

ON RECURRENCE FORMULAS AND GAPS OF LUCKY NUMBERS

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Abstract

In this article, we use elementary number theory techniques to investigate a sequence of integers called lucky numbers. Ulam et al. introduced lucky numbers as a sieve-based analog of prime numbers. First, we derive a recurrence formula for the nth lucky number. We prove an analog of Bertrand's postulate for lucky numbers for all $x \geq 4$. Finally, we use our formula for the nth lucky number to prove a new result on the order of magnitude of the gaps between consecutive lucky numbers. We obtain an asymptotic bound that is slightly stronger than an analog of Cramér's conditional bound for gaps between consecutive primes.

1. Introduction

The prime numbers can be obtained from the sieve of Eratosthenes. Given the importance of prime numbers in number theory, it is in our interest to investigate other sequences of integers similar to prime numbers. One such sequence is called the lucky numbers [4]. Lucky numbers are obtained by using a sieve similar to the sieve of Eratosthenes.

Let A_1 be the sequence of natural numbers in ascending order. Remove every second number from A_1. We call the remaining sequence of numbers A_2. Thus, $A_2 = 1, 3, 5, 7, \ldots$. For $n \geq 2$, we obtain the sequence A_{n+1} by removing every $A_n(n)$th term from A_n (where $A_n(n)$ is the nth term of the sequence A_n). For example, we have $A_2(2) = 3$. Therefore, remove every third number in A_2 to obtain $A_3 = 1, 3, 7, 9, \ldots$. Repeat this process, and the remaining sequence of integers, A_∞, is the sequence of *lucky numbers*. Therefore, we have

$$A_\infty = 1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, 49, 51, 63, \ldots.$$

Although 1 is the first number in A_∞, we consider 2 to be the first lucky number instead of 1, as 2 is the first number used in the sieve. The lucky number theorem

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[6] is an analog of the prime number theorem and is equivalent to the statement that the number of lucky numbers up to x is asymptotic to $\frac{x}{\log x}$. Hawkins and Briggs [6] also provided a recurrence relation and an analog of Mertens' theorem for lucky numbers.

Theorem 1 (Analog of Mertens' Theorem [6])**.** *Let l denote a lucky number. Then*

$$\lim_{n\to\infty} \log n \prod_{l\le n} \left(1 - \frac{1}{l}\right) = 1. \tag{1}$$

R. K. Guy mentioned lucky numbers in the book *Unsolved Problems in Number Theory* [5]. Apart from the lucky number theorem, relatively little is known about lucky numbers. R. K. Guy suggested that questions analogous to known results and conjectures about primes can be posed for lucky numbers. Many open questions about primes are dependent on the assumption that the Riemann hypothesis is true. One such result was proven by Cramér [2].

Theorem 2 ([2])**.** *Let p_n denote the nth prime number. Assuming that the Riemann hypothesis is true, the following holds:*

$$p_{n+1} - p_n = O(\sqrt{p_n}\log p_n).$$

In this article, we investigate lucky numbers to further our understanding of their distribution and develop new techniques to study their arithmetical structure. In Section 2, we iterate the recurrence relation for lucky numbers to find a recurrence formula for the counting function of lucky numbers. We highlight the relationship between the recurrence relation and the counting function for lucky numbers. Building on our formula for the counting function of lucky numbers, we prove a recurrence formula for the nth lucky number, l_n. In Section 3, we prove an analog of Bertrand's postulate for lucky numbers by using ideas similar to those used in the proof of the lucky number theorem. Although the lucky number theorem implies an analog of Bertrand's postulate for sufficiently large x, we directly prove an analog of Bertrand's postulate for all $x \ge 4$. In Section 4, we use our new formula for lucky numbers to provide a detailed proof of the following new upper bound on the gaps between consecutive lucky numbers.

Theorem 3. *Let l_n denote the nth lucky number. We have*

$$l_{n+1} - l_n = O\Big(\sqrt{l_n \log l_n}\Big),$$

or, equivalently,

$$l_{n+1} - l_n = O\big(\sqrt{n}\log n\big).$$

Therefore, in line with R. K. Guy's suggestion to investigate prime-type conjectures for lucky numbers, Theorem 3 proves a slightly stronger lucky number analog

of Cramér's conditional gap bound in Theorem 2 and does so unconditionally. However, this result remains much weaker than an analog of Cramér's conjectured bound for primes.

Throughout this paper, l denotes a lucky number and p denotes a prime. For $n \geq 1$, we define l_n as the nth lucky number, i.e., $l_1 = 2$, $l_2 = 3$, $l_3 = 7$, $l_4 = 9, \ldots$. The function log denotes the natural logarithm. Additionally, the symbol $\sim$ is used to indicate asymptotic equality as the parameter of interest increases without bound.

Definition 1. We define the function $L : [1, \infty) \to \mathbb{N} \cup \{0\}$ as

$$L(x) := |\{l \mid l \leq x \text{ and } l \text{ is a lucky number}\}|.$$

Thus, $L(x)$ is the number of lucky numbers less than or equal to x and will be referred to as the *counting function of lucky numbers*.

We denote a finite sequence as $(a_n)_{n=c_1}^{n=c_2}$ for some nonnegative integers c_1 and c_2 where $c_1 \leq c_2$. Here, a_{c_1} is the first term of the sequence, and a_{c_2} is the last.

2. Formulas for Lucky Numbers

In the proof of the lucky number theorem, Hawkins and Briggs [6] obtained the following recurrence relation:

$$R(n, x) = R(n-1, x) - \left\lfloor \frac{R(n-1, x)}{l_{n-1}} \right\rfloor, \text{ for } n \geq 2, \tag{2}$$

where

$$R(n, x) = |\{k \mid k \in A_n, k \leq x\}|.$$

Iterating this recurrence relation, we obtain a formula for $L(x)$.

Proposition 1. *Fix $x \geq 1$. Note that $l_{L(L(x))}$ is the largest lucky number less than or equal to $L(x)$. Then the following formula holds for $L(x)$:*

$$L(x) = \left\lceil \cdots \left\lceil \left\lceil \left\lceil \lfloor x \rfloor \left(1 - \frac{1}{l_1}\right) \right\rceil \left(1 - \frac{1}{l_2}\right) \right\rceil \left(1 - \frac{1}{l_3}\right) \right\rceil \cdots \left(1 - \frac{1}{l_{L(L(x))}}\right) \right\rceil.$$

Proof. First, note that there are $L(L(x))$ lucky numbers less than or equal to $L(x)$. Therefore, it follows that $l_{L(L(x))}$ is the largest lucky number less than or equal to $L(x)$. Consider the defined sequence $R(n, x)$, where $R(1, x) = \lfloor x \rfloor$. Note that by definition, $R(i, x)$ is a nonnegative integer for all i. In the first step of the sifting process, $\left\lfloor \frac{R(1,x)}{2} \right\rfloor$ numbers will be eliminated (this follows since every second integer will be eliminated). Hence, it follows that

$$R(2, x) = R(1, x) - \left\lfloor \frac{R(1, x)}{2} \right\rfloor = \left\lceil \frac{R(1, x)}{2} \right\rceil,$$

since $R(i,x)$ is an integer and

$$R(i,x) - \left\lfloor \frac{R(i,x)}{2} \right\rfloor = R(i,x) + \left\lceil -\frac{R(i,x)}{2} \right\rceil$$
$$= \left\lceil R(i,x) - \frac{R(i,x)}{2} \right\rceil = \left\lceil R(i,x)\left(1-\frac{1}{2}\right) \right\rceil, \text{ for all } i.$$

Similarly, in the second step of the sifting process, $\left\lfloor \frac{R(1,x)}{l_2} \right\rfloor$ numbers will be eliminated. Hence,

$$R(3,x) = R(2,x) - \left\lfloor \frac{R(2,x)}{l_2} \right\rfloor = \left\lceil R(2,x)\left(1-\frac{1}{l_2}\right) \right\rceil.$$

Repeat this process to get the result

$$R(i+1,x) = R(i,x) - \left\lfloor \frac{R(i,x)}{l_{i+1}} \right\rfloor = \left\lceil R(i,x)\left(1-\frac{1}{l_{i+1}}\right) \right\rceil. \tag{3}$$

This result is similar to the recurrence relation given by Equation (2) by Hawkins and Briggs [6]. Note that if $l_{i+1} > R(i,x)$ for some $R(i,x)$, then

$$\lceil R(i,x)(1-1/l_{i+1}) \rceil = \lceil R(i,x) - R(i,x)/l_{i+1} \rceil = R(i,x)$$

(since $R(i,x)$ is a nonnegative integer and $\frac{R(i,x)}{l_{i+1}} < 1$). Also note that this would imply that $R(i,x) = L(x)$ for every $i \geq L(L(x))$. Hence, it is only necessary to use the lucky numbers in the sifting process up to the largest $l_{L(L(x))}$ such that $l_{L(L(x))} \leq L(x)$. Intuitively, this implies that if there are fewer than $l_{L(L(x))+1}$ numbers remaining that are less than or equal to x, then there is no $l_{L(L(x))+1}$th number that can be eliminated. Hence, it follows that

$$L(x) = R(L(L(x)),x). \tag{4}$$

Using the recurrence relation given by Equation (3),

$$L(x) = R(L(L(x)),x) = \left\lceil R(L(L(x))-1,x)\left(1-\frac{1}{l_{L(L(x))}}\right) \right\rceil$$
$$= \left\lceil \left\lceil R(L(L(x))-2,x)\left(1-\frac{1}{l_{L(L(x))-1}}\right) \right\rceil \left(1-\frac{1}{l_{L(L(x))}}\right) \right\rceil$$
$$\vdots$$
$$= \left\lceil \cdots \left\lceil \left\lceil \left\lceil R(1,x)\left(1-\frac{1}{l_1}\right) \right\rceil \left(1-\frac{1}{l_2}\right) \right\rceil \left(1-\frac{1}{l_3}\right) \right\rceil \cdots \left(1-\frac{1}{l_{L(L(x))}}\right) \right\rceil$$
$$= \left\lceil \cdots \left\lceil \left\lceil \left\lceil \lfloor x \rfloor \left(1-\frac{1}{l_1}\right) \right\rceil \left(1-\frac{1}{l_2}\right) \right\rceil \left(1-\frac{1}{l_3}\right) \right\rceil \cdots \left(1-\frac{1}{l_{L(L(x))}}\right) \right\rceil. \qquad \square$$

Note that Equation (4) highlights the relationship between $L(x)$ and $R(i,x)$.

Example 1. We apply Proposition 1 to $x = 10$. We have $\lceil 10/2 \rceil = 5$ and $\lceil 5(2/3) \rceil = 4$. The next lucky number is 7, but $7 > 4$, hence the process ends. It follows that $L(10) = 4$, and the corresponding largest lucky number less than 4 is $l_{L(L(x))} = 3$.

The following proposition relates l_n to the lucky numbers less than or equal to n. The formula therefore provides a useful method to compute l_n without iterating the entire lucky sieve.

Proposition 2. *Let $n \geq 2$. Let l_β be the largest lucky number less than or equal to n. Then*

$$l_n = \left\lceil \left\lceil \left\lceil \cdots \left\lceil \left\lceil n \left(\frac{l_\beta}{l_\beta - 1} \right) - 1 \right\rceil \left(\frac{l_{\beta-1}}{l_{\beta-1} - 1} \right) - 1 \right\rceil \cdots \right\rceil \left(\frac{l_2}{l_2 - 1} \right) - 1 \right\rceil \left(\frac{l_1}{l_1 - 1} \right) - 1 \right\rceil \tag{5}$$

and

$$l_n = \lfloor \lfloor \cdots \lfloor (\lfloor (n l_\beta - 1)/(l_\beta - 1) \rfloor l_{\beta-1} - 1)/(l_{\beta-1} - 1) \rfloor \cdots \rfloor l_1 - 1)/(l_1 - 1) \rfloor. \tag{6}$$

Erdős and Jabotinsky [3] obtained a similar formula for sequences of integers defined by sieves. Note that, as defined in Proposition 2, we have $\beta = L(n)$.

Proof of Proposition 2. Note that the two formulas in Equations (5) and (6) are equivalent. This is a result of basic manipulations of the floor and ceiling functions and the fact that for $a, b \in \mathbb{N}$, we have $\left\lfloor \frac{a}{b} \right\rfloor = \left\lceil \frac{a+1}{b} \right\rceil - 1$ or equivalently $\left\lfloor \frac{a-1}{b} \right\rfloor = \left\lceil \frac{a}{b} \right\rceil - 1$. Applying this to each ceiling function in Equation (5), we obtain Equation (6). Therefore, it suffices to prove only one of the formulas. We now give a proof of Equation (5). First, define the *preimage* of the counting function

$$L^{-1}(n) = \{k \in \mathbb{N} \mid L(k) = n\}.$$

Note that

$$L(l_n) = L(l_n + 1) = \cdots = L(l_{n+1} - 1) = n.$$

Therefore, it follows that $\min\left(L^{-1}(n)\right) = l_n$. Thus, it is necessary to show that the formula in Equation (5) maps n to the minimum of the preimage $L^{-1}(n)$. We first construct a finite sequence of sets. Let

$$\begin{aligned}
B_1 &:= \left\{ y \in \mathbb{N} \mid \left\lceil y \left(1 - \frac{1}{l_\beta} \right) \right\rceil = n \right\} \\
B_2 &:= \left\{ y \in \mathbb{N} \mid \left\lceil y \left(1 - \frac{1}{l_{\beta-1}} \right) \right\rceil = \min(B_1) \right\} \\
B_3 &:= \left\{ y \in \mathbb{N} \mid \left\lceil y \left(1 - \frac{1}{l_{\beta-2}} \right) \right\rceil = \min(B_2) \right\} \\
&\vdots \\
B_\beta &:= \left\{ y \in \mathbb{N} \mid \left\lceil y \left(1 - \frac{1}{l_1} \right) \right\rceil = \min(B_{\beta-1}) \right\}.
\end{aligned}$$

Noting that $L(l_n) = n$ and in Proposition 1, $l_{L(L(l_n))}$ is the largest lucky number less than or equal to $L(l_n)$, we have $l_\beta = l_{L(n)} = l_{L(L(l_n))}$. Therefore, by construction and Proposition 1, we have $L(B_\beta) = n$ and

$$\min\left(L^{-1}(n)\right) = \min\left(B_\beta\right) = l_n.$$

Hence, we show that the first step of Equation (5) maps to the minimum value of B_1, i.e., we will show that $\left\lceil n\left(\frac{l_\beta}{l_\beta - 1}\right) - 1\right\rceil = \min\left(B_1\right)$. Then we show that the second step maps to $\min\left(B_2\right)$, i.e., $\left\lceil \min\left(B_1\right)\left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right) - 1\right\rceil = \min\left(B_2\right)$. In general, we show that

$$\left\lceil \min\left(B_i\right)\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right) - 1\right\rceil = \min\left(B_{i+1}\right).$$

Take some $y \in B_{\beta-i+1}$. Thus, y would be an arbitrary integer such that

$$\left\lceil y\left(1 - \frac{1}{l_i}\right)\right\rceil = \min(B_{\beta-i}).$$

Note that based on the definition in Proposition 1, this would imply $y \geq l_i$. It is possible to write $y = ql_i + r$ for some quotient q and some remainder $0 \leq r < l_i$. Use this to simplify the expression for the ith step:

$$\left\lceil y\left(1 - \frac{1}{l_i}\right)\right\rceil = y - \left\lfloor \frac{y}{l_i}\right\rfloor = y - \left\lfloor \frac{ql_i + r}{l_i}\right\rfloor = y - q - \left\lfloor \frac{r}{l_i}\right\rfloor = y - q. \tag{7}$$

We have

$$\left\lceil y\left(1 - \frac{1}{l_i}\right)\right\rceil \neq \left\lceil (y+2)\left(1 - \frac{1}{l_i}\right)\right\rceil,$$

since

$$\begin{aligned}\left\lceil (y+2)\left(1 - \frac{1}{l_i}\right)\right\rceil &= y + 2 - \left\lfloor \frac{y+2}{l_i}\right\rfloor = y + 2 - \left\lfloor \frac{ql_i + r + 2}{l_i}\right\rfloor \\ &= y + 2 - q - \left\lfloor \frac{r+2}{l_i}\right\rfloor > y - q.\end{aligned}$$

This is true since $l_i \geq 2$ and $r < l_i$. It follows that $r + 2 < 2l_i$ implies $\frac{r+2}{l_i} < 2$ and hence $\left\lfloor \frac{r+2}{l_i}\right\rfloor \leq 1$. Therefore, we can conclude that if $y \in B_{\beta-i+1}$, then $y + 2 \notin B_{\beta-i+1}$. Without loss of generality, this is equivalent to saying: if $y \in B_{\beta-i+1}$, then $y - 2 \notin B_{\beta-i+1}$. We can ask the question, if $y \in B_{\beta-i+1}$, then when is $y - 1 \in B_{\beta-i+1}$? This is relevant because it is necessary to show that in the case where both $y, y - 1 \in B_{\beta-i+1}$, then the ith step of Formula (5) only maps to $y - 1 = \min\left(B_{\beta-i+1}\right)$. Using a similar method to the one used to simplify $\left\lceil (y+2)\left(1 - \frac{1}{l_i}\right)\right\rceil$, we can show that

$$\left\lceil (y-1)\left(1 - \frac{1}{l_i}\right)\right\rceil = y - 1 - q - \left\lfloor \frac{r-1}{l_i}\right\rfloor.$$

Therefore, it is easy to see that $\left\lceil y\left(1-\frac{1}{l_i}\right)\right\rceil = \left\lceil (y-1)\left(1-\frac{1}{l_i}\right)\right\rceil$ only if $r=0$. Thus, we obtain the following:

$$\min\left(B_{\beta-i+1}\right) = \begin{cases} y, & \text{if } r \neq 0; \\ y-1, & \text{if } r = 0. \end{cases}$$

Consider the two cases.

`Case 1:` $r \neq 0$. Then

$$\left\lceil \left\lceil y\left(1-\frac{1}{l_i}\right)\right\rceil \left(\frac{l_i}{l_i-1}\right) - 1\right\rceil = \left\lceil \left(\frac{(y-q)l_i}{l_i-1}\right) - 1\right\rceil = \left\lceil \left(\frac{(yl_i - ql_i}{l_i-1}\right) - 1\right\rceil$$

which follows by applying Equation (7). Using the fact that $y = ql_i + r$, hence $ql_i = y - r$, we have

$$\begin{aligned} \left\lceil \left(\frac{(yl_i - ql_i}{l_i-1}\right) - 1\right\rceil &= \left\lceil \left(\frac{(yl_i - y + r}{l_i-1}\right) - 1\right\rceil = \left\lceil \left(\frac{(y(l_i-1) + r}{l_i-1}\right) - 1\right\rceil \\ &= y - 1 + \left\lceil \frac{r}{l_i-1}\right\rceil = y = \min\left(B_{\beta-i+1}\right), \end{aligned}$$

since $r > 0$, $\left\lceil \frac{r}{l_i-1}\right\rceil = 1$.

`Case 2:` $r = 0$. Then the preimage of the ith step is $\{y-1, y\}$. Hence, the smallest element is $y-1$ and therefore Equation (5) should map to $y-1$. We have

$$\left\lceil \left\lceil y\left(1-\frac{1}{l_i}\right)\right\rceil \left(\frac{l_i}{l_i-1}\right) - 1\right\rceil = \left\lceil \left(\frac{(y-q)l_i}{l_i-1}\right) - 1\right\rceil = \left\lceil \left(\frac{(yl_i - ql_i}{l_i-1}\right) - 1\right\rceil.$$

Since $r = 0, y = ql_i$, we get

$$\begin{aligned} \left\lceil \left(\frac{(yl_i - ql_i}{l_i-1}\right) - 1\right\rceil &= \left\lceil \left(\frac{(yl_i - y}{l_i-1}\right) - 1\right\rceil = \left\lceil \left(\frac{(y(l_i-1)}{l_i-1}\right) - 1\right\rceil \\ &= y - 1 = \min\left(B_{\beta-i+1}\right). \end{aligned}$$

This concludes the proof. □

Example 2. We use the formula from Equation (5) to calculate the third lucky number. First, we have $\lceil 3(3/2) - 1\rceil = 4$. Next, we have $\lceil 4(2/1) - 1\rceil = 7 = l_3$. We also use Formula (6) to find the eighth lucky number. First, we have $\lfloor (8 \times 7 - 1)/6\rfloor = 9$. Now $\lfloor (9 \times 3 - 1)/2\rfloor = 13$ and finally $\lfloor (13 \times 2 - 1)/1\rfloor = 25 = l_8$.

3. An Analog of Bertrand's Postulate for Lucky Numbers

We prove the following theorem.

Theorem 4 (Bertrand's Postulate for Lucky Numbers)**.** *For every $x \geq 4$, there is always a lucky number in the interval $[x, 2x]$.*

We first need to prove the following lemma.

Lemma 1. *For every $x \in [1, \infty)$,*

$$x \prod_{i=1}^{L(L(x))} \left(1 - \frac{1}{l_i}\right) \leq L(x) < x \prod_{i=1}^{L(L(x))} \left(1 - \frac{1}{l_i}\right) + L(L(x)).$$

Proof. The lower bound follows directly by applying the formula provided by Proposition 1. For the upper bound, consider the sequence $R(i, x)$. Using the fact that $\lceil y \rceil \leq y + 1$ for all $y \geq 0$, it follows that

$$R(i+1, x) = \lceil R(i, x) - R(i, x)/l_{i+1} \rceil \leq R(i, x)(1 - 1/l_{i+1}) + 1 \text{ for all } i.$$

Applying this at each step of Proposition 1 gives

$$\begin{aligned}
L(x) &= \left\lceil \cdots \left\lceil \left\lceil \left\lceil \lfloor x \rfloor \left(1 - \frac{1}{l_1}\right) \right\rceil \left(1 - \frac{1}{l_2}\right) \right\rceil \left(1 - \frac{1}{l_3}\right) \right\rceil \cdots \left(1 - \frac{1}{l_{L(L(x))}}\right) \right\rceil \\
&\leq \left\lceil \cdots \left\lceil \left\lceil \left(\lfloor x \rfloor \left(1 - \frac{1}{l_1}\right) + 1 \right) \left(1 - \frac{1}{l_2}\right) \right\rceil \left(1 - \frac{1}{l_3}\right) \right\rceil \cdots \left(1 - \frac{1}{l_{L(L(x))}}\right) \right\rceil \\
&\leq \left\lceil \cdots \left\lceil \left(\left(\lfloor x \rfloor \left(1 - \frac{1}{l_1}\right) + 1 \right) \left(1 - \frac{1}{l_2}\right) + 1 \right) \left(1 - \frac{1}{l_3}\right) \right\rceil \cdots \left(1 - \frac{1}{l_{L(L(x))}}\right) \right\rceil \\
&\quad \vdots \\
&\leq \left(\cdots \left(\left(\left(\lfloor x \rfloor \left(1 - \frac{1}{l_1}\right) + 1 \right) \left(1 - \frac{1}{l_2}\right) + 1 \right) \left(1 - \frac{1}{l_3}\right) + 1 \right) \right. \\
&\qquad \left. \cdots \left(1 - \frac{1}{l_{L(L(x))}}\right) \right) + 1 \\
&\quad \vdots \\
&= \lfloor x \rfloor \prod_{i=1}^{L(L(x))} \left(1 - \frac{1}{l_i}\right) + \sum_{i=1}^{L(L(x))} \prod_{j=i}^{L(L(x))} \left(1 - \frac{1}{l_j}\right).
\end{aligned}$$

Note that $\prod_{j=i}^{L(L(x))} \left(1 - \frac{1}{l_j}\right) < 1$ since each term in the product is less than 1. Hence,

$$\begin{aligned}
L(x) &< x \prod_{i=1}^{L(L(x))} \left(1 - \frac{1}{l_i}\right) + \sum_{i=1}^{L(L(x))} 1 \\
&= x \prod_{i=1}^{L(L(x))} \left(1 - \frac{1}{l_i}\right) + L(L(x)).
\end{aligned}$$

□

Proof of Theorem 4. Assume for the sake of contradiction that for some integer value of x, there is no lucky number between x and $2x$. Thus, $L(x) = L(2x)$. We will then show that for large enough x, we have $L(2x) - L(x) > 0$, contradicting that there is no lucky number between x and $2x$. Based on the assumption $L(x) = L(2x)$, we have $L(L(x)) = L(L(2x))$. Using the lower bound in Lemma 1 for $L(2x)$, we have

$$L(2x) \geq \frac{2x}{2} \prod_{i=2}^{L(L(x))} \left(1 - \frac{1}{l_i}\right) = x \prod_{i=2}^{L(L(x))} \left(1 - \frac{1}{l_i}\right).$$

We now use the upper bound in Lemma 1 for $L(x)$:

$$L(x) < \frac{x}{2} \prod_{i=2}^{L(L(x))} \left(1 - \frac{1}{l_i}\right) + L(L(x)).$$

Subtracting the previous two inequalities gives the following:

$$\begin{aligned} L(2x) - L(x) &> \frac{2x}{2} \prod_{i=2}^{L(L(x))} \left(1 - \frac{1}{l_i}\right) - \frac{x}{2} \prod_{i=2}^{L(L(x))} \left(1 - \frac{1}{l_i}\right) - L(L(x)) \\ &> \frac{x}{2} \prod_{i=2}^{L(L(x))} \left(1 - \frac{1}{l_i}\right) - L(L(x)). \end{aligned} \tag{8}$$

Rearranging the upper bound in Lemma 1,

$$L(x) - L(L(x)) < \frac{x}{2} \prod_{i=2}^{L(L(x))} \left(1 - \frac{1}{l_i}\right)$$

and applying this to Inequality (8), we obtain

$$\begin{aligned} L(2x) - L(x) &> (L(x) - L(L(x))) - L(L(x)) \\ &= L(x) - 2L(L(x)). \end{aligned} \tag{9}$$

Let $y = L(x)$. Note that for large enough y, by applying the formula from Proposition 1, it follows that $l_{L(L(y))} \geq 3$ for $y \geq 7$. Thus, for $y \geq 10$:

$$\begin{aligned} L(y) &\leq \lceil \lceil y/2 \rceil 2/3 \rceil \leq \lceil (y/2 + 1)2/3 \rceil = \lceil y/3 + 2/3 \rceil \\ &\leq y/3 + 2/3 + 1 \leq y/2. \end{aligned}$$

Hence, for $y \geq 10$, we have $L(y) \leq y/2 = L(x)/2$. Note that since $y = L(x)$, we have $L(y) = L(L(x))$. Thus, $L(L(x)) = L(y) \leq L(x)/2$. Apply this to Inequality (9):

$$L(2x) - L(x) > L(x) - 2(L(x)/2) = 0,$$

contradicting the assumption. This happens whenever $y \geq \max\{7, 10\} = 10$. Thus, the contradiction is when $L(x) \geq 10$. This implies $x \geq 33$ since $L(33) = 10$. By inspection of the lower cases, it follows that there is always a lucky number between x and $2x$ for all $x \geq 4$. □

4. On the Order of Magnitude of the Difference between Consecutive Lucky Numbers

In this section, we give a detailed proof of Theorem 3. We construct a recurrence sequence whose iteration yields our recurrence Formula (6). The central difficulty lies in bounding the error sequence that forms when separating terms in the recurrence sequence for the $(n+1)$th lucky number to obtain the recurrence sequence for the nth lucky number. Each term in the recurrence corresponds to a quotient q associated with a lucky number l_i where $1 \leq i \leq \beta$, and these satisfy

$$q(l_i - 1) \leq l_n.$$

We first find an upper bound for the error terms for each of the first

$$N = \left\lceil \frac{\sqrt{l_n}}{\sqrt{\log(l_n)}} \right\rceil$$

distinct quotients. We then use the inequality $q(l_i - 1) \leq l_n$ and the lucky number theorem to show that there are

$$O(N) = O\left(\left\lceil \frac{\sqrt{l_n}}{\sqrt{\log(l_n)}} \right\rceil \right)$$

terms left in the recurrence sequence for the nth lucky number. Finally, applying the analog of Mertens' theorem for lucky numbers, we obtain the desired bound. This is similar but not identical to the "square root trick" that is often used in sieves.

Let $n \in \mathbb{N}$. Our goal is to find an upper bound for $l_{n+1} - l_n$. We define $\beta = L(n)$. Hence, l_β is the largest lucky number less than or equal to n (as in Proposition 2). With this in mind, we define a new sequence. We define the finite sequence $(n_i)_{i=0}^{i=\beta}$, where $n_0 = n$ and

$$n_{i+1} = \left\lfloor \frac{n_i l_{\beta-i} - 1}{l_{\beta-i} - 1} \right\rfloor. \tag{10}$$

Remark 1. The sequence $(n_i)_{i=0}^{i=\beta}$ is an increasing sequence. Thus, $n_{i+1} \geq n_i$. Iterating this sequence results in Formula (6). Therefore, it follows that

$$n_\beta = l_n.$$

Proposition 3. *The sequence (n_i), defined as in Equation (10), is a strictly increasing sequence.*

Proof. We have

$$\begin{aligned} n_{i+1} &= \left\lfloor \frac{n_i l_{\beta-i} - 1}{l_{\beta-i} - 1} \right\rfloor = \left\lfloor \frac{n_i (l_{\beta-i} - 1) + n_i - 1}{l_{\beta-i} - 1} \right\rfloor \\ &= n_i + \left\lfloor \frac{n_i - 1}{l_{\beta-i} - 1} \right\rfloor. \end{aligned}$$

Since $n_i \geq n_0 = n \geq l_\beta \geq l_{\beta-i}$, it follows that $\left\lfloor \frac{n_i-1}{l_{\beta-i}-1} \right\rfloor \geq 1$. The result follows. □

Example 3. If $n = 10$, then $L(10) = 4$. Hence, $\beta = 4$ and $l_\beta = 9$ is the largest lucky number not greater than 10. Now, $n_0 = n = 10$, and

$$n_1 = \left\lfloor \frac{10(9)-1}{9-1} \right\rfloor = 11.$$

Next, we have

$$n_2 = \left\lfloor \frac{11(7)-1}{7-1} \right\rfloor = 12.$$

Continuing this way, we have

$$n_3 = \left\lfloor \frac{12(3)-1}{3-1} \right\rfloor = 17,$$
$$n_4 = n_\beta = \left\lfloor \frac{17(2)-1}{2-1} \right\rfloor = 33 = l_{10}.$$

We consider Equation (10) applied to $n+1$. We define the sequence $((n+1)_i)_{i=0}^{i=\beta'}$ (similar to $(n_i)_{i=0}^{i=\beta}$). Here, $(n+1)_0 = n+1$ and $\beta' = L(n+1)$. Thus, $l_{\beta'}$ is the largest lucky number less than or equal to $n+1$. We now consider the relationship between β and β'. If $l_{\beta'} = n+1$, then $\beta' = \beta + 1$. Using Equation (10), the fact that $n+1 = (n+1)_0$, and $l_{\beta'} = n+1$, we have

$$(n+1)_1 = \left\lfloor \frac{(n+1)(n+1)-1}{(n+1)-1} \right\rfloor = \left\lfloor \frac{n^2+2n}{n} \right\rfloor = n+2.$$

Therefore, it follows that

$$(n+1)_2 = \left\lfloor \frac{(n+2)l_\beta - 1}{l_\beta - 1} \right\rfloor = \left\lfloor \frac{nl_\beta - 1}{l_\beta - 1} + \frac{2l_\beta}{l_\beta - 1} \right\rfloor, \text{ since } \beta' - 1 = \beta.$$

We now define a sequence of errors of our choice. First, we choose $e_1 = 2$, so that $(n+1)_1 \leq n_0 + e_1 = n+2$. For $i \geq 2$, this pattern continues in such a way that e_i depends on e_{i-1}. Note that e_i is not necessarily well defined. The values of (e_i) are our own choice and can be anything that gives an upper bound, as in the following inequalities. Recall the definition of (n_i), $((n+1)_i)$ in Equation (10). We

will choose $e_2, e_3, \ldots$ so that

$$\begin{aligned}
(n+1)_2 &\leq \left\lfloor \frac{(n_0+e_1)l_\beta - 1}{l_\beta - 1} \right\rfloor = \left\lfloor \frac{n_0 l_\beta - 1}{l_\beta - 1} + \frac{2l_\beta}{l_\beta - 1} \right\rfloor \leq \left\lfloor \frac{n_0 l_\beta - 1}{l_\beta - 1} \right\rfloor + e_2 \\
&= n_1 + e_2, \\
(n+1)_3 &\leq \left\lfloor \frac{(n_1+e_2)l_{\beta-1} - 1}{l_{\beta-1} - 1} \right\rfloor = \left\lfloor \frac{n_1 l_{\beta-1} - 1}{l_{\beta-1} - 1} + \frac{e_2 l_{\beta-1}}{l_{\beta-1} - 1} \right\rfloor \leq \left\lfloor \frac{n_1 l_{\beta-1} - 1}{l_{\beta-1} - 1} \right\rfloor + e_3 \\
&= n_2 + e_3, \\
&\vdots \\
(n+1)_i &\leq \left\lfloor \frac{(n_{i-2}+e_{i-1})l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} \right\rfloor = \left\lfloor \frac{n_{i-2} l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} + \frac{e_{i-1} l_{\beta-i+2}}{l_{\beta-i+2} - 1} \right\rfloor \\
&\leq \left\lfloor \frac{n_{i-2} l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} \right\rfloor + e_i = n_{i-1} + e_i.
\end{aligned}$$

Therefore, it follows that

$$e_i \geq \left\lfloor \frac{(n_{i-2}+e_{i-1})l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} \right\rfloor - n_{i-1}.$$

The important part is that e_i depends on e_{i-1} to ensure that the upper bound $(n+1)_i \leq n_{i-1} + e_i$ still holds (no matter our choice for e_{i-1}). However, notice that if $n+1 \neq l_{\beta'}$, then $\beta = \beta'$. The sequence of errors can be defined in the same manner except where $e_1 = 1$ and we define e_i as the error when comparing $(n+1)_i$ with n_i. Hence, the inequality $(n+1)_i \leq n_i + e_i$ should hold instead of $(n+1)_i \leq n_{i-1} + e_i$. Since $\beta' = \beta < \beta + 1$ and $e_1 = 1 < 2$, we will only consider $n+1 = l_{\beta'}$ because the proof for this case yields the largest upper bound.

Note that the floor function satisfies the following property: for every $x, y \in \mathbb{R}$, we have

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1. \tag{11}$$

We will use this property to split the floor function.

Remark 2. By splitting the floor function in the definition of the errors and by Inequality (11), we can choose the errors so that either

$$e_i = \left\lfloor \frac{e_{i-1} l_{\beta-i+2}}{l_{\beta-i+2} - 1} \right\rfloor,$$

or

$$e_i = \left\lfloor \frac{e_{i-1} l_{\beta-i+2}}{l_{\beta-i+2} - 1} \right\rfloor + 1.$$

Let

$$n_{i-2} = q_{i-2}(l_{\beta-i+2} - 1) + r_{i-2}, \text{ where } 0 \leq r_{i-2} \leq l_{\beta-i+2} - 2,\ i \geq 2. \tag{12}$$

Remark 3. If $r_{i-2} = 0$, then we will simply choose

$$e_i = \left\lfloor \frac{e_{i-1} l_{\beta-i+2}}{l_{\beta-i+2} - 1} \right\rfloor + 1.$$

Proposition 4. *Consider the case where $r_{i-2} \neq 0$. If*

$$\left\lfloor \frac{r_{i-2} - 1 + e_{i-1}}{l_{\beta-i+2} - 1} \right\rfloor = \left\lfloor \frac{e_{i-1}}{l_{\beta-i+2} - 1} \right\rfloor,$$

then we can choose $e_i = \left\lfloor \frac{e_{i-1} l_{\beta-i+2}}{l_{\beta-i+2}-1} \right\rfloor$. However, if

$$\left\lfloor \frac{r_{i-2} - 1 + e_{i-1}}{l_{\beta-i+2} - 1} \right\rfloor = \left\lfloor \frac{e_{i-1}}{l_{\beta-i+2} - 1} \right\rfloor + 1, \tag{13}$$

then we have to choose $e_i = \left\lfloor \frac{e_{i-1} l_{\beta-i+2}}{l_{\beta-i+2}-1} \right\rfloor + 1$.

Proof. `Case 1:`

$$\left\lfloor \frac{r_{i-2} - 1 + e_{i-1}}{l_{\beta-i+2} - 1} \right\rfloor = \left\lfloor \frac{e_{i-1}}{l_{\beta-i+2} - 1} \right\rfloor.$$

Apply Equation (12) to obtain

$$\begin{aligned}
(n+1)_i &\leq \left\lfloor \frac{n_{i-2} l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} + \frac{e_{i-1} l_{\beta-i+2}}{l_{\beta-i+2} - 1} \right\rfloor \\
&= \left\lfloor \frac{(q_{i-2}(l_{\beta-i+2} - 1) + r_{i-2}) l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} + \frac{e_{i-1} l_{\beta-i+2}}{l_{\beta-i+2} - 1} \right\rfloor \\
&= q_{i-2} l_{\beta-i+2} + \left\lfloor \frac{r_{i-2} l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} + \frac{e_{i-1}(l_{\beta-i+2} - 1) + e_{i-1}}{l_{\beta-i+2} - 1} \right\rfloor \\
&= q_{i-2} l_{\beta-i+2} + e_{i-1} + \left\lfloor \frac{r_{i-2}(l_{\beta-i+2} - 1) + r_{i-2} - 1}{l_{\beta-i+2} - 1} + \frac{e_{i-1}}{l_{\beta-i+2} - 1} \right\rfloor \\
&= q_{i-2} l_{\beta-i+2} + r_{i-2} + e_{i-1} + \left\lfloor \frac{r_{i-2} - 1 + e_{i-1}}{l_{\beta-i+2} - 1} \right\rfloor. \qquad (14)
\end{aligned}$$

Using the definition of (n_i), we obtain

$$\begin{aligned}
n_{i-1} &= \left\lfloor \frac{n_{i-2} l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} \right\rfloor \\
&= \left\lfloor \frac{(q_{i-2}(l_{\beta-i+2} - 1) + r_{i-2}) l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} \right\rfloor \\
&= q_{i-2} l_{\beta-i+2} + \left\lfloor \frac{r_{i-2}(l_{\beta-i+2} - 1) + r_{i-2} - 1}{l_{\beta-i+2} - 1} \right\rfloor \\
&= q_{i-2} l_{\beta-i+2} + r_{i-2} + \left\lfloor \frac{r_{i-2} - 1}{l_{\beta-i+2} - 1} \right\rfloor \\
&= q_{i-2} l_{\beta-i+2} + r_{i-2}, \text{ since } 1 \leq r_{i-2} \leq l_{\beta-i+2} - 2. \qquad (15)
\end{aligned}$$

By Equation (15), we see that

$$\left\lfloor \frac{n_{i-2}l_{\beta-i+2}-1}{l_{\beta-i+2}-1} \right\rfloor + \left\lfloor \frac{e_{i-1}l_{\beta-i+2}}{l_{\beta-i+2}-1} \right\rfloor = q_{i-2}l_{\beta-i+2} + r_{i-2} + e_{i-1} + \left\lfloor \frac{e_{i-1}}{l_{\beta-i+2}-1} \right\rfloor. \quad (16)$$

With this in mind, the next objective is to determine when

$$\left\lfloor \frac{n_{i-2}l_{\beta-i+2}-1}{l_{\beta-i+2}-1} + \frac{e_{i-1}l_{\beta-i+2}}{l_{\beta-i+2}-1} \right\rfloor = \left\lfloor \frac{n_{i-2}l_{\beta-i+2}-1}{l_{\beta-i+2}-1} \right\rfloor + \left\lfloor \frac{e_{i-1}l_{\beta-i+2}}{l_{\beta-i+2}-1} \right\rfloor. \quad (17)$$

Notice that if Equation (17) holds, we have $(n+1)_i \leq n_{i-1} + \left\lfloor \frac{e_{i-1}l_{\beta-i+2}}{l_{\beta-i+2}-1} \right\rfloor$. By Equations (14) and (16), Equation (17) holds when

$$\left\lfloor \frac{r_{i-2}-1+e_{i-1}}{l_{\beta-i+2}-1} \right\rfloor = \left\lfloor \frac{e_{i-1}}{l_{\beta-i+2}-1} \right\rfloor.$$

Hence, by Equation (17), it follows that we can choose $e_i = \left\lfloor \frac{e_{i-1}l_{\beta-i+2}}{l_{\beta-i+2}-1} \right\rfloor$. This concludes the proof of Case 1.

Case 2:

$$\left\lfloor \frac{r_{i-2}-1+e_{i-1}}{l_{\beta-i+2}-1} \right\rfloor = \left\lfloor \frac{e_{i-1}}{l_{\beta-i+2}-1} \right\rfloor + 1.$$

Case 2 can be shown similarly to Case 1, using Remark 2. □

Lemma 2. *Excluding the value $r_{i-2} = 0$, the size of the interval for the possible values of r_{i-2} for which Equation (13) holds is at most $e_{i-1} - 1$.*

Proof. This can be split into two simple cases.

Case A: $e_{i-1} < l_{\beta-i+2} - 1$. Then clearly

$$\left\lfloor \frac{e_{i-1}}{l_{\beta-i+2}-1} \right\rfloor + 1 = 1.$$

Then Equation (13) can only hold when

$$r_{i-2} - 1 + e_{i-1} \geq l_{\beta-i+2} - 1$$

so that

$$r_{i-2} \geq l_{\beta-i+2} - e_{i-1}.$$

Combining this with Equation (12) yields

$$l_{\beta-i+2} - e_{i-1} \leq r_{i-2} \leq l_{\beta-i+2} - 2. \quad (18)$$

Case B: $e_{i-1} \geq l_{\beta-i+2} - 1$. Then simply use the fact that Equation (13) can hold only for the remainders where

$$0 \leq r_{i-2} \leq l_{\beta-i+2} - 2 \leq e_{i-1} - 1. \quad (19)$$

By Cases A and B, the remainder clearly falls in a range that has a size of at most $e_{i-1} - 1$. □

Recall Equation (12). If we have $q_{i-2} = q_{i-1} = \cdots = q_{i+j-2}$ for some integers i, j. Then, applying Equation (12) to the index values, $i-2, i-1, \ldots, i+j-2$, we have

$$\begin{aligned}
n_{i-2} &= q_{i-2}(l_{\beta-i+2} - 1) + r_{i-2}, \text{ where } 0 \le r_{i-2} \le l_{\beta-i+2} - 2,\ i \ge 2.\\
n_{i-1} &= q_{i-1}(l_{\beta-i+1} - 1) + r_{i-1}, \text{ where } 0 \le r_{i-1} \le l_{\beta-i+1} - 2.\\
&\vdots\\
n_{i+j-2} &= q_{i+j-2}(l_{\beta-i-j+2} - 1) + r_{i+j-2}, \text{ where } 0 \le r_{i+j-2} \le l_{\beta-i-j+2} - 2.
\end{aligned}$$

We therefore define the finite sequence of quotients $(k_i)_{i=1}^{i=N}$, where

$$N := \left\lceil \frac{\sqrt{l_n}}{\sqrt{\log(l_n)}} \right\rceil. \tag{20}$$

Let $k_1 = q_0 = q_1 = \cdots = q_{m_1}$, $k_2 = q_{m_1+1} = q_{m_1+2} = \cdots = q_{m_2}$, $k_3 = q_{m_2+1} = q_{m_2+2} = \cdots = q_{m_3}$, $\ldots$, $k_N = q_{m_{N-1}+1} = q_{m_{N-1}+2} = \cdots = q_{m_N}$. Here, $q_{m_1} \neq q_{m_1+1}$, $q_{m_2} \neq q_{m_2+1}$, $\ldots$, $q_{m_{N-1}} \neq q_{m_{N-1}+1}$.

Remark 4. We have that (k_i) is a strictly increasing sequence of integers. Therefore, $k_i \ge i$, for all i.

Proof. We know that $n_i \le n_{i+1}$ and $l_{\beta-i} > l_{\beta-i-1}$. Hence, $q_i \le q_{i+1}$. By setting $i = m_j$, we obtain $q_{m_j} \le q_{m_j+1}$. Thus, by construction, we see that $k_j < k_{j+1}$, since $q_{m_j} \neq q_{m_j+1}$. □

Also note that (k_i) is a subsequence of (q_i).

Definition 2. The *corresponding remainders* for each quotient k_i are the remainders in the set $\{r_{m_{i-1}+1}, r_{m_{i-1}+2}, r_{m_{i-1}+3}, \ldots, r_{m_i}\}$. Here, we have $m_0 = -1$. Similarly, the same logic is used when we talk about the *corresponding errors* or the *corresponding lucky numbers* or *corresponding terms in* (q_i).

Notice that every quotient k_i can have multiple corresponding lucky numbers and remainders.

Proposition 5. *Let $1 \le j \le N$. Let $r_{i_j-1} > 0$ be the smallest remainder corresponding to k_j for which Equation (13) holds (assuming such a remainder exists). Then, there are at most*

$$2 + \left\lfloor \frac{e_{i_j} - 1}{1 + 2k_j} \right\rfloor \tag{21}$$

remainders corresponding to k_j for which Equation (13) holds.

Proof. We have

$$k_j = q_{m_{j-1}+1} = q_{m_{j-1}+2} = q_{m_{j-1}+3} = \cdots = q_{m_j}.$$

Therefore, we have

$$\begin{aligned}
n_{m_{j-1}+1} &= k_j(l_{\beta-m_{j-1}-1} - 1) + r_{m_{j-1}+1}, \text{ where } 0 \le r_{m_{j-1}+1} \le l_{\beta-m_{j-1}-1} - 2.\\
n_{m_{j-1}+2} &= k_j(l_{\beta-m_{j-1}-2} - 1) + r_{m_{j-1}+2}, \text{ where } 0 \le r_{m_{j-1}+2} \le l_{\beta-m_{j-1}-2} - 2.\\
&\vdots\\
n_{m_j} &= k_j(l_{\beta-m_j} - 1) + r_{m_j}, \text{ where } 0 \le r_{m_j} \le l_{\beta-m_j} - 2.
\end{aligned}$$

Subtract the first two equations to obtain

$$n_{m_{j-1}+2} - n_{m_{j-1}+1} = k_j\big(l_{\beta-m_{j-1}-2} - l_{\beta-m_{j-1}-1}\big) + r_{m_{j-1}+2} - r_{m_{j-1}+1}.$$

By Proposition 3, (n_i) is a strictly increasing sequence. Hence, $n_{m_{j-1}+2} - n_{m_{j-1}+1} \ge 1$. We also have $l_{\beta-m_{j-1}-2} - l_{\beta-m_{j-1}-1} \le -2$. It follows that $1 + 2k_j \le r_{m_{j-1}+2} - r_{m_{j-1}+1}$. Hence, $r_{m_{j-1}+1} + 1 + 2k_j \le r_{m_{j-1}+2}$. The same can be done by subtracting the second equation from the third, or the third from the fourth, and so on. Therefore, for every τ such that $m_{j-1} + 1 \le \tau \le m_j - 1$, we have

$$r_{\tau-1} + 1 + 2k_j \le r_\tau. \tag{22}$$

Therefore, the remainders corresponding to each quotient are strictly increasing. Assume r_{i_j-1} is the smallest remainder greater than 0 corresponding to k_j for which Equation (13) holds (assuming that such a remainder exists; otherwise we immediately obtain a result smaller than Expression (21)). Iterating the recurrence relation from Inequality (22) yields

$$\begin{aligned}
r_{m_j} &\ge r_{m_j-1} + 1 + 2k_j\\
&\ge \big(r_{m_j-2} + 1 + 2k_j\big) + 1 + 2k_j = r_{m_j-2} + 2(1 + 2k_j)\\
&\ge r_{m_j-3} + 3(1 + 2k_j)\\
&\vdots\\
&\ge r_{i_j-1} + (m_j - i_j + 1)(1 + 2k_j).
\end{aligned} \tag{23}$$

Therefore, Equation (13) holds in at least two cases: either the remainder is $r = 0$, or it is r_{i_j-1}. Recall the proof of Lemma 2. Consider Case A and Case B again.

`Case A:` $e_{i-1} < l_{\beta-i+2} - 1$. Apply Inequality (18) by replacing the indexes to obtain

$$l_{\beta-i_j+1} - e_{i_j} \le r_{i_j-1} \le l_{\beta-i_j+1} - 2.$$

Using $r_{m_j} \leq l_{m_j} - 2 \leq l_{\beta - i_j + 1} - 2$ and Inequality (23), we have

$$\begin{aligned} l_{\beta-i_j+1} - e_{i_j} &\leq r_{i_j-1} \leq r_{i_j-1} + (m_j - i_j + 1)(1 + 2k_j) \\ &\leq r_{m_j} \leq l_{m_j} - 2 \leq l_{\beta-i_j+1} - 2. \end{aligned}$$

Therefore, we have

$$l_{\beta-i_j+1} - e_{i_j} \leq r_{i_j-1} + (m_j - i_j + 1)(1 + 2k_j) \leq l_{\beta-i_j+1} - 2.$$

Hence, we obtain

$$m_j - i_j + 1 \leq \left\lfloor \frac{e_{i_j} - 1}{1 + 2k_j} \right\rfloor. \tag{24}$$

Notice that $i_j - 1$ is the smallest index corresponding to k_j for which Equation (13) holds, and m_j is the last index. Therefore, there are at most $m_j - i_j + 1$ remainders for which Equation (13) holds strictly greater than r_{i_j-1}. Hence, including r_{i_j-1} and the possibility that $r = 0$ is a corresponding remainder for k_j, we can conclude that Proposition 5 holds.

`Case B:` $e_{i-1} \geq l_{\beta-i+2} - 1$. Apply Inequality (19). Using the same logic as in Case A, we obtain the same desired result. □

Note that by Equation (17), in all other cases than those in Proposition 5, the following can be applied:

$$\begin{aligned} (n+1)_i &\leq \left\lfloor \frac{n_{i-2} l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} + \frac{e_{i-1} l_{\beta-i+2}}{l_{\beta-i+2} - 1} \right\rfloor \\ &\leq \left\lfloor \frac{n_{i-2} l_{\beta-i+2} - 1}{l_{\beta-i+2} - 1} \right\rfloor + \left\lfloor \frac{e_{i-1} l_{\beta-i+2}}{l_{\beta-i+2} - 1} \right\rfloor = n_{i-1} + e_i. \end{aligned} \tag{25}$$

We define the sequence $(s_i)_{i=1}^{i=N+1}$ so that $s_1 = 2 = e_1$ and for $i \geq 2$, we set

$$s_i = 2 + s_{i-1} + \frac{s_{i-1} + 1}{2(k_i - 1) + 1}. \tag{26}$$

Lemma 3. *Let N be as in Equation (20). For every j where $1 \leq j \leq N$, we have*

$$(n+1)_{m_j+2} \leq n_{m_j+1} + \left\lfloor s_{j+1} \prod_{i=0}^{m_j} \left(\frac{l_{\beta-i}}{l_{\beta-i} - 1} \right) \right\rfloor.$$

Proof. We create a subsequence (e_{i_j}) where e_{i_j} is the first term in (e_i) in which Equation (13) holds for the quotient k_j (as in Proposition 5). We start with k_1. We must account for the possibility that the first remainder is 0. Hence, we apply Remark 3 and add 1 when splitting the floor function in the first step to obtain an upper bound:

$$(n+1)_2 \leq n_1 + \left\lfloor e_1 \left(\frac{l_\beta}{l_\beta - 1} \right) + 1 \right\rfloor.$$

Therefore, in this case, we can choose $e_2 = \left\lfloor e_1\left(\frac{l_\beta}{l_\beta - 1}\right) + 1 \right\rfloor$. If $2 \neq i_1$, then we can use Inequality (25) to obtain

$$\begin{aligned}(n+1)_3 &\leq n_2 + \left\lfloor e_2\left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right)\right\rfloor \\ &= n_2 + \left\lfloor \left\lfloor e_1\left(\frac{l_\beta}{l_\beta - 1}\right) + 1\right\rfloor \left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right)\right\rfloor \\ &\leq n_2 + \left\lfloor e_1\left(\frac{l_\beta}{l_\beta - 1}\right)\left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right) + \left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right)\right\rfloor.\end{aligned}$$

Hence, we choose $e_3 = \left\lfloor e_1\left(\frac{l_\beta}{l_\beta-1}\right)\left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right) + \left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right)\right\rfloor$. If $3 \neq i_1$, then we use Inequality (25) and repeat the process. Thus, we obtain

$$\begin{aligned}(n+1)_4 &\leq n_3 + \left\lfloor e_3\left(\frac{l_{\beta-2}}{l_{\beta-2}-1}\right)\right\rfloor \\ &= n_3 + \left\lfloor \left\lfloor e_1\left(\frac{l_\beta}{l_\beta - 1}\right)\left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right) + \left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right)\right\rfloor \left(\frac{l_{\beta-2}}{l_{\beta-2}-1}\right)\right\rfloor \\ &\leq n_3 + \left\lfloor e_1\left(\frac{l_\beta}{l_\beta - 1}\right)\left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right)\left(\frac{l_{\beta-2}}{l_{\beta-2}-1}\right)\right. \\ &\qquad \left. + \left(\frac{l_{\beta-1}}{l_{\beta-1}-1}\right)\left(\frac{l_{\beta-2}}{l_{\beta-2}-1}\right)\right\rfloor.\end{aligned}$$

Repeat this process for all $i < i_1$. Hence, we have

$$\begin{aligned}(n+1)_{i_1} &\leq n_{i_1-1} + \left\lfloor e_1 \prod_{i=0}^{i_1-2}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right) + \prod_{i=1}^{i_1-2}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right\rfloor \\ &\leq n_{i_1-1} + \left\lfloor (e_1+1)\prod_{i=0}^{i_1-2}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right\rfloor.\end{aligned}$$

We can therefore set $e_{i_1} = \left\lfloor (e_1+1)\prod_{i=0}^{i_1-2}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right\rfloor$. Next, we apply Equation (13) and add 1 when splitting the floor function to obtain an upper bound:

$$\begin{aligned}(n+1)_{i_1+1} &\leq n_{i_1} + \left\lfloor \frac{e_{i_1} l_{\beta-i_1+1}}{l_{\beta-i_1+1}-1}\right\rfloor + 1 \\ &\leq n_{i_1} + \left\lfloor \left\lfloor (e_1+1)\prod_{i=0}^{i_1-2}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right\rfloor \left(\frac{l_{\beta-i_1+1}}{l_{\beta-i_1+1}-1}\right)\right\rfloor + 1 \\ &\leq n_{i_1} + \left\lfloor (e_1+1)\prod_{i=0}^{i_1-1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right\rfloor + 1.\end{aligned}$$

Therefore, we can choose the error $e_{i_1+1} = \left\lfloor (e_1+1)\prod_{i=0}^{i_1-1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right\rfloor + 1$. This is the first of at most $1+\left\lfloor \frac{e_{i_1}-1}{1+2k_1}\right\rfloor$ cases by Expression (21). By repeating this process, we obtain

$$\begin{aligned}(n+1)_{i_1+2} &\leq n_{i_1+1} + \left\lfloor (e_1+1)\prod_{i=0}^{i_1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right) + \left(\frac{l_{\beta-i_1}}{l_{\beta-i_1}-1}\right) + 1\right\rfloor \\ &\leq n_{i_1+1} + \left\lfloor (e_1+1)\prod_{i=0}^{i_1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right) + 2\left(\frac{l_{\beta-i_1}}{l_{\beta-i_1}-1}\right)\right\rfloor.\end{aligned}$$

Similarly, we have

$$\begin{aligned}(n+1)_{i_1+3} &\leq n_{i_1+2} + \left\lfloor (e_1+1)\prod_{i=0}^{i_1+1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right) + 3\left(\frac{l_{\beta-i_1}}{l_{\beta-i_1}-1}\right)\left(\frac{l_{\beta-i_1-1}}{l_{\beta-i_1-1}-1}\right)\right\rfloor \\ &\vdots \\ (n+1)_{m_1+2} &\leq n_{m_1+1} + \left\lfloor (e_1+1)\prod_{i=0}^{m_1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right. \\ &\qquad \left. +(m_1+2-i_1)\prod_{i=i_1}^{m_1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right\rfloor. \qquad (27)\end{aligned}$$

In Inequality (27), m_1 is the largest index corresponding to the quotient k_1. We apply Inequality (24) to obtain

$$m_1 - i_1 + 2 \leq 1 + \left\lfloor \frac{e_{i_1}-1}{1+2k_1}\right\rfloor.$$

Substitute the error $e_{i_1} = \left\lfloor (e_1+1)\prod_{i=0}^{i_1-2}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right\rfloor$ into the preceding equation to see that

$$\begin{aligned}m_1 - i_1 + 2 &\leq 1 + \left\lfloor \left((e_1+1)\prod_{i=0}^{i_1-2}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right)\left(\frac{1}{1+2k_1}\right) - \frac{1}{1+2k_1}\right\rfloor \\ &\leq 1 + \left\lfloor \left(\frac{e_1+1}{1+2k_1}\right)\prod_{i=0}^{i_1-1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right\rfloor. \qquad (28)\end{aligned}$$

Substitute the upper bound for $m_1 - i_1 + 2$ from Inequality (28) into Inequality (27)

to obtain

$$\begin{aligned}(n+1)_{m_1+2} &\le n_{m_1+1} + \Bigg\lfloor (e_1+1)\prod_{i=0}^{m_1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right) \\ &\qquad + \left(1+\left(\frac{e_1+1}{1+2k_1}\right)\prod_{i=0}^{i_1-1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right)\prod_{i=i_1}^{m_1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\Bigg\rfloor \\ &= n_{m_1+1} + \Bigg\lfloor (e_1+1)\prod_{i=0}^{m_1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right) \\ &\qquad + \left(1+\left(\frac{e_1+1}{1+2k_1}\right)\right)\prod_{i=0}^{m_1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\Bigg\rfloor \\ &= n_{m_1+1} + \left\lfloor \left(e_1+2+\left(\frac{e_1+1}{1+2k_1}\right)\right)\prod_{i=0}^{m_1}\left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right\rfloor. \end{aligned} \tag{29}$$

Using induction on j and applying the same process as before (as can be seen in Inequality (29)), the result follows. □

We now have a lemma that gives us an upper bound for s_{N+1}.

Lemma 4. *We have*

$$s_{N+1} = O(N). \tag{30}$$

Proof. We start with Equation (26) to see that

$$\begin{aligned} s_i &= 2 + s_{i-1} + \frac{s_{i-1}+1}{2(k_i-1)+1} \le 2 + s_{i-1} + \frac{s_{i-1}+1}{2(i-1)+1}, \text{ by Remark 4} \\ &\le s_{i-1} + 2 + \frac{s_{i-1}+2}{2(i-1)+1} \\ &= (s_{i-1}+2)\left(1+\frac{1}{2(i-1)+1}\right). \end{aligned} \tag{31}$$

Starting at s_{N+1} and applying Inequality (31) repeatedly, we obtain

$$\begin{aligned} S_{N+1} &\le (s_N+2)\left(1+\frac{1}{2N+1}\right) \\ &= 2\left(1+\frac{1}{2N+1}\right) + s_N\left(1+\frac{1}{2N+1}\right) \\ &\le 2\left(1+\frac{1}{2N+1}\right) + (s_{N-1}+2)\left(1+\frac{1}{2(N-1)+1}\right)\left(1+\frac{1}{2N+1}\right) \\ &= 2\left(\frac{2N+2}{2N+1}\right) + 2\left(\frac{2(N-1)+2}{2(N-1)+2}\right)\left(\frac{2N+1}{2N+1}\right) \\ &\qquad + s_{N-1}\left(\frac{2(N-1)+2}{2(N-1)+2}\right)\left(\frac{2N+1}{2N+1}\right). \end{aligned}$$

Repeat this process to obtain

$$\begin{aligned} s_{N+1} &\leq 2\left(\left(\frac{2N+2}{2N+1}\right)+\left(\frac{2(N-1)+2}{2(N-1)+2}\right)\left(\frac{2N+1}{2N+1}\right)+\cdots+\prod_{j=1}^{N}\left(\frac{2j+2}{2j+1}\right)\right) \\ &\quad + s_1\prod_{j=1}^{N}\left(\frac{2j+2}{2j+1}\right) \\ &= 2\sum_{c=1}^{N}\prod_{j=c}^{N}\left(\frac{2j+2}{2j+1}\right)+2\prod_{j=1}^{N}\left(\frac{2j+2}{2j+1}\right),\ s_1=2. \end{aligned}$$

Since $2\prod_{j=1}^{N}\left(\frac{2j+2}{2j+1}\right)$ is just a term of the preceding sum, it follows that

$$s_{N+1} \leq 4\sum_{c=1}^{N}\prod_{j=c}^{N}\left(\frac{2j+2}{2j+1}\right). \tag{32}$$

To find an upper bound for Equation (32), consider the product:

$$P(c,N) := \prod_{j=c}^{N}\left(\frac{2j+2}{2j+1}\right) = \prod_{j=c}^{N}\left(1+\frac{1}{2j+1}\right),\ c \geq 1. \tag{33}$$

Taking the natural log on both sides and using the inequality $\log(1+x) \leq x$ for $x \geq 0$, we obtain

$$\begin{aligned} \log P(c,N) &= \sum_{j=c}^{N}\log\left(1+\frac{1}{2j+1}\right) \leq \sum_{j=c}^{N}\frac{1}{2j+1} \\ &= \sum_{j=1}^{N}\frac{1}{2j+1}-\sum_{j=1}^{c-1}\frac{1}{2j+1}. \end{aligned} \tag{34}$$

Notice that $\frac{1}{2j+1} < \frac{1}{2j}$. Thus, we have

$$\sum_{j=1}^{N}\frac{1}{2j} > \sum_{j=1}^{N}\frac{1}{2j+1}.$$

Adding $\sum_{j=1}^{N}\frac{1}{2j+1}$ to both sides, we obtain

$$\frac{1}{2}\sum_{j=2}^{2N+1}\frac{1}{j} > \sum_{j=1}^{N}\frac{1}{2j+1}.$$

Using the fact that $\sum_{i=1}^{n} \frac{1}{i} = \log n + \gamma + O\left(\frac{1}{n}\right)$, [1] where γ is Euler's constant, we obtain

$$\sum_{j=1}^{N} \frac{1}{2j+1} < \frac{1}{2}\left(\log(2N+1) + \gamma - 1 + O\left(\frac{1}{2N+1}\right)\right). \tag{35}$$

Note that $\frac{1}{2j-1} > \frac{1}{2j}$ for $1 \le j \le c$. Thus, we see that

$$\sum_{j=2}^{c} \frac{1}{2j} < \sum_{j=1}^{c-1} \frac{1}{2j+1}.$$

Once again, adding $\sum_{j=1}^{c-1} \frac{1}{2j+1}$ to both sides, we find

$$\sum_{j=1}^{c-1} \frac{1}{2j+1} > \frac{1}{2}\sum_{j=3}^{2c} \frac{1}{j} = \frac{1}{2}\left(\log(2c) + \gamma - \frac{3}{2} + O\left(\frac{1}{2c}\right)\right). \tag{36}$$

Combining Inequality (34), Inequality (35), and Inequality (36) we obtain the upper bound:

$$\begin{aligned}\log P(c,N) &\le \frac{1}{2}\left(\log\left(\frac{2N+1}{2c}\right) + \frac{1}{2} + O\left(\frac{1}{2c}\right) + O\left(\frac{1}{2N+1}\right)\right)\\ &\le \log\left(\left(\frac{2N+1}{2c}\right)^{\frac{1}{2}}\right) + O(1), \text{ since } c, N \ge 1.\end{aligned}$$

Therefore, we have

$$P(c,N) \le \left(\frac{2N+1}{2c}\right)^{\frac{1}{2}} e^{O(1)} = \frac{(2N+1)^{1/2}}{(2c)^{1/2}} O(1). \tag{37}$$

Apply the upper bound in Equation (37) to Inequality (32) to obtain

$$\begin{aligned}s_{N+1} &\le 4\sum_{c=1}^{N} P(c,N), \text{ using Equation (33)}\\ &\le 4\sum_{c=1}^{N} \frac{(2N+1)^{1/2}}{(2c)^{1/2}} O(1)\\ &= O\left(N^{1/2}\sum_{c=1}^{N} \frac{1}{c^{1/2}}\right).\end{aligned}$$

Shifting the integral yields:

$$\sum_{c=1}^{N} \frac{1}{c^{1/2}} \le \int_{0}^{N} \frac{1}{c^{1/2}} dt = O\left(N^{1/2}\right).$$

Finally, we can conclude that $s_{N+1} = O(N)$. □

Lemma 5. *We have*

$$\beta - m_N - 1 = O\left(\frac{l_n^{1/2}}{\log^{1/2}(l_n)}\right).$$

Proof. By Equation (12), we have $n_{m_N} = q_{m_N}(l_{\beta-m_N}-1)+r_{m_N}$. Recall that based on the definition of m_N, we have $q_{m_N} = k_N \geq N = \left\lceil \frac{\sqrt{l_n}}{\sqrt{\log(l_n)}} \right\rceil$ by Remark 4. Thus, we have

$$n_{m_N} \geq \left\lceil \frac{\sqrt{l_n}}{\sqrt{\log(l_n)}} \right\rceil (l_{\beta-m_N} - 1) + r_{m_N}. \tag{38}$$

Since (n_i) is an increasing sequence with $n_\beta = l_n$, we have $n_{m_N} \leq n_\beta = l_n$. Applying this to Inequality (38), we have

$$l_n / \left\lceil \frac{\sqrt{l_n}}{\sqrt{\log(l_n)}} \right\rceil \geq l_{\beta-m_N} - 1 + r_{m_N} / \left\lceil \frac{\sqrt{l_n}}{\sqrt{\log(l_n)}} \right\rceil.$$

Hence, we see that

$$l_n / \left\lceil \frac{\sqrt{l_n}}{\sqrt{\log(l_n)}} \right\rceil + 1 \geq l_{\beta-m_N}, \text{ since } r_{m_N} \geq 0.$$

Therefore, we have

$$l_{\beta-m_N} = O\Big(l_n^{1/2} \log^{1/2} l_n\Big).$$

Hence, for a fixed constant $M > 0$ and for sufficiently large n, we obtain

$$l_{\beta-m_N} \leq M l_n^{1/2} \log^{1/2} l_n.$$

Since the lucky counting function is a monotonically increasing function, we have

$$\beta - m_N \leq L\Big(M l_n^{1/2} \log^{1/2} l_n\Big) = O\Big(L\Big(l_n^{1/2} \log^{1/2} l_n\Big)\Big).$$

Applying the lucky number theorem, we see that

$$\begin{aligned}
\beta - m_N &= O\left(\frac{l_n^{1/2} \log^{1/2} l_n}{\log\left(l_n^{1/2} \log^{1/2} l_n\right)}\right) \\
&= O\left(\frac{l_n^{1/2} \log^{1/2} l_n}{\log l_n + \log\log l_n}\right) \\
&= O\left(\frac{l_n^{1/2}}{\log^{1/2}(l_n)}\right).
\end{aligned}$$

The result follows. □

By Lemma 5, there are $O\left(\frac{l_n^{1/2}}{\log^{1/2}(l_n)}\right)$ terms left in the sequence $((n+1)_i)_{i=0}^{i=\beta'}$ after the term $(n+1)_{m_N+2}$. This is because $\beta' - (m_N+2) = \beta - m_N - 1$. We can now use all the results proven so far to complete the proof of Theorem 3.

Proof of Theorem 3. Substituting $j = N$ in Lemma 3, we have

$$(n+1)_{m_N+2} \leq n_{m_N+1} + \left\lfloor (s_{N+1}) \prod_{i=0}^{m_N} \left(\frac{l_{\beta-i}}{l_{\beta-i}-1} \right) \right\rfloor. \tag{39}$$

Inequality (39) corresponds to the quotient:

$$k_N \geq N = \left\lceil \frac{\sqrt{l_n}}{\sqrt{\log(l_n)}} \right\rceil.$$

By the definition of m_N, Inequality (39) is also the last step corresponding to the quotient k_N. Apply Equation (30) to Inequality (39) to see that

$$(n+1)_{m_N+2} \leq n_{m_N-1} + \left\lfloor O(N) \prod_{i=0}^{m_N} \left(\frac{l_{\beta-i}}{l_{\beta-i}-1} \right) \right\rfloor.$$

We use the fact that $\lfloor x+y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$ for all the remaining terms of the sequence to obtain

$$\begin{aligned}(n+1)_{m_N+3} &\leq n_{m_N+2} + \left\lfloor O(N) \prod_{i=0}^{m_N+1} \left(\frac{l_{\beta-i}}{l_{\beta-i}-1} \right) \right\rfloor + 1 \\ &\leq n_{m_N+2} + \left\lfloor (O(N)+1) \prod_{i=0}^{m_N+1} \left(\frac{l_{\beta-i}}{l_{\beta-i}-1} \right) \right\rfloor.\end{aligned}$$

Repeat the process up to $(n+1)_{\beta+1} = l_{n+1}$ for $\beta - m_N - 1$ steps and apply Lemma 5 to obtain

$$\begin{aligned}(n+1)_{m_N+4} &\leq n_{m_N+3} + \left\lfloor (O(N)+2) \prod_{i=0}^{m_N+2} \left(\frac{l_{\beta-i}}{l_{\beta-i}-1} \right) \right\rfloor \\ (n+1)_{m_N+5} &\leq n_{m_N+4} + \left\lfloor (O(N)+3) \prod_{i=0}^{m_N+3} \left(\frac{l_{\beta-i}}{l_{\beta-i}-1} \right) \right\rfloor \\ &\vdots \\ (n+1)_{\beta+1} &\leq n_\beta + \left\lfloor \left(O(N) + O\left(\frac{l_n^{1/2}}{\log^{1/2}(l_n)} \right) \right) \prod_{i=0}^{\beta-1} \left(\frac{l_{\beta-i}}{l_{\beta-i}-1} \right) \right\rfloor.\end{aligned}$$

Substituting $(n+1)_{\beta+1} = l_{n+1}$, $n_\beta = l_n$, and $N = \left\lceil \frac{l_n^{1/2}}{\log^{1/2}(l_n)} \right\rceil = O\Big(\frac{l_n^{1/2}}{\log^{1/2}(l_n)}\Big)$, it follows that

$$l_{n+1} = l_n + O\left(\frac{l_n^{1/2}}{\log^{1/2}(l_n)} \prod_{i=0}^{\beta-1} \left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right)\right). \tag{40}$$

By Equation (1) (Mertens' theorem), we obtain

$$\prod_{i=0}^{\beta-1} \left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right) = \prod_{l \le n} \left(\frac{l}{l-1}\right) = O(\log n). \tag{41}$$

Clearly, since $n \le l_n$, we have $\prod_{i=0}^{\beta-1} \left(\frac{l_{\beta-i}}{l_{\beta-i}-1}\right) = O(\log(l_n))$. Therefore, we have

$$l_{n+1} = l_n + O\Big(l_n^{1/2} \log^{1/2} l_n\Big).$$

Rearranging the above equation concludes the proof of the first part of the theorem. Applying Equation (41) to Equation (40) and using the lucky number theorem for large enough n, we have

$$\begin{aligned} l_{n+1} &= l_n + O\left(\frac{l_n^{1/2}}{\log^{1/2}(l_n)} \log n\right) \\ &= l_n + O\left(\frac{n^{1/2}\log^{1/2} n}{\log^{1/2}(n \log n)} \log n\right) \\ &= l_n + O\Big(n^{1/2} \log n\Big). \end{aligned}$$

This concludes the proof of the theorem. □

As a direct consequence of this theorem, we have the following corollary.

Corollary 1. *There exists a constant $M \in \mathbb{N}$ and a constant c such that for all $a \in \mathbb{N}$, if $a \ge M$, then there is always a lucky number between a^2 and $(a + c\log a)^2$.*

For primes, Legendre's conjecture asks: is there always a prime between a^2 and $(a+1)^2$? An analog of this conjecture would be: is there always a lucky number between a^2 and $(a+1)^2$? If this is true, it would imply that $l_{n+1} - l_n = O(l_n^{1/2})$. Therefore, the result in Corollary 1 is still slightly weaker than an analog of Legendre's conjecture.

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