

OFF-DIAGONAL RADO NUMBERS FOR $x + y + c = z$ AND $x + y + k = z$

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Abstract

In the two-color off-diagonal setting, one fixes a pair of linear equations $(\mathcal{E}_1, \mathcal{E}_2)$ and seeks the least integer N such that every red–blue coloring of $\{1, 2, \ldots, N\}$ contains either a red solution to $\mathcal{E}_1$ or a blue solution to $\mathcal{E}_2$. This threshold integer is referred to as the off-diagonal Rado number of the system $(\mathcal{E}_1, \mathcal{E}_2)$. In this work, we study the discrete and continuous two-color off-diagonal Rado numbers for the nonhomogeneous linear equations $x+y+c = z$ and $x+y+k = z$, where $c \leq k$. In the discrete setting, c and k are nonnegative integers, whereas in the continuous setting, they are nonnegative real numbers. We determine the exact discrete and continuous two-color off-diagonal Rado numbers for this pair of shifted Schur equations.

1. Introduction

The study of Ramsey-type problems for linear equations originated with Schur's theorem and was later placed in a systematic framework by Richard Rado. Schur's classical theorem [11] establishes that every finite coloring of the positive integers $\mathbb{N}$

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must contain a monochromatic solution to the equation $x+y=z$. Rado [8] generalized Schur's theorem by characterizing the homogeneous systems of linear equations that are partition regular over the positive integers. Such a system is *partition regular* if every finite coloring of the positive integers admits a monochromatic solution. While the early literature largely concentrated on diagonal settings, where a single equation must have a monochromatic solution, attention has more recently shifted toward other variants, such as off-diagonal, disjunctive, or simultaneous versions.

In the off-diagonal setting, rather than seeking a monochromatic solution to a single equation, one seeks to guarantee that at least one of two equations admits a monochromatic solution in its prescribed color class. This viewpoint naturally extends classical Schur-type problems by allowing different equations to be associated with different colors. As a result, it leads to a richer framework that reveals new combinatorial behavior and provides a deeper structural understanding of these problems.

Definition 1 (Rado number)**.** Let $\mathcal{E}$ be a linear equation with integer coefficients. The *t-color Rado number* of $\mathcal{E}$, denoted by $R_t(\mathcal{E})$, is the smallest positive integer N such that every t-coloring of the set $\{1, 2, \ldots, N\}$ contains a monochromatic solution to $\mathcal{E}$. If no such integer N exists, then $R_t(\mathcal{E})$ is said to be infinite.

Definition 2 (Off-diagonal Rado number)**.** Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be linear equations with integer coefficients. The *two-color off-diagonal Rado number* of the system of equations $(\mathcal{E}_1, \mathcal{E}_2)$, denoted by $R_2(\mathcal{E}_1, \mathcal{E}_2)$, is the smallest positive integer N such that every red–blue coloring of the set $\{1, 2, \ldots, N\}$ contains either a red solution to $\mathcal{E}_1$ or a blue solution to $\mathcal{E}_2$. If no such integer N exists, then $R_2(\mathcal{E}_1, \mathcal{E}_2)$ is said to be infinite.

Definition 3 (Off-diagonal continuous Rado number)**.** Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be linear equations with real coefficients, and let $\alpha \in \mathbb{R}$. The *two-color off-diagonal continuous Rado number* of the system of equations $(\mathcal{E}_1, \mathcal{E}_2)$ on $[\alpha, \infty)$, denoted by $R_2(\mathcal{E}_1, \mathcal{E}_2; \alpha)$, is the smallest real number $N \geq \alpha$ such that every red–blue coloring of the interval $[\alpha, N]$ contains either a red solution to $\mathcal{E}_1$ or a blue solution to $\mathcal{E}_2$. If no such real number N exists, then $R_2(\mathcal{E}_1, \mathcal{E}_2; \alpha)$ is said to be infinite.

Throughout the paper, variables occurring in a solution are not required to be distinct.

2. Previous Work

The study of Rado numbers has its origins in classical Ramsey theory. Schur [11] proved that the equation $x + y = z$ is regular, introducing what are now known as Schur numbers. Rado [8] generalized this work by characterizing the homogeneous

linear systems that are partition regular over the positive integers, thereby identifying precisely which systems admit monochromatic solutions under every finite coloring.

Burr and Loo [4] initiated the systematic study of exact Rado numbers for specific linear equations. In particular, they determined the two-color Rado number of the shifted Schur equation

$$R_2(x + y + c = z) = 4c + 5$$

for every integer $c \geq -1$. Beutelspacher and Brestovansky [2] proved that, for every positive integer k, the two-color Rado number of the generalized Schur equation is

$$R_2(x_1 + x_2 + \cdots + x_k = x_0) = k^2 + k - 1.$$

A natural extension of this theory is the off-diagonal setting, where instead of a single equation, one considers a pair of equations $(\mathcal{E}_1, \mathcal{E}_2)$ and seeks the smallest integer N such that every red–blue coloring of $[1, N]$ yields either a red solution to $\mathcal{E}_1$ or a blue solution to $\mathcal{E}_2$. This framework generalizes classical Rado numbers by allowing different equations to be associated with different colors.

The first systematic study in this direction was carried out by Robertson and Schaal [9], who introduced off-diagonal generalized Schur numbers. For positive integers $k \geq 3$ and $l \geq 3$, they determined the smallest integer $S(k, l)$ such that every red–blue coloring of $[1, S(k, l)]$ contains either a red solution to

$$x_1 + x_2 + \cdots + x_{k-1} = x_k$$

or a blue solution to

$$x_1 + x_2 + \cdots + x_{l-1} = x_l.$$

Further developments were made by Myers and Robertson [7], who established general bounds for homogeneous linear equations ($x + qy = z$ and $x + sy = z$) and obtained exact values in several special cases. Yao and Xia [14] subsequently obtained formulas for other homogeneous families, including

$$R_2(3x + 3y = z, 3x + 3qy = z)$$

and

$$R_2(2x + 3y = z, 2x + 2qy = z).$$

More recently, Jing and Mei [5] determined

$$R_2(2x + qy = 2z, 2x + sy = 2z)$$

for odd integers $q > s \geq 1$.

Sabo, Schaal, and Tokaz [10] studied shifted Schur equations such as $x_1 + x_2 + c = x_3$ and determined the disjunctive Rado number. In 2025, Vestal and Glackin [12] studied the same equations in the context of simultaneous solutions. In this paper, we determine the corresponding two-color off-diagonal Rado number for the same equations $x + y + c = z$ and $x + y + k = z$, thereby covering another natural Ramsey-type variant for this shifted Schur family. More recently, an off-diagonal Rado number for a combination of homogeneous and nonhomogeneous equations has also been studied [1].

3. Main Result

Throughout this paper, we denote the two-color off-diagonal Rado number by

$$R_2(c,k) := R_2(x+y+c=z,\ x+y+k=z),$$

where $x+y+c=z$ is called the *c-equation* and is associated with red color, and $x+y+k=z$ is called the *k-equation* and is associated with blue color.

3.1. The Discrete Version

Theorem 1. *Let c and k be nonnegative integers with $c \le k$. Then the two-color off-diagonal Rado number $R_2(c,k)$ for the system*

$$\begin{aligned} x+y+c &= z \quad \textit{(red)},\\ x+y+k &= z \quad \textit{(blue)} \end{aligned}$$

is given by

$$R_2(c,k) = \begin{cases} \infty, & \textit{if } c \not\equiv k \pmod 2,\\ k+3c+5, & \textit{if } c \equiv k \pmod 2 \textit{ and } k \le 2c,\\ 2k+c+4, & \textit{if } c \equiv k \pmod 2 \textit{ and } k > 2c. \end{cases}$$

Proof. We evaluate $R_2(c,k)$ case by case.

`Case 1.` c and k have opposite parity. First, suppose c is even, and k is odd. Color every odd integer red and every even integer blue. If x, y, and z formed a red solution to $x+y+c=z$, then x, y, and z would all be odd. However, the sum of two odd numbers is even, and since c is even, z would also be even, a contradiction.

Similarly, if x, y, and z formed a blue solution to $x+y+k=z$, then x, y, and z would all be even. However, the sum of two even numbers and an odd number is odd, so z would be odd, again a contradiction. Hence, this coloring avoids both types of monochromatic solutions.

If c is odd and k is even, reverse the coloring: color every even integer red and every odd integer blue. In this case, any red solution to $x+y+c=z$ would require x, y, and z to be even, but the sum of two even numbers and an odd number is odd; therefore, this is impossible. Likewise, any blue solution to $x+y+k=z$ would require x, y, and z to be odd, but the sum of two odd numbers and an even number is even; therefore, this is again impossible.

Thus, for every N there exists a coloring of $[1,N]$ with no red solution to $x+y+c=z$ and no blue solution to $x+y+k=z$. Therefore, the off-diagonal Rado number is infinite.

`Case 2.` c and k have the same parity with $k \le 2c$. We show that $R_2(c,k) \ge k+3c+5$.

Let

$$R = \{1, 2, \ldots, c+1\} \cup \{k+2c+4, \ldots, k+3c+4\},$$
$$B = \{c+2, c+3, \ldots, k+2c+3\}.$$

Consider the two-coloring of $[1, k+3c+4]$ defined by

$$\chi(n) = \begin{cases} \text{Red}, & n \in R, \\ \text{Blue}, & n \in B. \end{cases}$$

Under this coloring, we will show that there is no red solution to the c-equation and no blue solution to the k-equation with all variables in $[1, k+3c+4]$.

If x and y are blue, then $c+2 \le x, y \le k+2c+3$, and hence

$$z = x + y + k \ge (c+2) + (c+2) + k = k + 2c + 4.$$

Thus, for any x and y in B, the value of z is either in the second red block or outside the colored interval. Therefore, there is no blue solution to the k-equation.

If x and y are red and belong to the first block, then

$$z = x + y + c \ge 1 + 1 + c = c + 2$$

and

$$z \le (c+1) + (c+1) + c = 3c + 2.$$

Since $c \le k$, we have $3c + 2 < k + 2c + 3$. Therefore, z belongs to B and is blue. If x comes from the first block and y from the second block of the red interval, then $z \ge 1 + (k + 2c + 4) + c = k + 3c + 5$, so z lies outside the colored interval $[1, k + 3c + 4]$. Similarly, if both x and y come from the second red block, then $z = x + y + c \ge 2(k + 2c + 4) + c > k + 3c + 4$, so z lies outside the colored interval. Hence, there is no red solution to the c-equation. Consequently, this coloring avoids a red solution to the c-equation and a blue solution to the k-equation. Therefore, the off-diagonal Rado number satisfies $R_2(c, k) \ge k + 3c + 5$.

We now show that $R_2(c, k) \le k+3c+5$. Consider an arbitrary red–blue coloring of the set $[1, k+3c+5]$. We claim that it contains either a red solution to $x+y+c = z$ or a blue solution to $x+y+k = z$. Suppose, for a contradiction, that it contains neither a red solution to $x + y + c = z$ nor a blue solution to $x + y + k = z$. We start with the color of 1 and successively force the colors of other numbers up to $k+3c+5$. In each case, avoiding both types of monochromatic solutions leads to a contradiction. We represent a solution to the c-equation or the k-equation by an ordered triple (x, y, z), with a subscript indicating the equation. When monochromatic solutions are forbidden, the colors already assigned to entries of such a triple may force the color of another entry. For example, if 1 is red, then taking $x = y = 1$ in the c-equation forces $c + 2$ to be blue. We record this as

$$(1, 1, c+2)_c \text{ forces } c+2 \text{ to be blue.}$$

Note that $k + 3c + 5 > 2k + c + 4$ as $k \le 2c$ and $2c - k + 1 > 0$. Also, $\frac{k+c}{2}$ is an integer because c and k have the same parity.

At boundary values, some forcing points may coincide. Each branch is followed only until a contradiction occurs. A repeated same-color assignment is redundant, whereas an opposite-color assignment terminates the branch. This includes the case when $c = k = 0$.

`Subcase 2.1.` The color of 1 is red. The following sequence of observations shows that there is a blue solution to the k-equation, which gives a contradiction:

$$\begin{aligned}
(1, 1, c+2)_c &\text{ forces } c+2 \text{ to be blue;}\\
(c+2, c+2, k+2c+4)_k &\text{ forces } k+2c+4 \text{ to be red;}\\
(\tfrac{k+c}{2}+2, \tfrac{k+c}{2}+2, k+2c+4)_c &\text{ forces } \tfrac{k+c}{2}+2 \text{ to be blue;}\\
\left(\tfrac{k+c}{2}+2, \tfrac{k+c}{2}+2, 2k+c+4\right)_k &\text{ forces } 2k+c+4 \text{ to be red;}\\
(1, k+2c+4, k+3c+5)_c &\text{ forces } k+3c+5 \text{ to be blue;}\\
(2c+3, c+2, k+3c+5)_k &\text{ forces } 2c+3 \text{ to be red;}\\
(1, 2c+3, 3c+4)_c &\text{ forces } 3c+4 \text{ to be blue;}\\
(2c-k+2, c+2, 3c+4)_k &\text{ forces } 2c-k+2 \text{ to be red;}\\
(1, 2c-k+2, 3c-k+3)_c &\text{ forces } 3c-k+3 \text{ to be blue;}\\
(k+2, k+2, 2k+c+4)_c &\text{ forces } k+2 \text{ to be blue.}
\end{aligned}$$

Thus, $(k+2, 3c-k+3, k+3c+5)$ is a blue solution to the k-equation, a contradiction.

`Subcase 2.2.` The color of 1 is blue. The following sequence of observations shows that there is a blue solution to the k-equation, which gives a contradiction:

$$\begin{aligned}
(1, 1, k+2)_k &\text{ forces } k+2 \text{ to be red;}\\
(k+2, k+2, 2k+c+4)_c &\text{ forces } 2k+c+4 \text{ to be blue;}\\
\left(\tfrac{k+c}{2}+2, \tfrac{k+c}{2}+2, 2k+c+4\right)_k &\text{ forces } \tfrac{k+c}{2}+2 \text{ to be red;}\\
\left(\tfrac{k+c}{2}+2, \tfrac{k+c}{2}+2, k+2c+4\right)_c &\text{ forces } k+2c+4 \text{ to be blue;}\\
(c+2, c+2, k+2c+4)_k &\text{ forces } c+2 \text{ to be red;}\\
(c+2, c+2, 3c+4)_c &\text{ forces } 3c+4 \text{ to be blue;}\\
(1, 3c+4, k+3c+5)_k &\text{ forces } k+3c+5 \text{ to be red;}\\
(k+c+3, c+2, k+3c+5)_c &\text{ forces } k+c+3 \text{ to be blue.}
\end{aligned}$$

Thus, $(1, k+c+3, 2k+c+4)_k$ is a blue solution to the k-equation, a contradiction. Thus, we arrive at a contradiction in both subcases. Therefore, $R_2(c,k) \le k+3c+5$. Combining the lower and upper bounds, we obtain $R_2(c,k) = k+3c+5$ when $c \le k \le 2c$.

`Case 3.` c and k have the same parity with $k > 2c$. We show that $R_2(c,k) \ge 2k+c+4$.

Let

$$\begin{aligned}
R &= \{k+2, \ldots, 2k+c+3\},\\
B &= \{1, 2, \ldots, k+1\}.
\end{aligned}$$

Consider the 2-coloring of $[1, 2k + c + 3]$ defined by

$$\chi(n) = \begin{cases} \text{Red}, & n \in R, \\ \text{Blue}, & n \in B. \end{cases}$$

We show that under this coloring, there is neither a red solution to the c-equation nor a blue solution to the k-equation. If both x and y belong to B, then $z = x + y + k \geq 1 + 1 + k = k + 2$. Thus, if $z \leq 2k + c + 3$, then $z \in R$, whereas if $z > 2k + c + 3$, then z lies outside the colored interval. Hence, there is no blue solution to the k-equation.

If both x and y belong to R, then $z = x+y+c \geq (k+2)+(k+2)+c = 2k+c+4$, which is outside the colored interval. Therefore, no red solution to the c-equation exists. This gives the lower bound $R_2(c, k) > 2k+c+3$. Hence, $R_2(c, k) \geq 2k+c+4$. We show that $R_2(c, k) \leq 2k + c + 4$. Suppose, for a contradiction, that the set $\{1, 2, \ldots, 2k + c + 4\}$ admits a red–blue coloring containing neither a red solution to the c-equation nor a blue solution to the k-equation. We derive a contradiction by analyzing the color assigned to 1.

`Subcase 3.1.` The color of 1 is red. The following observations apply:

$$\begin{aligned} (1, 1, c + 2)_c &\text{ forces } c + 2 \text{ to be blue;} \\ (c + 2, c + 2, k + 2c + 4)_k &\text{ forces } k + 2c + 4 \text{ to be red;} \\ (\frac{k + c}{2} + 2, \frac{k + c}{2} + 2, k + 2c + 4)_c &\text{ forces } \frac{k + c}{2} + 2 \text{ to be blue.} \end{aligned}$$

Since $k > 2c$, we have $k - 2c \geq 1$. Hence, $k - 2c \in [1, 2k + c + 4]$, and it is valid to consider its color.

If $k - 2c$ is red, then the following observations apply:

$$\begin{aligned} (1, k - 2c, k - c + 1)_c &\text{ forces } k - c + 1 \text{ to be blue;} \\ (k - c + 1, c + 2, 2k + 3)_k &\text{ forces } 2k + 3 \text{ to be red;} \\ (1, 2k + 3, 2k + c + 4)_c &\text{ forces } 2k + c + 4 \text{ to be blue.} \end{aligned}$$

Thus, $(\frac{k+c}{2} + 2, \frac{k+c}{2} + 2, 2k + c + 4)_k$ is a blue solution, a contradiction.

If $k - 2c$ is blue, then the following observations apply:

$$\begin{aligned} (c + 2, k - 2c, 2k - c + 2)_k &\text{ forces } 2k - c + 2 \text{ to be red;} \\ (k - c + 1, k - c + 1, 2k - c + 2)_c &\text{ forces } k - c + 1 \text{ to be blue;} \\ (k - c + 1, c + 2, 2k + 3)_k &\text{ forces } 2k + 3 \text{ to be red;} \\ (1, 2k + 3, 2k + c + 4)_c &\text{ forces } 2k + c + 4 \text{ to be blue.} \end{aligned}$$

Thus, $(\frac{k+c}{2} + 2, \frac{k+c}{2} + 2, 2k + c + 4)_k$ is a blue solution, a contradiction.

`Subcase 3.2.` The color of 1 is blue. The following observations apply:

$$\begin{aligned} (1, 1, k + 2)_k &\text{ forces } k + 2 \text{ to be red;} \\ (k + 2, k + 2, 2k + c + 4)_c &\text{ forces } 2k + c + 4 \text{ to be blue.} \end{aligned}$$

If $k-2c$ is red, then the following observations apply:

$$(k-2c, k+2, 2k-c+2)_c \text{ forces } 2k-c+2 \text{ to be blue;}$$
$$(\frac{k-c}{2}+1, \frac{k-c}{2}+1, 2k-c+2)_k \text{ forces } \frac{k-c}{2}+1 \text{ to be red.}$$

Thus, $(\frac{k-c}{2}+1, \frac{k-c}{2}+1, k+2)_c$ is a red solution, a contradiction.

If $k-2c$ is blue, then the following observations apply:

$$(\frac{k+c}{2}+2, \frac{k+c}{2}+2, 2k+c+4)_k \text{ forces } \frac{k+c}{2}+2 \text{ to be red;}$$
$$(k-2c, 3c+4, 2k+c+4)_k \text{ forces } 3c+4 \text{ to be red;}$$
$$(c+2, c+2, 3c+4)_c \text{ forces } c+2 \text{ to be blue;}$$
$$(c+2, c+2, k+2c+4)_k \text{ forces } k+2c+4 \text{ to be red.}$$

Thus, $(\frac{k+c}{2}+2, \frac{k+c}{2}+2, k+2c+4)_c$ is a red solution, a contradiction. Thus, we arrive at a contradiction in both subcases. Therefore, $R_2(c,k) \leq 2k+c+4$. Combining the lower and upper bounds, we obtain $R_2(c,k) = 2k+c+4$ when $k > 2c$. □

Thus, we have determined the two-color off-diagonal Rado number in all three cases. We now extend the result to the continuous setting.

3.2. The Continuous Version

Theorem 2. *Let c, k, and α be nonnegative real numbers with $c \leq k$. Then the two-color off-diagonal continuous Rado number $R_2(c,k;\alpha)$ on the interval $[\alpha,\infty)$ for the system*

$$x+y+c=z \quad \textit{(red)},$$
$$x+y+k=z \quad \textit{(blue)}$$

is given by

$$R_2(c,k;\alpha) = \begin{cases} k+3c+5\alpha, & \text{if } k \leq 2c+\alpha, \\ 2k+c+4\alpha, & \text{if } k > 2c+\alpha. \end{cases}$$

Proof. The boundary-point convention used in the discrete proof also applies below, including when $\alpha = 0$.

`Case 1.` $k \leq 2c+\alpha$. We show that $R_2(c,k;\alpha) = k+3c+5\alpha$. The lower bound is given by the following coloring.

$$R = [\alpha, c+2\alpha) \cup [k+2c+4\alpha, k+3c+5\alpha),$$
$$B = [c+2\alpha, k+2c+4\alpha).$$

This coloring of the interval $[\alpha, k+3c+5\alpha)$ contains neither a red solution to the c-equation nor a blue solution to the k-equation. The verification is analogous

to that of the corresponding discrete lower-bound construction. Hence, for every $N < k + 3c + 5\alpha$, its restriction to $[\alpha, N]$ avoids both forbidden monochromatic solutions. Therefore,

$$R_2(c, k; \alpha) \geq k + 3c + 5\alpha.$$

For the upper bound, suppose, for a contradiction, that there is a red–blue coloring of $[\alpha, k + 3c + 5\alpha]$ containing neither a red solution to the c-equation nor a blue solution to the k-equation.

`Subcase 1.1.` The color of α is red. The following sequence of observations shows that there is a blue solution to the k-equation, which gives a contradiction:

$$\begin{aligned}
(\alpha, \alpha, c + 2\alpha)_c &\text{ forces } c + 2\alpha \text{ to be blue;}\\
(c + 2\alpha, c + 2\alpha, k + 2c + 4\alpha)_k &\text{ forces } k + 2c + 4\alpha \text{ to be red;}\\
(\tfrac{k+c}{2} + 2\alpha, \tfrac{k+c}{2} + 2\alpha, k + 2c + 4\alpha)_c &\text{ forces } \tfrac{k+c}{2} + 2\alpha \text{ to be blue;}\\
\left(\tfrac{k+c}{2} + 2\alpha, \tfrac{k+c}{2} + 2\alpha, 2k + c + 4\alpha\right)_k &\text{ forces } 2k + c + 4\alpha \text{ to be red;}\\
(\alpha, k + 2c + 4\alpha, k + 3c + 5\alpha)_c &\text{ forces } k + 3c + 5\alpha \text{ to be blue;}\\
(2c + 3\alpha, c + 2\alpha, k + 3c + 5\alpha)_k &\text{ forces } 2c + 3\alpha \text{ to be red;}\\
(\alpha, 2c + 3\alpha, 3c + 4\alpha)_c &\text{ forces } 3c + 4\alpha \text{ to be blue;}\\
(2c - k + 2\alpha, c + 2\alpha, 3c + 4\alpha)_k &\text{ forces } 2c - k + 2\alpha \text{ to be red;}\\
(\alpha, 2c - k + 2\alpha, 3c - k + 3\alpha)_c &\text{ forces } 3c - k + 3\alpha \text{ to be blue;}\\
(k + 2\alpha, k + 2\alpha, 2k + c + 4\alpha)_c &\text{ forces } k + 2\alpha \text{ to be blue.}
\end{aligned}$$

Thus, $(k + 2\alpha, 3c - k + 3\alpha, k + 3c + 5\alpha)$ is a blue solution to the k-equation, a contradiction.

`Subcase 1.2.` The color of α is blue. The following sequence of observations shows that there is a blue solution to the k-equation, which gives a contradiction:

$$\begin{aligned}
(\alpha, \alpha, k + 2\alpha)_k &\text{ forces } k + 2\alpha \text{ to be red;}\\
(k + 2\alpha, k + 2\alpha, 2k + c + 4\alpha)_c &\text{ forces } 2k + c + 4\alpha \text{ to be blue;}\\
\left(\tfrac{k+c}{2} + 2\alpha, \tfrac{k+c}{2} + 2\alpha, 2k + c + 4\alpha\right)_k &\text{ forces } \tfrac{k+c}{2} + 2\alpha \text{ to be red;}\\
\left(\tfrac{k+c}{2} + 2\alpha, \tfrac{k+c}{2} + 2\alpha, k + 2c + 4\alpha\right)_c &\text{ forces } k + 2c + 4\alpha \text{ to be blue;}\\
(c + 2\alpha, c + 2\alpha, k + 2c + 4\alpha)_k &\text{ forces } c + 2\alpha \text{ to be red;}\\
(c + 2\alpha, c + 2\alpha, 3c + 4\alpha)_c &\text{ forces } 3c + 4\alpha \text{ to be blue;}\\
(\alpha, 3c + 4\alpha, k + 3c + 5\alpha)_k &\text{ forces } k + 3c + 5\alpha \text{ to be red;}\\
(k + c + 3\alpha, c + 2\alpha, k + 3c + 5\alpha)_c &\text{ forces } k + c + 3\alpha \text{ to be blue.}
\end{aligned}$$

Thus, $(\alpha, k+c+3\alpha, 2k+c+4\alpha)$ is a blue solution to the k-equation, a contradiction. Thus, we arrive at a contradiction in both subcases. Therefore, $R_2(c, k; \alpha) \leq k + 3c + 5\alpha$.

`Case 2.` $k > 2c + \alpha$. We show that $R_2(c, k; \alpha) = 2k + c + 4\alpha$.

The lower bound is given by the following coloring.

$$R = [k + 2\alpha, 2k + c + 4\alpha),$$
$$B = [\alpha, k + 2\alpha).$$

The above coloring of the interval $[\alpha, 2k+c+4\alpha)$ avoids both red solutions to the c-equation and blue solutions to the k-equation. Therefore, $R_2(c, k; \alpha) \geq 2k+c+4\alpha$. For the upper bound, assume, for a contradiction, that there is a red–blue coloring of $[\alpha, 2k + c + 4\alpha]$ containing neither a red solution to the c-equation nor a blue solution to the k-equation.

`Subcase 2.1.` The color of α is red. The following observations apply:

$$(\alpha, \alpha, c + 2\alpha)_c \text{ forces } c + 2\alpha \text{ to be blue;}$$
$$(c + 2\alpha, c + 2\alpha, k + 2c + 4\alpha)_k \text{ forces } k + 2c + 4\alpha \text{ to be red;}$$
$$(\frac{k + c}{2} + 2\alpha, \frac{k + c}{2} + 2\alpha, k + 2c + 4\alpha)_c \text{ forces } \frac{k + c}{2} + 2\alpha \text{ to be blue.}$$

Since $k > 2c + \alpha$, we have $k - 2c > \alpha$. Hence, the point $k - 2c$ lies in the interval $[\alpha, 2k + c + 4\alpha]$, and it is valid to consider its color.

If $k - 2c$ is red, then the following observations apply:

$$(\alpha, k - 2c, k - c + \alpha)_c \text{ forces } k - c + \alpha \text{ to be blue;}$$
$$(k - c + \alpha, c + 2\alpha, 2k + 3\alpha)_k \text{ forces } 2k + 3\alpha \text{ to be red;}$$
$$(\alpha, 2k + 3\alpha, 2k + c + 4\alpha)_c \text{ forces } 2k + c + 4\alpha \text{ to be blue.}$$

Thus, $(\frac{k+c}{2} + 2\alpha, \frac{k+c}{2} + 2\alpha, 2k + c + 4\alpha)_k$ is a blue solution, a contradiction.

If $k - 2c$ is blue, then the following observations apply:

$$(c + 2\alpha, k - 2c, 2k - c + 2\alpha)_k \text{ forces } 2k - c + 2\alpha \text{ to be red;}$$
$$(k - c + \alpha, k - c + \alpha, 2k - c + 2\alpha)_c \text{ forces } k - c + \alpha \text{ to be blue;}$$
$$(k - c + \alpha, c + 2\alpha, 2k + 3\alpha)_k \text{ forces } 2k + 3\alpha \text{ to be red;}$$
$$(\alpha, 2k + 3\alpha, 2k + c + 4\alpha)_c \text{ forces } 2k + c + 4\alpha \text{ to be blue.}$$

Thus, $(\frac{k+c}{2} + 2\alpha, \frac{k+c}{2} + 2\alpha, 2k + c + 4\alpha)_k$ is a blue solution, a contradiction.

`Subcase 2.2.` The color of α is blue. The following observations apply:

$$(\alpha, \alpha, k + 2\alpha)_k \text{ forces } k + 2\alpha \text{ to be red;}$$
$$(k + 2\alpha, k + 2\alpha, 2k + c + 4\alpha)_c \text{ forces } 2k + c + 4\alpha \text{ to be blue.}$$

If $k - 2c$ is red, then the following observations apply:

$$(k - 2c, k + 2\alpha, 2k - c + 2\alpha)_c \text{ forces } 2k - c + 2\alpha \text{ to be blue;}$$
$$(\frac{k - c}{2} + \alpha, \frac{k - c}{2} + \alpha, 2k - c + 2\alpha)_k \text{ forces } \frac{k - c}{2} + \alpha \text{ to be red.}$$

Thus, $(\frac{k-c}{2}+\alpha, \frac{k-c}{2}+\alpha, k+2\alpha)_c$ is a red solution, a contradiction.

If $k-2c$ is blue, then the following observations apply:

$$\begin{aligned}
(\frac{k+c}{2}+2\alpha, \frac{k+c}{2}+2\alpha, 2k+c+4\alpha)_k &\text{ forces } \frac{k+c}{2}+2\alpha \text{ to be red;}\\
(k-2c, 3c+4\alpha, 2k+c+4\alpha)_k &\text{ forces } 3c+4\alpha \text{ to be red;}\\
(c+2\alpha, c+2\alpha, 3c+4\alpha)_c &\text{ forces } c+2\alpha \text{ to be blue;}\\
(c+2\alpha, c+2\alpha, k+2c+4\alpha)_k &\text{ forces } k+2c+4\alpha \text{ to be red.}
\end{aligned}$$

Thus, $(\frac{k+c}{2}+2\alpha, \frac{k+c}{2}+2\alpha, k+2c+4\alpha)_c$ is a red solution, a contradiction. Thus, we arrive at a contradiction in both subcases. Therefore, $R_2(c,k;\alpha) \leq 2k+c+4\alpha$. □

Remark 1. The parity condition appearing in the discrete version has no analog in the continuous version. Its role in the proof of the discrete version is to ensure that certain midpoint expressions are integers. Since the continuous problem allows x, y, and z to range over real intervals, these midpoints are admissible real numbers, and the continuous formula depends only on whether $k \leq 2c+\alpha$ or $k > 2c+\alpha$.

Remark 2. For the shifted Schur pair $x+y+c=z$ and $x+y+k=z$, with nonnegative real numbers c and k satisfying $c \leq k$, and with $\alpha \geq 0$, the continuous two-color off-diagonal Rado number is always finite. This contrasts with the discrete case, where opposite parity between c and k makes the off-diagonal Rado number infinite by admitting a valid two-coloring with no forbidden monochromatic solution.

4. Conclusion

The theory of Rado numbers involves numerous open problems, whose complexity increases significantly with the number of colors. As a result, much of the literature has focused on the two-color case. In this setting, several variants, including disjunctive [6], off-diagonal [5, 7, 9], and continuous [3, 13, 12] Rado numbers, have been studied for various classes of linear equations and systems. Nevertheless, the existing work on off-diagonal Rado numbers is largely limited to homogeneous equations, with the nonhomogeneous case remaining comparatively underexplored.

In this work, we determined the two-color off-diagonal Rado numbers $R_2(c,k)$ and $R_2(c,k;\alpha)$ for $x+y+c=z$ and $x+y+k=z$ in the discrete and continuous settings, respectively. There are several directions open for future investigation, including extending the present framework to broader families of equations encompassing both homogeneous and nonhomogeneous cases, as well as generalizing the known results to colorings with more than two colors.

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