbb{E}(e^{-kX'_u}).$$

Therefore, it follows from $\mathbb{E}(e^{-kX_u}) = (k+1)^{-u}$ that

$$\mathbb{E}(e^{-kX_{s-u}}) = \frac{\mathbb{E}(e^{-kX_s})}{\mathbb{E}(e^{-kX_u})} = (k+1)^u \mathbb{E}(e^{-kX_s}).$$

Summing both sides over $k = 0, \dots, n-1$ and applying Lemma 1 completes the proof. $\square$

Lemma 2 shows that decreasing the parameter of the generalized harmonic numbers by u is equivalent to multiplying each expectation by the factor $(k+1)^u$. The following identity expresses a weighted sum of expectations in terms of generalized harmonic numbers.

Corollary 1. *For every integer $m \geq 0$ and $s > m+1$, we have*

$$\frac{1}{(m+1)!} \sum_{j=0}^{m+1} (-1)^{m+1-j} \begin{bmatrix} m+1 \\ j \end{bmatrix} H_n^{(s-j)} = \sum_{k=0}^{n-1} \binom{k+1}{m+1} \mathbb{E}(e^{-kX_s}). \quad (8)$$

Proof. Equation (4) yields

$$\begin{aligned} \binom{k+1}{m+1} &= \frac{(k+1)_{m+1}}{(m+1)!} \\ &= \frac{1}{(m+1)!} \sum_{j=0}^{m+1} (-1)^{m+1-j} \begin{bmatrix} m+1 \\ j \end{bmatrix} (k+1)^j. \end{aligned}$$

Multiplying both sides by $\mathbb{E}(e^{-kX_s})$ and summing over $k = 0, \dots, n-1$, we obtain Equation (8) using Equations (5) and (7). Here, for $1 \leq j \leq m+1$, the assumption $s > m+1$ ensures that $0 < j < s$. $\square$

3. Proof of Proposition 1

The proof uses the representation of generalized harmonic numbers obtained in Section 2.

Proof of Proposition 1. We first prove the identity for $\alpha = s > m+1$. The following identity is well-known [3, Table 174] (it is sometimes referred to as the “hockey-stick identity”):

$$\sum_{j=m}^{n-1} \binom{j}{m} = \binom{n}{m+1}.$$

It follows that

$$\sum_{j=k+1}^{n-1} \binom{j}{m} = \binom{n}{m+1} - \binom{k+1}{m+1} \quad \text{for } 0 \leq k \leq n-1. \quad (9)$$

Hence,

$$\begin{aligned} \sum_{j=1}^{n-1} \binom{j}{m} H_j^{(s)} &\stackrel{(5)}{=} \sum_{k=0}^{n-1} \mathbb{E}(e^{-kX_s}) \sum_{j=k+1}^{n-1} \binom{j}{m} \\ &\stackrel{(9)}{=} \binom{n}{m+1} \sum_{k=0}^{n-1} \mathbb{E}(e^{-kX_s}) - \sum_{k=0}^{n-1} \binom{k+1}{m+1} \mathbb{E}(e^{-kX_s}) \\ &\stackrel{(8)}{=} \binom{n}{m+1} H_n^{(s)} - \frac{1}{(m+1)!} \sum_{j=0}^{m+1} (-1)^{m+1-j} \begin{bmatrix} m+1 \\ j \end{bmatrix} H_n^{(s-j)}, \end{aligned}$$

which proves Equation (3) for $\alpha = s > m + 1$. Since both sides of Equation (3), regarded as functions of α , are real-analytic, the identity theorem implies that Equation (3) holds for all real α . $\square$

4. Concluding Remarks

Although this note has been developed in terms of the gamma distribution, an equivalent probabilistic interpretation can also be obtained by using a distribution on $(0, 1)$. Indeed, Peterson [8] used the exponential distribution, whereas Nakata [6] showed that an equivalent argument can be obtained via the corresponding uniform distribution by exploiting the beta function.

To obtain an analogous representation for the gamma distribution, let $X_s \sim \text{Gamma}(s, 1)$ with $s > 0$, and define

$$V_s = e^{-X_s}. \quad (10)$$

It follows that the probability density function of V_s is

$$f_{V_s}(v) = \begin{cases} \frac{(-\log v)^{s-1}}{\Gamma(s)} & \text{for } 0 < v < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The generalized harmonic number $H_n^{(s)}$ admits the representation

$$H_n^{(s)} = \sum_{k=0}^{n-1} \mathbb{E}(V_s^k) = \mathbb{E}\left(\frac{1 - V_s^n}{1 - V_s}\right).$$

The random variable V_s also satisfies the multiplicative semigroup property, for $0 < u < s$,

$$V_s \stackrel{d}{=} V_{s-u} V'_u,$$

where $V'_u = e^{-X'_u}$ and V'_u is independent of V_{s-u} . Thus, the transformation defined by Equation (10) provides an equivalent probabilistic model. However, the gamma

distribution itself seems to be the more natural framework, since both its Laplace transform and its reproductive property are available directly.

Acknowledgements. The author would like to thank the referee for helpful suggestions. This research was supported by KAKENHI JP24K06847 of Japan Society for the Promotion of Science.

References

- [1] R. Farhadian, A probabilistic proof that $\sum_{j=1}^n H_j^{(s)} = (n+1)H_n^{(s)} - H_n^{(s-1)}$, *Integers* **23** (2023), #A33, 3 pp.
- [2] R. Farhadian, A probabilistic proof of a recursion formula for sums of integer powers, *Integers* **23** (2023), #A88, 4 pp.
- [3] R. Graham, D. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, Second Edition, Addison-Wesley, Reading, MA, 1994.
- [4] G. Grimmett and D. Stirzaker, *Probability and Random Processes*, Fourth Edition, Oxford University Press, New York, 2020.
- [5] X. Hu and Y. Zhong, A probabilistic proof of a recursion formula for sums of powers, *Amer. Math. Monthly* **127** (2) (2020), 166–168.
- [6] T. Nakata, Another probabilistic proof of a binomial identity, *The Fibonacci Quarterly* **52.2** (2014), 139–140.
- [7] T. Nakata, Probabilistic proofs of Pascal’s recursion formulas for sums of powers, *Integers* **25** (2025), #A49, 5 pp.
- [8] J. Peterson, A probabilistic proof of a binomial identity, *Amer. Math. Monthly* **120.6** (2013), 558–562.
- [9] S. Ross, *A First Course in Probability*, Tenth Edition, Pearson Education Limited, London, 2020.
- [10] J. Spiess, Some identities involving harmonic numbers, *Math. Comp.* **55** (1990), 839–863.