

ON THE SYMMETRIC SQUARE COEFFICIENTS

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Abstract

Let $\omega(n)$ denote the number of distinct prime factors of n. For the second moment of the coefficients $\lambda_{\mathrm{Sym}^2 f}(n)$ associated with the symmetric square lift of a Hecke cusp form f, we obtain an asymptotic formula for the distribution of integers with $\omega(n) = k$ in short intervals $(X, X+Y]$, valid uniformly over a wide range of Y. Furthermore, we prove a central limit theorem for $\lambda^2_{\mathrm{Sym}^2 f}(n)$ and show that the associated error term is optimal.

1. Introduction

For a positive integer n, let $\omega(n)$ denote the number of distinct prime divisors of n. The study of the distribution of $\omega(n)$ goes back to classical results in analytic number theory. Turán [35] proved that

$$\sum_{n\le X}\big(\omega(n)-\log_2 X\big)^2 \ll X\log_2 X,$$

where $\log_2 X := \log\log X$ denotes the 2-*fold iterated logarithm*. Given arithmetic functions g and G, we say that g has *normal order* G if, for each $\epsilon>0$, we have

$$|g(n)-G(n)| \le \epsilon |G(n)|$$

on a set of integers n of density 1. The essence of Turán's argument was probabilistic, and as a corollary, he recovered a result of Hardy and Ramanujan [15] that the normal order of $\omega(n)$ is $\log_2 n$. The natural next step was to ask about the distribution of the values of $\omega(n)$ around its normal order. In their celebrated 1940 paper, Erdős and Kac [8] showed that $\omega(n)$ follows a Gaussian law. More precisely, for each $\lambda\in\mathbb{R}$,

$$\frac{1}{X}\sum_{\substack{n\le X\\ \omega(n)-\log_2 X\le \lambda(\log_2 X)^{1/2}}} 1 \longrightarrow \Phi(\lambda), \quad \text{as } X\to\infty,$$

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where $\Phi(\lambda)$ is the *standard normal distribution function* given by

$$\Phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-t^2/2}\, dt.$$

The Erdős-Kac theorem marked the beginning of probabilistic number theory. For further developments, we refer to the monograph of Kubilius [21]. Later refinements were given by Halberstam [14], who introduced the method of "all moments," and by Billingsley [1], who provided a probabilistically natural approach via auxiliary independent random variables.

Since then, many extensions of the Erdős–Kac theorem have been obtained for other arithmetic functions. Murty, Murty, and Pujahari [26] developed a general axiomatic framework for sequences $\{a_n\}$, showing that $\omega(a_n)$ satisfies an Erdős–Kac type law under broad hypotheses. Elliott [6, 7] proved weighted central limit theorems for coefficients of GL_2-forms, including the case of $d(n)^\alpha$, where $d(n)$ denotes the divisor function. In particular, for

$$D_\alpha(X) := \sum_{n \le X} d(n)^\alpha,$$

Elliott established

$$\frac{1}{D_\alpha(X)} \sum_{\substack{n \le X \\ \omega(n) - 2^\alpha \log_2 X \le \lambda (2^\alpha \log_2 X)^{1/2}}} d(n)^\alpha \;\longrightarrow\; \Phi(\lambda), \qquad X \to \infty,$$

which recovers the Erdős–Kac theorem when $\alpha = 0$. Recently, Liu and Wu [24] extended these results to short intervals. Defining

$$D_\alpha(X, Y) := \sum_{X < n \le X+Y} d(n)^\alpha,$$

they proved that

$$\frac{1}{D_\alpha(X, Y)} \sum_{\substack{X < n \le X+Y \\ \omega(n) - 2^\alpha \log_2 X \le \lambda (2^\alpha \log_2 X)^{1/2}}} d(n)^\alpha = \Phi(\lambda) + O_{\alpha,\epsilon}\left(\frac{1}{\sqrt{\log_2 X}} \right),$$

uniformly for $X^{7/12+\epsilon} \le Y \le X$, with optimal error term. They also studied the corresponding local distribution

$$\pi_{k,\alpha}(X, Y) := \sum_{\substack{X < n \le X+Y \\ \omega(n) = k}} d(n)^\alpha,$$

and derived sharp asymptotics in the range $1 \le k \le B 2^\alpha \log_2 X$ for some $B > 0$.

In parallel with these developments, attention has also turned to the distribution of Hecke eigenvalues. Let $f \in \mathcal{S}_r(1)$ be a Hecke cusp form of even integral weight r on the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Its Fourier expansion at the cusp at infinity is given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)\, n^{\frac{r-1}{2}}\, e(nz), \quad e(z) := e^{2\pi i z}, \qquad z \in \mathbb{H},$$

where the normalization is given by $\lambda_f(1) = 1$, the *upper half-plane* is defined by $\mathbb{H} := \{x + iy \in \mathbb{C} : y > 0\}$, and the coefficients $\{\lambda_f(n)\}_{n\geq 1}$ form a real-valued multiplicative sequence. The associated *Hecke L-function* is defined as

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},$$

which converges absolutely for $\Re(s) > 1$ and admits the Euler product

$$L(f,s) = \prod_p L_p(f,s), \qquad L_p(f,s) = \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1},$$

with Satake parameters satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \qquad \alpha_f(p)\beta_f(p) = 1, \qquad |\alpha_f(p)| = |\beta_f(p)| = 1.$$

For $n > 1$, let $P^+(n)$ denote the largest prime factor of n, and set $u = \log X/\log Y$. Wang [36] proved that for large Y and $1 \leq u \leq \exp\left(\frac{1}{a}\log Y\right)$,

$$\sum_{\substack{n\leq X \\ P^+(n)\leq Y}} \lambda_f^2(n) = \frac{X}{\log X}\, V(Y) \left(\varrho(u) + O\left(\frac{\log(u+1)}{u\log Y}\right)\right),$$

where ϱ is the *Dickman function* (see, [36]) and

$$V(Y) = \prod_{p\leq Y} \left(1 + \sum_{j\geq 1} \frac{\lambda_f^2(p^j)}{p^j}\right).$$

Building upon this framework, Yao [38] analyzed it over short intervals. For any $\epsilon > 0$, real number $B > 0$, and $2 \leq k \leq B\log_2 X$, Yao proved that

$$\sum_{\substack{X<n\leq X+Y \\ \omega(n)=k}} \lambda_f^2(n) = \frac{Y}{\log X}\, \frac{(\log_2 X)^{k-1}}{(k-1)!} \left(\lambda\left(\frac{k-1}{\log_2 X}\right) + O_{B,\epsilon}\left(\frac{k-1}{(\log_2 X)^2}\right)\right),$$

valid for Y in an appropriate range, where

$$\lambda(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \sum_{i\geq 1} \frac{\lambda_f^2(p^i) z}{p^i}\right) \left(1 - \frac{1}{p}\right)^z.$$

For the special case $k = 1$, this yields

$$\sum_{X<p\le X+Y} \lambda_f^2(p) = \frac{Y}{\log X}\left(1 + O\left(\frac{\log_2 X}{\log Y}\right)\right).$$

The *symmetric power L-function associated with* $f \in \mathcal{S}_r(1)$ is defined, for $j \ge 0$, by

$$L(\mathrm{Sym}^j f, s) = \prod_p \prod_{i=0}^{j} \left(1 - \alpha_f^{j-i}(p)\beta_f^i(p)\, p^{-s}\right)^{-1} =: \sum_{n=1}^{\infty} \lambda_{\mathrm{Sym}^j f}(n) n^{-s},$$

which converges absolutely for $\Re(s) > 1$, with coefficients $\{\lambda_{\mathrm{Sym}^i f}(n)\}_{n\ge1}$ real and multiplicative.

The coefficients $\lambda_{\mathrm{Sym}^j f}(n)$ have been investigated extensively, particularly regarding their asymptotic behavior when the indices are restricted to specific arithmetic sequences. Sharma and Sankaranarayanan initiated the study of moments of the coefficients of symmetric power L-functions over sums of squares. They established asymptotic formulas for the second, third, and fourth moments of $\lambda_{\mathrm{Sym}^2 f}(n)$ over sums of four and six squares [30, 31, 32]. These results have subsequently been refined and generalized by Hua [16, 17], P. Wang and T. Wang [37], Liu and Huang [25], Feng [9], and Tiwari and Godara [12, 34]. More recently, we [11] extended this line of research by investigating the third and fourth moments of the coefficients $\lambda_{\mathrm{Sym}^j f}(n)$.

1.1. Main Results

The principal contributions of this paper are twofold: we establish an asymptotic formula for the weighted sum over integers n in a short interval and prove a Gaussian law for $\omega(n)$ with respect to the probability measure induced by $\lambda^2_{\mathrm{Sym}^2 f}(n)$.

Theorem 1. *Let $B > 0$ and let $f \in \mathcal{S}_r(1)$ be a Hecke eigenform. Then for $\epsilon > 0$, we have*

$$\sum_{\substack{X<n\le X+Y\\ \omega(n)=k}} \lambda^2_{\mathrm{Sym}^2 f}(n) = \frac{Y}{\log X}\frac{(\log_2 X)^{k-1}}{(k-1)!} \times \left(\lambda\left(\frac{k-1}{\log_2 X}\right) + O_{f,B,\epsilon}\left(\frac{k-1}{(\log_2 X)^2} + \frac{\log_2 X}{k\log X}\right)\right),$$

uniformly for $X^{\frac{238+663B}{420+663B}+\epsilon} \le Y \le X$, and $2 \le k \le B\log_2 X$ with sufficiently large X, where

$$\lambda(z) := \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \sum_{j\ge1} \frac{\lambda^2_{\mathrm{Sym}^2 f}(p^j) z}{p^j}\right)\left(1 - \frac{1}{p}\right)^z.$$

In the special case $k = 1$, we obtain

$$\sum_{X<p\leq X+Y} \lambda^2_{\mathrm{Sym}^2 f}(p) = \frac{Y}{\log X} + O\left(\frac{Y \log_2 X}{(\log X)^2}\right).$$

Theorem 1 refines the result of Yao [38] in the case of $\lambda^2_{\mathrm{Sym}^2 f}(n)$. Next, we establish a central limit theorem for $\omega(n)$ with respect to the probability measure induced by $\lambda^2_{\mathrm{Sym}^2 f}(n)$.

Theorem 2. *Let $B > 0$ and let $f \in \mathcal{S}_r(1)$ be a Hecke eigenform. For*

$$S(X, Y) := \sum_{X<n\leq X+Y} \lambda^2_{\mathrm{Sym}^2 f}(n),$$

we have, for each $\epsilon > 0$ and $\lambda \in \mathbb{R}$,

$$\frac{1}{S(X,Y)} \sum_{\substack{X<n\leq X+Y \\ \omega(n)-\log_2 X\leq\lambda\sqrt{\log_2 X}}} \lambda^2_{\mathrm{Sym}^2 f}(n) = \Phi(\lambda) + O_{f,B,\epsilon}\left(\frac{1}{\sqrt{\log_2 X}}\right), \quad (1)$$

uniformly for $X^{\frac{238+663B}{420+663B}+\epsilon} \leq Y \leq X$ with sufficiently large X, where the implied constant depends only on f, B, and ϵ. Here, $\Phi(\lambda)$ denotes the standard normal distribution. Moreover, the error term in Equation (1) *is optimal.*

Throughout the paper, we adopt the following notation. We write $U(X) \ll V(X)$ (or equivalently $U(X) = O(V(X))$), if there exist constants $C > 0$ and $X_0 > 0$ such that $|U(X)| \leq C|V(X)|$ for all $X > X_0$. For functions $U(X)$ and $V(X)$, we write $U(X) = o(V(X))$ as $X \to \infty$ if $\lim_{X\to\infty} U(X)/V(X) = 0$. We write $U(X) \sim V(X)$ if $\lim_{X\to\infty} U(X)/V(X) = 1$. All asymptotics are taken as $X \to \infty$. The symbol ϵ always denotes a small positive constant, which may vary from one occurrence to another. Throughout, $\zeta(s)$ stands for the Riemann zeta function, while $\Gamma(s)$ denotes the Euler Gamma function. For a complex number $s \in \mathbb{C}$, we write $s = \sigma + it$ with $\Re(s) = \sigma$ and $\Im(s) = t$. We also use the standard notation

$$\log_2 X := \log\log X, \qquad \omega(n) := \#\{p \mid n : p \text{ prime}\}.$$

2. Preliminaries

Let $\mathbb{A}_{\mathbb{Q}}$ denote the *adele ring* of $\mathbb{Q}$, namely the restricted direct product of the completions $\mathbb{Q}_v$ over all places v of $\mathbb{Q}$, taken with respect to the rings of integers $\mathbb{Z}_p$ at the finite places. For this notion, and for the notions of an automorphic representation and of cuspidality used below, we refer the reader to Bump [2, Chapter

3]. For each $f \in \mathcal{S}_r(1)$, there exists a cuspidal automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ such that $L(\pi_f, s) = L(f, s)$.

We shall use the following case of the Langlands Functoriality Conjecture; for the general formulation in terms of L-groups, we refer to Langlands [22].

Conjecture 1 ([22])**.** Let $j \geq 1$ be an integer and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, with Satake parameters $\alpha_\pi(p)$ and $\beta_\pi(p)$ at an unramified prime p. Then π admits a *symmetric power lift* $\mathrm{Sym}^j\pi$, that is, an automorphic representation of $\mathrm{GL}_{j+1}(\mathbb{A}_{\mathbb{Q}})$ whose Satake parameters at every such p are $\alpha_\pi^{j-i}(p)\beta_\pi^i(p)$ for $0 \leq i \leq j$.

For $\pi = \pi_f$ with $f \in \mathcal{S}_r(1)$, Conjecture 1 is now a theorem, and comparing Euler products gives

$$L(\mathrm{Sym}^j\pi_f, s) = L(\mathrm{Sym}^j f, s).$$

The cases $j = 2$, $j = 3$, and $j = 4$ are due to Gelbart and Jacquet [10], Kim and Shahidi [20], and Kim [19], respectively. Newton and Thorne [27, 28] recently established that $\mathrm{Sym}^j\pi_f$ is a cuspidal automorphic representation of $\mathrm{GL}_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j \geq 1$. As a consequence, each $L(\mathrm{Sym}^j f, s)$ for $j \geq 1$ extends analytically to the whole complex plane and satisfies a functional equation analogous to that of the Riemann zeta function (see [3, Section 3.2.1] and [18, Chapter 5]).

Remark 1. From the above discussion, for every $j \geq 1$, the symmetric power L-function $L(\mathrm{Sym}^j f, s)$ belongs to the class of general L-functions in the sense of Perelli [29].

We shall use the following subconvexity bound, which is the second estimate in Equation (1.7) of [5].

Lemma 1 ([5])**.** *For any $\epsilon > 0$ and $|t| \geq 1$, we have*

$$L\big(\mathrm{Sym}^2 f, \tfrac{1}{2} + it\big) \ll (1 + |t|)^{\frac{4}{7}+\epsilon}.$$

2.1. Dirichlet Series of Type $\mathcal{P}(z, \mu, \alpha, \delta, A, B, C, M)$

In analytic number theory, the Selberg–Delange method furnishes asymptotic formulas for partial sums of arithmetic functions whose Dirichlet series can be expressed as the product of a suitably regular analytic function and a complex power of the Riemann zeta function of the form $\zeta(s)^w$, where $w \in \mathbb{C}$. We now recall the notion of Dirichlet series of type $\mathcal{P}$, which provides the analytic framework for applying Selberg–Delange type methods in short intervals.

Let $f(n)$ be an arithmetic function, and consider its Dirichlet series

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \qquad s = \sigma + it, \quad \sigma \gg 1.$$

Throughout, we take the principal branch of the logarithm.

Definition 1. Let $z, \mu \in \mathbb{C}$, $\alpha > 0$, $\delta, A \geq 0$, and $B, C, M > 0$. We say that the Dirichlet series $\mathcal{F}(s)$ is of type $\mathcal{P}(z, \mu, \alpha, \delta, A, B, C, M)$ if the following conditions hold.

(i) Growth of coefficients: for every $\epsilon > 0$,

$$f(n) \ll n^{\epsilon}.$$

(ii) Convergence near $\sigma = 1$: for $\sigma > 1$,

$$\sum_{n=1}^{\infty} f(n) n^{-\sigma} \ \ll \ (\sigma - 1)^{-\alpha}.$$

(iii) Analytic continuation: define

$$\mathcal{G}(s; z, \mu) := \mathcal{F}(s)\, \zeta(s)^{-z}\, \zeta(2s)^{-\mu}.$$

Then $\mathcal{G}(s; z, \mu)$ admits analytic continuation to $\sigma > \frac{1}{2}$ and satisfies the bound

$$|\mathcal{G}(s; z, \mu)| \leq M(1 + |t|)^{\max\{\delta(1-\sigma), 0\}} \log^{A}(1 + |t|),$$

uniformly for $|z| \leq B$ and $|\mu| \leq C$.

The following two lemmas are crucial in proving Theorem 1.

Lemma 2. *For $\Re(s) > 1$, the Dirichlet series*

$$\mathcal{H}(s; z, f) = \sum_{n=1}^{\infty} \frac{\lambda^2_{\mathrm{Sym}^2 f}(n) z^{\omega(n)}}{n^s}$$

is of type

$$\mathcal{P}\left(z, \frac{1}{2}(z - 3z^2),\, 1,\, \frac{51}{14}B,\, 0,\, B,\, \frac{B}{2}(3B + 1),\, M_B\right),$$

for some suitable constants $B > 0$ and $M_B > 0$.

Proof. From Remark 1 and Perelli's result on general L-functions [29], one has that, for each $j \geq 1$, the convexity bound

$$L(\mathrm{Sym}^j f, \sigma + it) \ \ll (1 + |t|)^{\max\{\frac{1}{2}(j+1)(1-\sigma), 0\} + \epsilon}, \tag{2}$$

is valid uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ and $|t| \geq 1$. In the special case $j = 2$, the Dirichlet series defining $L(\mathrm{Sym}^2 f, s)$ converges absolutely on the line $\sigma = 1 + \epsilon$, so that $L(\mathrm{Sym}^2 f, 1 + \epsilon + it) \ll_{f,\epsilon} 1$ there. Interpolating this bound with Lemma 1 by means of the Phragmén–Lindelöf principle yields the sharper estimate

$$L(\mathrm{Sym}^2 f, \sigma + it) \ \ll \ (1 + |t|)^{\frac{8}{7}(1-\sigma) + \epsilon}, \tag{3}$$

which improves upon the earlier bound of Lin, Nunes, and Qi [23]. From Lemma 2.4 in [38], for

$$\mathcal{G}(s;z,\mu) := \mathcal{H}(s;z,f)\,\zeta(s)^{-z}\,\zeta(2s)^{-\mu},$$

we have

$$\mathcal{G}\left(s;z,\frac{1}{2}(z-3z^2)\right) = L(\mathrm{Sym}^2 f,s)^z L(\mathrm{Sym}^4 f,s)^z V(s;z,f),$$

where $V(s;z,f)$ is absolutely convergent for $\Re(s) > \frac{1}{2}$. Moreover, for $\sigma > \frac{1}{2}$ and $|z| \le B$, one has

$$V(\sigma+it;z,f) \ll_{\epsilon,f} M_1(B),$$

with $M_1(B)$ a positive constant depending only on B. Applying the estimate from Equation (2) to $L(\mathrm{Sym}^j f,s)$ with $j=4$ and the refined estimate from Equation (3) to $j=2$, we obtain

$$\mathcal{G}(\sigma+it;z,\frac{1}{2}(z-3z^2)) \le M_1(B)M_2(B)\,(1+|t|)^{\frac{51}{14}B(1-\sigma)},$$

uniformly for $|z| \le B$, where $M_1(B)$ and $M_2(B)$ are positive constants. For $\sigma > 1$ and $|z| \le B$, the Euler product defining $\mathcal{H}(s;z,f)$ converges absolutely. Indeed, using $|z^{\omega(n)}| \le B^{(1+o(1))\frac{\log n}{\log_2 n}} \ll_B n^{\epsilon}$ for $|z| \le B$, we have

$$\sum_{n\ge 1} \frac{|\lambda_{\mathrm{Sym}^2 f}(n)|^4 |z|^{\omega(n)}}{n^{\sigma}} \ll_B \int_1^{\infty} \frac{dt}{t^{\sigma-\epsilon}} \ll_{B,\epsilon} \frac{1}{\sigma-1}.$$

For $|z| \le B$, we have

$$|\mu(z)| = \left|\frac{1}{2}(z-3z^2)\right| \le \frac{B}{2}(3B+1).$$

Setting $M(B) := M_1(B)M_2(B)$, we obtain that $\mathcal{H}(s;z,f)$ is of type

$$\mathcal{P}\left(z,\frac{1}{2}(z-3z^2),\,1,\,\frac{51}{14}B,\,0,\,B,\,\frac{B}{2}(3B+1),\,M_B\right).$$

□

Lemma 3. *For any $\epsilon > 0$, we have*

$$\sum_{X<n\le X+Y} \lambda^2_{\mathrm{Sym}^2 f}(n) z^{\omega(n)} = Y(\log X)^{z-1}\left(z\lambda(z) + O_{f,B,\epsilon}\left(\frac{1}{\log X}\right)\right),$$

uniformly for $X \ge 3$ and $X^{\frac{238+663B}{420+663B}+\epsilon} \le Y \le X$, where the implied constant depends at most on $f, B,$ and ϵ. Here,

$$\lambda(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \sum_{j\ge 1} \frac{\lambda^2_{\mathrm{Sym}^2 f}(p^j)\,z}{p^j}\right)\left(1-\frac{1}{p}\right)^z.$$

Proof. Let $N(\sigma,T)$ denote the number of zeros $\rho=\beta+it$ of the Riemann zeta function $\zeta(s)$ lying in the rectangle $\sigma\le\beta\le1$, $|t|\le T$. A refined form of the Selberg–Delange method in short intervals, due to Cui, Lü, and Wu [4, Corollary 1.2], combined with the zero-density estimate

$$N(\sigma,T)\ll T^{\frac{30}{13}(1-\sigma)+\epsilon},\qquad \sigma\ge\frac{1}{2},\ T\ge2,$$

established by Guth and Maynard [13], and using Lemma 2, we get the desired result. □

Proof of Theorem 1. The asymptotic formula in the range $2\le k\le B\log_2 X$ follows from [38, Theorem 1.2] together with Lemmas 2 and 3. □

A function $F:\mathbb{R}\to\mathbb{R}$ is said to be a *distribution function* if it is non-decreasing and satisfies

$$F(-\infty)=0,\qquad F(+\infty)=1.$$

The *characteristic function* of F is the Fourier–Stieltjes transform, defined by

$$f(\tau):=\int_{-\infty}^{+\infty}e^{i\tau x}\,dF(x).$$

Now, we state the Berry–Esseen inequality [33, Theorem II.7.16], first proved by Berry (1941) and independently by Esseen (1942).

Lemma 4 (Berry–Esseen inequality)**.** *Let F and G be two distribution functions with respect to characteristic functions f and g. Suppose G is differentiable and G' is bounded on $\mathbb{R}$. Then, for all $T>0$,*

$$\|F-G\|_\infty\ \le\ 16\,\frac{\|G'\|_\infty}{T}\ +\ 6\int_{-T}^{T}\left|\frac{f(\tau)-g(\tau)}{\tau}\right|d\tau,$$

where $\|F\|_\infty:=\sup_{\lambda\in\mathbb{R}}|F(\lambda)|$.

Proof of Theorem 2. We begin by defining the normalized distribution function

$$F_{X,Y}(f,\lambda):=\frac{1}{S(X,Y)}\sum_{\substack{X<n\le X+Y\\ \omega(n)-\log_2 X\le\lambda\sqrt{\log_2 X}}}\lambda^2_{\mathrm{Sym}^2 f}(n),$$

and its characteristic function

$$\begin{aligned}\phi_{X,Y}(f,\tau)&=\int_{-\infty}^{\infty}e^{i\tau\lambda}\,dF_{X,Y}(f,\lambda)\\&=\frac{1}{S(X,Y)}\sum_{X<n\le X+Y}\lambda^2_{\mathrm{Sym}^2 f}(n)\,\exp\left(\frac{i\tau\big(\omega(n)-\log_2 X\big)}{\sqrt{\log_2 X}}\right).\end{aligned}$$

Setting $T := \sqrt{\log_2 X}$, the function $\phi_{X,Y}(f,\tau)$ can be written as

$$\phi_{X,Y}(f,\tau) = \frac{e^{-i\tau T}}{S(X,Y)} \sum_{X<n\le X+Y} \lambda^2_{\mathrm{Sym}^2 f}(n)\, e^{i(\tau/T)\omega(n)}.$$

By the Berry–Esseen inequality (Lemma 4),

$$\|F_{X,Y} - \Phi\|_\infty \le \frac{16}{\sqrt{2\pi}T} + 6\int_{-T}^{T} \left| \frac{\phi_{X,Y}(f,\tau) - e^{-\tau^2/2}}{\tau} \right| d\tau.$$

Thus, the theorem reduces to proving

$$\int_{-T}^{T} \left| \frac{\phi_{X,Y}(f,\tau)}{\tau} - \frac{e^{-\tau^2/2}}{\tau} \right| d\tau \ll \frac{1}{T}. \tag{4}$$

We first expand $\phi_{X,Y}(f,\tau)$. From Lemma 3, inserting $z = e^{it}$ yields

$$\frac{1}{S(X,Y)} \sum_{X<n\le X+Y} \lambda^2_{\mathrm{Sym}^2 f}(n) e^{it\omega(n)} = (\log X)^{(e^{it}-1)} \Big(A(e^{it}) + O(1/\log X) \Big),$$

uniformly for $t \in \mathbb{R}$, $X \ge 3$ and $X^{\frac{238+663B}{420+663B}+\epsilon}$, where $A(z) = z\lambda(z)/\lambda(1)$ is holomorphic in $|z| \le B$ with $A(1) = 1$. Setting $t = \tau/T$, we obtain

$$\phi_{X,Y}(f,\tau) = (\log X)^{(e^{i\tau/T}-1)} e^{-i\tau T} \Big(A(e^{i\tau/T}) + O(1/\log X) \Big), \tag{5}$$

uniformly for $|\tau| \le T$.

Next, we split the Berry–Esseen integral. Since $T = \sqrt{\log_2 X}$, for large X, we have

$$\frac{1}{\log X} < T^{1/3} = (\log_2 X)^{1/6} < T = (\log_2 X)^{1/2}.$$

Thus, the cutoffs

$$0 < \frac{1}{\log X} < T^{1/3} < T$$

define three natural scales for τ. Accordingly, we decompose

$$[-T, T] = I_1 \cup I_2 \cup I_3,$$

where

$$\begin{aligned} I_1 &= [-1/\log X, 1/\log X], \\ I_2 &= [-T^{1/3}, -1/\log X] \cup [1/\log X, T^{1/3}], \\ I_3 &= [-T, -T^{1/3}] \cup [T^{1/3}, T]. \end{aligned}$$

We first treat the region I_1 of very small τ, where $|\tau| \le 1/\log X$. Here,

$$\exp\left(\frac{i\tau\left(\omega(n)-\log_2 X\right)}{\sqrt{\log_2 X}}\right) = 1 + O\left(\frac{|\tau|\log X}{T}\right),$$

while $e^{-\tau^2/2} = 1 + O(\tau^2)$. Therefore, we have

$$\int_{I_1}\left|\frac{\phi_{X,Y}(f,\tau) - e^{-\tau^2/2}}{\tau}\right| d\tau \ll \frac{1}{T}.$$

We turn to the region I_2 of intermediate τ, where $1/\log X \le |\tau| \le T^{1/3}$. Using the expansions

$$A(e^{i\tau/T}) = 1 + O(\tau/T)$$

and

$$e^{i\tau/T} = 1 + i\tau/T - (1/2)(\tau/T)^2 + O((\tau/T)^3)$$

for small τ/T, and inserting them into Equation (5), we find

$$\phi_{X,Y}(f,\tau) = e^{-\tau^2/2}\left(1 + O\left(\frac{|\tau| + |\tau|^3}{T}\right)\right)\left(1 + O(1/\log X)\right).$$

Consequently, we have

$$\int_{I_2}\left|\frac{\phi_{X,Y}(f,\tau) - e^{-\tau^2/2}}{\tau}\right| d\tau \ll \frac{1}{T}.$$

Finally, we treat the region I_3 of large τ, where $T^{1/3} \le |\tau| \le T$. For $|t| \le 1$, we have $\cos t - 1 \le -2(t/\pi)^2$, and therefore it follows that

$$|(\log X)^{(e^{i\tau/T}-1)}e^{-i\tau T}| = e^{(\cos(\tau/T)-1)T^2} \le e^{-2(\tau/\pi)^2}.$$

Thus, $\phi_{X,Y}(f,\tau) \ll e^{-2(\tau/\pi)^2}$, which implies

$$\int_{I_3}\left|\frac{\phi_{X,Y}(f,\tau) - e^{-\tau^2/2}}{\tau}\right| d\tau \ll \frac{1}{T}.$$

Combining the bounds for I_1, I_2, and I_3, we establish Equation (4). This completes the initial part of the proof.

It remains to be proved that the error term is optimal. To quantify the error precisely, set

$$E_{X,Y}(\lambda) := F_{X,Y}(f,\lambda) - \Phi(\lambda), \qquad E(X,Y) := \sup_{\lambda\in\mathbb{R}} |E_{X,Y}(\lambda)|.$$

Let $k = \lfloor \log_2 X \rfloor$ and $\varphi := k - \log_2 X \in (-1, 0]$. We define

$$S_k(X,Y) := \sum_{\substack{X<n\leq X+Y \\ \omega(n)=k}} \lambda^2_{\mathrm{Sym}^2 f}(n).$$

Choose a constant c with $0 < c < 1$. Then, by the definition of $F_{X,Y}$, we have

$$F_{X,Y}\left(f, \frac{\varphi}{\sqrt{\log_2 X}}\right) = \frac{1}{S(X,Y)} \sum_{\substack{X<n\leq X+Y \\ \omega(n)\leq k}} \lambda^2_{\mathrm{Sym}^2 f}(n)$$

and

$$F_{X,Y}\left(f, \frac{\varphi - c}{\sqrt{\log_2 X}}\right) = \frac{1}{S(X,Y)} \sum_{\substack{X<n\leq X+Y \\ \omega(n)\leq k-c}} \lambda^2_{\mathrm{Sym}^2 f}(n).$$

Since $0 < c < 1$, the condition $\omega(n) \leq k-c$ is equivalent to $\omega(n) \leq k-1$. Subtracting these two equalities gives

$$\frac{S_k(X,Y)}{S(X,Y)} = F_{X,Y}\left(f, \frac{\varphi}{\sqrt{\log_2 X}}\right) - F_{X,Y}\left(f, \frac{\varphi - c}{\sqrt{\log_2 X}}\right).$$

Since $F_{X,Y}$ is close to Φ up to the error $E(X,Y)$, we obtain

$$\begin{aligned} \frac{S_k(X,Y)}{S(X,Y)} &\leq \Phi\left(\frac{\varphi}{\sqrt{\log_2 X}}\right) - \Phi\left(\frac{\varphi - c}{\sqrt{\log_2 X}}\right) + 2E(X,Y) \qquad (6) \\ &\leq \int_{\frac{\varphi-c}{\sqrt{\log_2 X}}}^{\frac{\varphi}{\sqrt{\log_2 X}}} \frac{1}{\sqrt{2\pi}}\, e^{-\tau^2/2}\, d\tau + 2E(X,Y) \\ &\leq \frac{c}{\sqrt{2\pi \log_2 X}} + 2E(X,Y). \end{aligned}$$

On the other hand, applying Stirling's formula together with Theorem 1 and the asymptotic in Lemma 3 at $z = 1$, we get

$$\frac{S_k(X,Y)}{S(X,Y)} \sim \frac{1}{\sqrt{2\pi \log_2 X}}.$$

Comparing it with the bound in Equation (6) forces

$$E(X,Y) \;\geq\; \frac{1-c}{2\sqrt{2\pi \log_2 X}}\,(1 + o(1)),$$

uniformly for $X \geq 3$ and $X^{\frac{238+663B}{420+663B}+\epsilon} \leq Y \leq X$. This shows that the error term cannot be improved beyond this order, and hence, it is optimal. □

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