

MAXIMA OF PARTIAL SUMS OVER THE SYMMETRIC GROUP

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Abstract

In this article, we study the maximum of $\sum_{k=1}^{n}(k-\sigma(k))^q$, with $q \in \mathbb{N} \setminus \{0\}$, over the symmetric group $\mathfrak{S}_n$. Two cases are considered. The first one corresponds to even powers $q = 2r$ of $(k - \sigma(k))$; the maximum can be found analytically using an "exchange argument", resulting from the fact that the function $\varphi(t) = t^{2r}$ (r being a natural number) is even and strictly convex. An exact expression is given, based on the Faulhaber formula and involving Bernoulli numbers. The second case, corresponding to odd powers $q = 2r + 1$ of $(k - \sigma(k))$ is more complicated. A conjecture is proposed, based on the numerical study of powers 3 and 5.

1. Introduction

Let $\mathfrak{S}_n$ be the symmetric group on n letters. For $\sigma \in \mathfrak{S}_n$, define

$$F_{q,n}(\sigma) = \sum_{k=1}^{n}(k-\sigma(k))^q.$$

The study of such dissimilarity measures on the symmetric group $\mathfrak{S}_n$ has important roots in non-parametric statistics. Indeed, for the quadratic case $q = 2$, the functional $F_{2,n}(\sigma)$ is intimately related to Spearman's footrule, a classical metric used to quantify the misalignment between rankings [1, 2]. Our approach can be viewed as a generalization of the permutation depth to higher powers, shedding light on symmetry breaking and large deviations in $\mathfrak{S}_n$ [4, 7]. In this work, we seek

$$F_{q,n}^{\max} = \max_{\sigma\in\mathfrak{S}_n} F_{q,n}(\sigma),$$

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the maximum of $F_{q,n}(\sigma)$, and the permutation(s) that achieve(s) this maximum. An explicit formula is given for even powers q in terms of Bernoulli numbers, and an example is detailed for the particular case $q = 4$. The more difficult case of odd powers is also discussed. The maximum value for even powers, expressed as two nested sums over Bernoulli numbers, is presented in Section 2. The odd-power case is studied in Section 3, starting from $q = 3$ and 5 and opening to a conjecture.

2. Maximum Value for an Even Power $q = 2r$

2.1. Expression in Terms of Bernoulli Numbers

Theorem 1. *Let $n \in \mathbb{N}^*$. We have*

$$F_{2r,n}^{\max} := \max_{\sigma\in\mathfrak{S}_n} \sum_{k=1}^{n} (k-\sigma(k))^{2r} = \sum_{k=1}^{n} (2k-(n+1))^{2r}$$
$$= (n+1)^{2r+1} \left(\sum_{p=0}^{2r} \frac{(-2)^p}{(p+1)} \binom{2r}{p} \sum_{i=0}^{p} \binom{p+1}{i} \frac{B_i}{(n+1)^i} - \frac{1}{(n+1)} \right),$$

where B_i is the ith Bernoulli number.

Proof. Let $q = 2r$ be an even natural number, and

$$\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}^+,$$

where $t \mapsto \varphi(t) = t^{2r}$. Note that φ is even and strictly convex on $\mathbb{R}$ (for $r \geq 1$). We wish to calculate

$$F_{2r,n}^{\max} = \max_{\sigma\in\mathfrak{S}_n} F_{2r,n}(\sigma) = \max_{\sigma\in\mathfrak{S}_n} \sum_{k=1}^{n} \varphi\big(k-\sigma(k)\big).$$

Consider two indices i and j such that $i < j$ and let $a = \sigma(i)$ and $b = \sigma(j)$. If $a < b$ (that is, if the elements of the pair (i, j) are not decreasing in their images), we compare the contribution of these two indices before and after swapping the images a and b. The increase obtained by replacing the pair (a, b) with (b, a) is

$$\Delta = \varphi(i-b) + \varphi(j-a) - \varphi(i-a) - \varphi(j-b).$$

Let $x = i - a$ and $y = j - b$. Then $i - b = x - (b - a)$ and $j - a = y + (b - a)$, so

$$\Delta = \varphi(x-(b-a)) + \varphi(y+(b-a)) - \varphi(x) - \varphi(y).$$

The convexity of φ (and the fact that it is even) immediately implies that for $b - a > 0$ we have $\Delta > 0$. Indeed, increasing the absolute distance of the two arguments by pushing them apart, increases in turn the sum for a convex even

function. Thus, any transposition that makes the sequence of images $\sigma(1), \ldots, \sigma(n)$ "more decreasing" does not reduce the sum, and these swaps can be iterated until we obtain a strictly decreasing sequence of images.

Hence, any permutation maximizing $F_{2r,n}$ must have $\sigma(1) \geq \sigma(2) \geq \cdots \geq \sigma(n)$. Since σ is bijective, the only possibility is the reversing permutation

$$\sigma_{\text{rev}}(k) = n + 1 - k,$$

which achieves the maximum.

The contribution for the reversing permutation

$$d_k := k - \sigma_{\text{rev}}(k) = k - (n + 1 - k) = 2k - (n + 1)$$

gives

$$F_{2r,n}^{\max} = \sum_{k=1}^{n} (2k - (n+1))^{2r}.$$

According to the binomial theorem, we have

$$(2k - n - 1)^{2r} = \sum_{p=0}^{2r} (-2)^p \binom{2r}{p} k^p (n+1)^{2r-p},$$

yielding

$$\begin{aligned} F_{2r,n}^{\max} &= \sum_{k=1}^{n} \sum_{p=0}^{2r} (-2)^p \binom{2r}{p} k^p (n+1)^{2r-p} \\ &= \sum_{p=0}^{2r} (-2)^p \binom{2r}{p} (n+1)^{2r-p} \left(\sum_{k=1}^{n} k^p \right). \end{aligned} \tag{1}$$

The Faulhaber formula [3, 5]

$$\sum_{k=0}^{n} k^p = \frac{1}{(p+1)} \sum_{i=0}^{p} \binom{p+1}{i} B_i \, n^{p+1-i},$$

where B_i is the ith Bernoulli number and $B_1 = -1/2$, can be recast into

$$\sum_{k=1}^{n} k^p = \frac{1}{(p+1)} \sum_{i=0}^{p} \binom{p+1}{i} B_i \, n^{p+1-i} - \delta_{p,0}, \tag{2}$$

where $\delta_{i,j}$ represents the Kronecker symbol. Inserting Equation (2) into Equation (1) gives

$$\begin{aligned} \max_{\sigma \in \mathfrak{S}_n} \sum_{k=1}^{n} (k - \sigma(k))^{2r} &= F_{2r,n}^{\max} \\ &= (n+1)^{2r+1} \\ &\quad \times \left(\sum_{p=0}^{2r} \frac{(-2)^p}{(p+1)} \binom{2r}{p} \sum_{i=0}^{p} \binom{p+1}{i} \frac{B_i}{(n+1)^i} - \frac{1}{(n+1)} \right), \end{aligned}$$

which completes the proof. □

For $r = 1$, this maximum can be elegantly expressed in terms of a binomial coefficient:

$$F_{2,n}^{\max} = 2\binom{n+1}{3} = \frac{n(n^2-1)}{3}.$$

This explicit formula generates the sequence $0, 2, 8, 20, 40, 70, \ldots$ for $n \geq 1$, which is well-documented in the Online Encyclopedia of Integer Sequences (OEIS) under the identifier A007290 [6].

It is convenient to distinguish between even and odd n. In the case where n is even ($n = 2m$), we can write

$$2k - (n+1) = 2k - (2m+1) \in \{\pm(2m-1), \pm(2m-3), \ldots, \pm 1\},$$

and each absolute value $2j-1$ (for $j = 1, \ldots, m$) appears twice (by symmetry of k and $n+1-k$). Thus, one has

$$\max_{\sigma \in \mathfrak{S}_n} \sum_{k=1}^{n} (k - \sigma(k))^{2r} = 2\sum_{j=1}^{m} (2j-1)^{2r}, \quad \text{where } n = 2m.$$

Conversely, in the case where $n = 2m+1$ is odd, then

$$2k - (n+1) = 2k - (2m+2) \in \{\pm 2m, \pm 2(m-1), \ldots, 0\},$$

where the value 0 (for $k = m+1$) appears once, and each value $\pm 2j$ (for $j = 1, \ldots, m$) appears twice. Therefore, we have

$$\begin{aligned} \max_{\sigma \in \mathfrak{S}_n} \sum_{k=1}^{n} (k - \sigma(k))^{2r} &= 2\sum_{j=1}^{m} (2j)^{2r} \\ &= 2^{2r+1} \sum_{j=1}^{m} j^{2r}, \quad \text{where } n = 2m+1. \end{aligned}$$

2.2. The Case $q = 4$

Let us set $r = 2$. We seek $F_{4,n}^{\max} = \max_{\sigma \in \mathfrak{S}_n} F_{4,n}(\sigma)$ and an explicit polynomial expression in n. For any even, strictly convex function (here $t \mapsto \varphi(t) = t^4$), the exchange argument shows that the permutation maximizing $\sum_k \varphi(k - \sigma(k))$ is the reversing permutation $\sigma_{\mathrm{rev}}(k) = n + 1 - k$. Therefore, one has

$$F_{4,n}^{\max} = \sum_{k=1}^{n} \big(k - \sigma_{\mathrm{rev}}(k)\big)^4 = \sum_{k=1}^{n} \big(2k - (n+1)\big)^4.$$

Let us expand the expression

$$\sum_{k=1}^{n}(2k-(n+1))^4 = \sum_{k=1}^{n}\big(16k^4 - 32(n+1)k^3 + 24(n+1)^2k^2 - 8(n+1)^3k + (n+1)^4\big).$$

Using the classical sums

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{k=1}^{n} k^3 = \Big(\frac{n(n+1)}{2}\Big)^2,$$
$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30},$$

or alternatively the expression in terms of Bernoulli numbers, we obtain, after algebraic manipulation, the compact polynomial form:

$$F_{4,n}^{\max} = \sum_{k=1}^{n}(2k-(n+1))^4 = \frac{n\big(3n^4-10n^2+7\big)}{15} = \frac{3n^5-10n^3+7n}{15}.$$

This formula is valid for all $n \geq 1$. When $n = 2m$ is even, one has

$$\max_{\sigma\in\mathfrak{S}_n}\sum_{k=1}^{n}(k-\sigma(k))^4 = 2\sum_{j=1}^{m}(2j-1)^4 = 2\sum_{j=1}^{m}\big(16j^4-32j^3+24j^2-8j+1\big),$$

and if $n = 2m+1$ is odd, we get

$$\max_{\sigma\in\mathfrak{S}_n}\sum_{k=1}^{n}(k-\sigma(k))^4 = 2\sum_{j=1}^{m}(2j)^4 = 2\cdot 16\sum_{j=1}^{m} j^4 = 32\sum_{j=1}^{m} j^4.$$

Expanding these sums and simplifying via Faulhaber's formula yields exactly

$$F_{4,n}^{\max} = \frac{n(3n^4-10n^2+7)}{15}.$$

For instance, for $n = 4$ ($m = 2$), the reversing permutation gives differences $(\pm 3, \pm 1)$, and

$$F_{4,n}^{\max} = 2(3^4+1^4) = 2(81+1) = 164.$$

Similarly, for $n = 5$ ($m = 2$), the reversing permutation gives differences $(\pm 4, \pm 2, 0)$, and

$$F_{4,n}^{\max} = 2(4^4+2^4) = 2(256+16) = 544.$$

3. Expressions for an Odd Power $q = 2r + 1$

For odd q, the function $t \mapsto t^q$ is odd, concave on $\mathbb{R}^-$ and convex on $\mathbb{R}^+$, and the analysis of the maxima changes completely. The exchange-permutation technique used for the even case does not apply anymore.

3.1. The $q = 3$ Case

The exchange-permutation technique used for even powers fails here because a complete reversal of the permutation would produce symmetric pairs of differences whose cubic powers cancel out, yielding a total sum of 0. For odd powers, the sum benefits from creating a few very large positive differences while keeping the magnitudes of negative differences as small and uniform as possible. For $n \geq 1$ and $\sigma \in \mathfrak{S}_n$, we have

$$F_{3,n}(\sigma) = \sum_{k=1}^{n} (k - \sigma(k))^3.$$

We want to determine the permutations that maximize $F_{3,n}$ and give a simple explicit expression that can be evaluated to find the maximum for any given n. For that purpose, let us consider, for an integer s with $1 \leq s \leq \lfloor n/2 \rfloor$, the permutation $\sigma_{n,s}$ defined as follows:

$$\sigma_{n,s}(k) = \begin{cases} k + s, & \text{if } 1 \leq k \leq n - s, \\ s + 1 - (k - (n - s)), & \text{if } n - s + 1 \leq k \leq n. \end{cases} \tag{3}$$

Actually, we have

$$(\sigma_{n,s}(1), \ldots, \sigma_{n,s}(n)) = (1 + s, 2 + s, \ldots, n,\ s, s - 1, \ldots, 1),$$

which means that the first $n - s$ elements were moved forward by s, and the block $(s, s - 1, \ldots, 1)$ was placed in the last s places.

For the permutation $\sigma_{n,s}$ the differences $d_k = k - \sigma_{n,s}(k)$ have the simple form:

$$d_k = \begin{cases} -s, & \text{if } 1 \leq k \leq n - s, \\ n - 2s + 1,\ n - 2s + 3,\ \ldots,\ n - 1, & \text{if } n - s < k \leq n. \end{cases}$$

Hence, the contribution of $\sigma_{n,s}$ is

$$F_{3,n}(\sigma_{n,s}) = \sum_{j=0}^{s-1} \big(n - 2s + 1 + 2j\big)^3 \ - \ (n - s)\, s^3. \tag{4}$$

For an odd power the sum benefits from creating a few very large positive differences while keeping the magnitudes of negative differences as small (and equal) as possible. This is actually exactly what the permutation $\sigma_{n,s}$ does: it produces $n - s$ equal negative differences $-s$ and s (strictly increasing) positive differences that are the

n	max value	maximizer
1	0	(1)
2	0	(1,2), (2,1)
3	6	(2,3,1)
4	24	(2,3,4,1)
5	60	(2,3,4,5,1)
6	120	(3,4,5,6,2,1)
7	240	(3,4,5,6,7,2,1)
8	420	(3,4,5,6,7,8,2,1)
9	672	(3,4,5,6,7,8,9,2,1)
10	1008	(3, 4, 5, 6, 7, 8, 9, 10, 2, 1)
11	1512	(4, 5, 6, 7, 8, 9, 10, 11, 3, 2, 1)
12	2160	(4, 5, 6, 7, 8, 9, 10, 11, 12, 3, 2, 1)

Table 1: Maximum $\max_{\sigma\in\mathfrak{S}_n} F_{3,n}(\sigma)$ and explicit maximizing permutation $\sigma_{n,s}$ for $n \leq 12$.

largest possible given the bijectivity constraint, namely the odd-spaced block $n-2s+1,\ldots,n-1$. Heuristically, swapping two images out of pattern tends to increase the value for odd powers, suggesting that any maximizer can be rearranged into a permutation of this form. A brute-force numerical check over the symmetric group $\mathfrak{S}_n$ for small n indicates that the global maximum indeed falls within this family for all tested cases.

A brute-force numerical verification over the symmetric group $\mathfrak{S}_n$ for a few values of n confirms that the global maximum seems to be always achieved within this family. However, contrary to what a limited analysis for small n might suggest, the optimal parameter s is not fixed but grows stepwise with n. Table 1 illustrates the first few transitions of the optimal parameter s_{opt} and the corresponding maximum value $F_{3,n}^{\max}$. The maximizer belongs to the family $\sigma_{n,s}$ with $s=1$ for $3\leq n\leq 5$, $s=2$ for $6\leq n\leq 10$ and $s=3$ for $n=11$ and 12.

To understand the systematic mechanism behind these transitions, let us introduce the continuous variable $\alpha = s/n$ with $0<\alpha\leq 1/2$. Approximating the discrete sum of the positive terms in Equation (4) by an integral yields the leading-order behavior:

$$\frac{F_{3,n}(\sigma_{n,s})}{n^4} \approx \mathcal{F}_3(\alpha),$$

where

$$\mathcal{F}_3(\alpha) = \int_0^{\alpha} (1-2\alpha+2x)^3 dx - (1-\alpha)\alpha^3 = \alpha(1-\alpha)^3.$$

The continuous function $\mathcal{F}_3(\alpha)$ on $(0,1/2]$ reaches its unique maximum where its derivative vanishes:

$$\mathcal{F}_3'(\alpha) = (1-4\alpha)(\alpha-1)^2,$$

n	max value	maximizer
1	0	(1)
2	0	(1,2), (2,1)
3	30	(2,3,1)
4	240	(2,3,4,1)
5	1020	(2,3,4,5,1)
6	3240	(3,4,5,6,2,1)
7	8640	(3,4,5,6,7,2,1)
8	19740	(3,4,5,6,7,8,2,1)
9	40320	(3,4,5,6,7,8,9,2,1)
10	77280	(4,5,6,7,8,9,10,3,2,1)
11	138600	(4,5,6,7,8,9,10,11,3,2,1)
12	234720	(4,5,6,7,8,9,10,11,12,3,2,1)

Table 2: Maximum $\max_{\sigma\in\mathfrak{S}_n} F_{5,n}(\sigma)$ and explicit maximizing permutation $\sigma_{n,s}$ for $n \le 12$.

yielding the exact optimal ratio

$$\alpha^* = \frac{1}{4} = 0.25.$$

Consequently, for large n, the optimal integer s satisfies the asymptotic behavior $s_{\text{opt}} \sim \lfloor 0.25\, n \rfloor$. The sequence of transitions observed numerically (such as the switch to $s = 3$ at $n = 11$, where $3/11 \approx 0.273$) is nothing but the discrete realization of this continuous regular growth towards the $\alpha^* = 1/4$ attractor. The global maximum thus obeys the scaling law:

$$F_{3,n}^{\max} = \max_{\sigma\in\mathfrak{S}_n} F_{3,n}(\sigma) \sim \lfloor \mathcal{F}_3(0.25)\, n^4 \rfloor = \left\lfloor \frac{27}{256}\, n^4 \right\rfloor \approx \lfloor 0.105\, n^4 \rfloor \text{ as } n \to \infty.$$

3.2. The $q = 5$ Case

To search for optimal permutations in the $q = 5$ case, we extend the heuristic approach developed for $q = 3$ and consider the family of candidate permutations $\sigma_{n,s}$ defined in Equation (3). Assuming the maximum is attained within this family, finding $F_{5,n}^{max}$ reduces to searching over $s = 1, 2, ..., \lfloor n/2 \rfloor$ for the value that maximizes $F_{5,n}(\sigma_{n,s})$.

It is instructive to look at small-n exact results. For each $1 \le n \le 12$, the maximum values of $F_{5,n}(\sigma_{n,s})$ for $1 \le s \le \lfloor n/2 \rfloor$, verified by brute force, are given in Table 2. In order to get (numerically) the exact maximum for any concrete n, one has to compute Equation (4) for each integer $s = 1, \ldots, \lfloor n/2 \rfloor$ and pick s that maximizes $F_{5,n}(\sigma_{n,s})$. The maximizer belongs to the family $\sigma_{n,s}$ with $s = 1$ for $3 \le n \le 5$, and $s = 2$ for $6 \le n \le 12$.

It is also useful to study how the optimal s grows with n. Set $\alpha = s/n$. Using

the Riemann summation, we obtain the leading order behavior

$$\frac{F_{5,n}(\sigma_{n,s})}{n^6} \;\approx\; \mathscr{F}_5(\alpha),$$

with

$$\mathscr{F}_5(\alpha) = \frac{1}{3}(1-\alpha)^3\alpha(3-6\alpha+13\alpha^2),$$

and $0 < \alpha \le \frac{1}{2}$. This can be obtained by replacing the discrete sum of the s odd-spaced positive terms by an integral. The continuous function $\mathscr{F}_5(\alpha)$ is smooth on $(0, 1/2]$ and attains a unique maximum at some $\alpha^* \in (0, 1/2)$. One has

$$\mathscr{F}_5'(\alpha) = (1-\alpha)^2(1-8\alpha+23\alpha^2-26\alpha^3).$$

Numerically, one gets $\alpha^* \approx 0.274$ and

$$\mathscr{F}_5(\alpha^*) = \max_{\alpha} \mathscr{F}_5(\alpha) \approx 0.0815.$$

Consequently, for large n the optimal integer s behaves like

$$s_{\text{opt}} \sim \lfloor \alpha^* n \rfloor = \lfloor 0.274n \rfloor,$$

and the maximal value satisfies

$$F_{5,n}^{\max} = \max_{\sigma \in \mathfrak{S}_n} F_{5,n}(\sigma) \sim \left\lfloor \left(\max_{\alpha} \mathscr{F}_5(\alpha) \right) n^6 \right\rfloor \approx \left\lfloor 0.0815\, n^6 \right\rfloor \text{ as } n \to \infty.$$

3.3. General Case for q Odd: Shift-and-Reverse Permutations

When q is odd, the strictly increasing function $t \mapsto t^q$ is odd and not convex on all of $\mathbb{R}$. The exchange argument used for even powers therefore fails. Intuitively, for odd q, it can be advantageous to create a few large positive differences (close to $n-1$) and many small negative ones (like -1). Cycles produce a very uniform pattern $(-r, \ldots, -r, n-r, \ldots, n-r)$, but one can often do slightly better by adjusting a few images to make the positive values unequal (for instance, $n-1$, $n-3$, $\ldots$), which increases the sum of odd powers. The permutation $\sigma_{n,s}$ defined in Equation (3) by a shift on the first block and a reversal (mirroring) on the remaining elements, implies that the differences $d_k = k - \sigma_{n,s}(k)$ are perfectly asymmetric. For the first $n-s$ indices, the shift yields a constant negative difference $d_k = -s$. For the last s indices, the reversal creates large positive differences, spanning from $(n-s)+1-s$ up to $n-1$.

Hence, for an odd power q, the functional $F_{n,q}(\sigma_{n,s})$ acts as a probe for this broken symmetry. Due to the rapid growth of the odd powers, the large positive differences in the reversed block dominate the constant negative contributions. A natural family of candidates for the global maximum magnitude is therefore obtained by maximizing over the parameter s:

$$\max_{1 \le s \le \lfloor n/2 \rfloor} \left| F_{n,q}(\sigma_{n,s}) \right|.$$

4. Conclusion

In this work, we studied the maximum of $\sum_{k=1}^{n}(k-\sigma(k))^q$ over the symmetric group $\mathfrak{S}_n$. For even q, convexity forces the reversal permutation to be optimal and we obtained, using simple algebraic manipulations and the Faulhaber formula, a general expression involving Bernoulli numbers. For odd $q \geq 3$, the symmetry breaks: the maximizing permutation depends on n and q, and often belongs to (or is close to) the family of circular shifts. The reason why reversal (which worked for even powers) fails here is simple: reversal produces symmetric pairs of differences $\pm a$ whose odd powers cancel, giving 0; but for odd powers one prefers to produce a small number of very large positive differences and many small negatives. This is exactly what the following permutations do:

$$\sigma_{n,s}(k) = \begin{cases} k+s, & \text{if } 1 \leq k \leq n-s, \\ s+1-(k-(n-s)), & \text{if } n-s+1 \leq k \leq n, \end{cases}$$

where the parameter s is not fixed but grows stepwise with n. It is hoped that this work will stimulate the development of a general and rigorously proved formula for odd powers.

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