

MOTZKIN PATHS WITH TWO VARIANTS OF LEVEL STEPS ON ODD LEVELS – A KERNEL METHOD APPROACH

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Abstract

The sequence A176677 in the Encyclopedia of Integer Sequences enumerates Motzkin paths where two types of horizontal steps may occur, but only on odd indexed levels. We show how to perform the enumeration, also dealing with partial such Motzkin paths leading to a particular level or to any level (open paths). The method is the kernel method where functional equations are manipulated in a suitable way. The coefficients of sequence A176677 satisfy a holonomic recursion that was recently discussed in an article by Tong Niu. We show how this can be established in an (almost) automatic fashion. Eventually we switch the roles of "odd" and "even". One could also allow more versions of horizontal steps but we leave this to the interested readers.

1. A Brief Synopsis of the Kernel Method

The kernel method is well-established in the realm of combinatorial enumeration and exists in the toolkit of many researchers. Some twenty years ago [1] the author provided a collection of popular examples, together with a historic introduction. Just recently, this collection was augmented by many more interesting examples [2], mostly with a lattice path flavor, as this short note has. Here is the most basic set up.

The goal is to enumerate a sequence f_n and the associated generating function $F(z) = \sum_{n\geq 0} f_n z^n$. Oftentimes, one can introduce a second parameter, and consider $F(z,u) = \sum_{n,k\geq 0} f_{n,k} z^n u^k$. We will see examples of such an additional parameter. It might or might not have some combinatorial significance. It does in this paper. Typically, one can solve for $F(z,u)$, but as a parameter, we have to

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include $F(z,0)$. Naively, one would like to set $u=0$ and solve. However, this does not work, as it produces a void equation. So, typically, one considers the denominator of $F(z,u)$ and factors it as $(u-s_1(z))(u-s_2(z))(u-s_3(z))\dots$. Now, some of the terms $(u-s_i(z))$ are "bad", as they do not allow a power series expansion around $(z,u)\sim(0,0)$. Consequently, these factors must cancel out, which leads to a simplification of the numerator, and the denominator of $F(z,u)$. After all this, one can plug in $u=0$, compute $F(z,0)$ and, as a bonus, $F(z,u)$. One can then expand the solution and find satisfactory answers for n and k.

This works well in the context of sequence A176677, and leads to straightforward derivation of related results. The sequence of interest is of the Motzkin type, and [2] contains many such examples. It is our opinion that this approach should add favorably to the tool kit of the working combinatorialist.

The necessary calculations can be done with a computer algebra system (Maple in our case); further developments can be achieved by gfun, a Maple package designed to deal with generating functions, [3]. It allows us to switch from algebraic equations to holonomic equations (linear differential equations with polynomial coefficients) and to recursions for the coefficients. In what follows, the sketched program will be seen in action.

2. Motzkin Paths with Variations on the Possible Level Steps

Motzkin paths have up steps $(1,1)$, down steps $(1,-1)$ and level steps $(1,0)$. Like Dyck paths, they do not go below the x-axis and return eventually to the x-axis. Partial paths that do not necessarily return to the x-axis are also of interest. If there are two versions of level steps allowed, but only on an odd level, the enumeration leads to the sequence A176677 in [4][1]. We show how to compute this using the kernel method, as explained in [1] and many other papers.

We use an automaton (Figure 1) to describe the Motzkin paths in question. We use two levels ("even" on top, "odd" at the bottom); as usual, there are loops, but for the bottom layer there are loops of two possible level steps.

Since not all the states are accessible, one might simplify the automaton (Figure 2):

However, we find it more straightforward to stay with the two layers of states even though not all of them can be reached. We use generating functions $f_i(z)$ and $g_i(z)$ leading to level i on top and bottom layer, and derive recursions according to what the last step was:

Notice that $f_1=f_3=f_5\cdots=0$, and $g_0=g_2=g_4\cdots=0$. Recursions are

[1]The encyclopedia provides 4 different references for this particular sequence

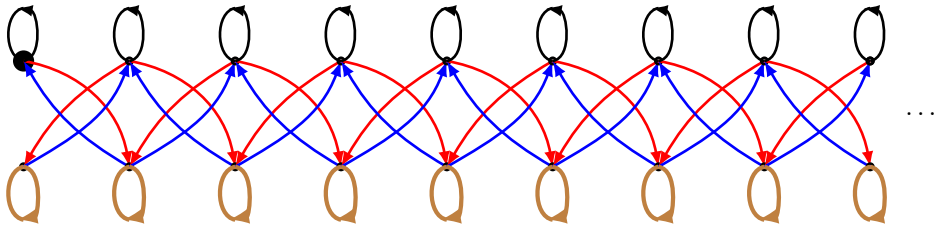

Figure 1: Distinguishing even and odd levels (even levels on top). On odd levels, two variants of level steps are possible (depicted in brown). Odd numbered states on top and even numbered states at the bottom cannot be reached.

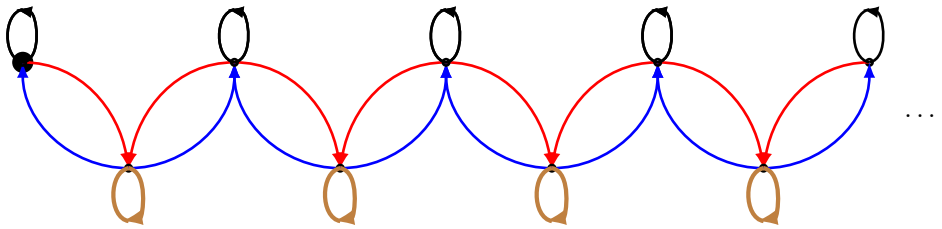

Figure 2: Simplified version

obtained by considering the last step and where it could come from,

$$\begin{aligned} f_0 &= 1 + zf_0 + zg_1, \\ f_i &= zf_i + zg_{i+1} + zg_{i-1}, \ i \geq 1, \\ g_i &= 2zg_i + zf_{i+1} + zf_{i-1}, \ i \geq 1. \end{aligned}$$

Then we need bivariate generating functions as well:

$$F(z,u) = \sum_{i\geq 0} u^i f_i(z), \quad G(z,u) = \sum_{i\geq 0} u^i g_i(z).$$

$F(z,u)$ has only even powers of u, and $G(z,u)$ has only odd powers of u. By summing up the recursions one finds

$$\begin{aligned} F(z,u) &= \sum_{i\geq 0} u^i f_i = 1 + zf_0 + zg_1 + \sum_{i\geq 1} u^i [zf_i + zg_{i+1} + zg_{i-1}] \\ &= 1 + \sum_{i\geq 0} u^i zf_i + \sum_{i\geq 0} u^i zg_{i+1} + \sum_{i\geq 1} u^i zg_{i-1} \\ &= 1 + zF(z,u) + \frac{z}{u}[G(z,u) - g_0] + zuG(z,u). \end{aligned}$$

Notice again that $g_0 = 0$;

$$\begin{aligned} G(z,u) &= \sum_{i\geq 0} u^i g_i = \sum_{i\geq 1} u^i [2zg_i + zf_{i+1} + zf_{i-1}] \\ &= 2zG(z,u) + \frac{z}{u}[F(z,u) - f_0] + zuF(z,u). \end{aligned}$$

For simplicity, we will occasionally write $F(u) = F(z,u)$, $G(u) = G(z,u)$, $F(0) = f_0$, $G(0) = g_0 = 0$. Solving the system,

$$F(u) = \frac{2zu^2 - u^2 + z^2u^2f_0 + f_0z^2}{z^2 + z^2u^4 - u^2 + 3zu^2},$$

$$G(u) = -\frac{z(u + uzf_0 + u^3 - uf_0)}{z^2 + z^2u^4 - u^2 + 3zu^2}.$$

We need to factor the denominator and abbreviate

$$W := \sqrt{(1-z)(1-2z)(1-3z-2z^2)}\,;$$

$$\begin{aligned} s_1 &= \frac{\sqrt{2}\sqrt{1-3z+W}}{2z}, \quad s_2 = -\frac{\sqrt{2}\sqrt{1-3z+W}}{2z}, \\ s_3 &= \frac{\sqrt{2}\sqrt{1-3z-W}}{2z}, \quad s_4 = -\frac{\sqrt{2}\sqrt{1-3z-W}}{2z}. \end{aligned}$$

Then

$$z^2 + z^2u^4 - u^2 + 3zu^2 = z^2(u-s_1)(u-s_2)(u-s_3)(u-s_4).$$

Plugging in $u = 0$ leads to a void equation, but the factors $u - s_3$ and $u - s_4$ are "bad" (in the spirit of the kernel method [1], as they do not have a power series expansion around $(u,z) \sim (0,0)$) and must cancel out. Then (the cancellation can be done with Maple, regardless of what s_3 and s_4 are; Maple has the quo command for that) the simplified generating functions really are

$$\begin{aligned} F(u) &= \frac{2z - 1 + f_0z^2}{z^2(u-s_1)(u-s_2)}, \\ G(u) &= \frac{-zu}{z^2(u-s_1)(u-s_2)}. \end{aligned}$$

Maple also has the rem command that we do not use here. Together, these two functions quo and rem provide the division with remainder. From this we find, by setting $u = 0$,

$$\begin{aligned} f_0 &= \frac{1 - 3z + 2z^2 - W}{2(1-z)z^2} \\ &= 1 + z + 2z^2 + 5z^3 + 14z^4 + 41z^5 + 123z^6 + 375z^7 + 1158z^8 + 3615z^9 + \ldots, \end{aligned}$$

which is the sequence A176677 in [4]. Note that

$$(u-s_1)(u-s_2) = u^2 - s_1^2.$$

Then

$$G(u) = \frac{-u}{z(u-s_1)(u-s_2)} = \frac{-u}{z(u^2-s_1^2)} = \frac{u}{zs_1^2(1-u^2/s_1^2)}$$

and

$$[u^{2k+1}]G(u) = [u^{2k}]\frac{1}{zs_1^2(1-u^2/s_1^2)} = \frac{1}{zs_1^{2k+2}}.$$

Further,

$$F(u) = \frac{2z-1+f_0z^2}{z^2(u-s_1)(u-s_2)} = \frac{2z-1+f_0z^2}{z^2}\frac{1}{u^2-s_1^2} = \frac{2z-1+f_0z^2}{z^2}\frac{-1}{s_1^2}\frac{1}{1-u^2/s_1^2},$$

and therefore

$$[u^{2k}]F(u) = \frac{2z-1+f_0z^2}{z^2}\frac{-1}{s_1^2}[u^{2k}]\frac{1}{1-u^2/s_1^2} = \frac{2z-1+f_0z^2}{z^2}\frac{-1}{s_1^{2k+2}}.$$

A direct computation yields

$$f_0 = \frac{1}{z^2} - \frac{2}{z} - \frac{1+s_1^2}{1-z}$$

and

$$-\frac{2z-1+f_0z^2}{z^2} = \frac{1+s_1^2}{1-z}.$$

Therefore,

$$[u^{2k}]F(u) = \frac{1+s_1^2}{1-z}\frac{1}{s_1^{2k+2}}.$$

Note that $F(1)+G(1)$ is the generating function of all *open* Motzkin paths (variations as discussed in this note). We find

$$\begin{aligned}&F(1)+G(1)\\ &= 1+2z+6z^2+19z^3+62z^4+205z^5+684z^6+2298z^7+7764z^8+26355z^9+\dots\end{aligned}$$

This sequence of coefficients $1, 2, 6, 19, 62, 205, 684, 2298, 7764, 26355\dots$ is not in the encyclopedia [4].

We summarize our findings.

Theorem 1. *Let*

$$s_1^2 = \frac{1-3z+\sqrt{(1-z)(1-2z)(1-3z-2z^2)}}{2z^2},$$

then the generating function of Motzkin paths with 2 variations of level steps on odd levels, leading to level $2k$ *is*

$$\frac{1+s_1^2}{1-z}\frac{1}{s_1^{2k+2}}$$

and to level $2k+1$ *is*

$$\frac{1}{zs_1^{2k+2}}.$$

The generating function of open Motzkin paths is

$$F(1)+G(1)$$
$$=1+2z+6z^2+19z^3+62z^4+205z^5+684z^6+2298z^7+7764z^8+26355z^9+\ldots;$$

an analytic expression for $F(1)+G(1)$ *is available from the text.*

3. Automatic Derivation of a Holonomic Recurrence for the Motzkin Numbers in Question

We want to derive a recursion for the coefficients of the generating function

$$f_0=\frac{1-3z+2z^2-\sqrt{(1-z)(1-2z)(1-3z-2z^2)}}{2(1-z)z^2};$$

the software gun [3] contains all the necessary tools for that. The following steps will be performed.

- Find an algebraic equation that f_0 satisfies,
- Translate the algebraic equation into a differential equation,
- Translate the differential equation into a recurrence for the coefficients.

$$>\mathsf{algfuntoalgeq}(f_0,y(z));$$

the result is the algebraic equation

$$z^2(-1+z)y^2+(1-z)(1-2z)y+2z-1=0.$$

The algebraic equation can be translated into a differential equation

$$>\mathsf{algeqtodiffeq}(y^2z^3-y^2z^2+2yz^2-3yz+2z+y-1=0,y(z));$$

the result is the differential equation

$$6z^2-7z+2+(10z-14z^2+4z^3+4z^4-2)y(z)$$
$$+(4z^5-9z^3-z+6z^2)y'(z)=0.$$

One can get a homogenous differential equation as well:

$$> \mathsf{diffeqtohomdiffeq}(6z^2-7z+2+(10z-14z^2+4z^3+4z^4-2)y(z) \\ +(4z^5-9z^3-z+6z^2)y'(z)=0);$$

the result is the homogenous differential equation as well:

$$2(24z^4-6z^3-18z^2+13z-3)(z-1)y(z) \\ +2(2z-1)(24z^5-17z^4-38z^3+54z^2-23z+3)y'(z) \\ +z(z-1)(3z-2)(2z^2+3z-1)(2z-1)^2y''(z)=0.$$

Now we translate the differential equation into a recursion for the (Taylor) coefficients:

$$> \mathsf{diffeqtorec}(6z^2-7z+2+(10z-14z^2+4z^3+4z^4-2)y(z) \\ +(4z^5-9z^3-z+6z^2)y'(z),y(z),m(n));$$

the result is the recursion

$$4(n+1)m(n)+4m(n+1)-(9n+32)m(n+2) \\ +2(3n+14)m(n+3)-(n+6)m(n+4)=0, \\ m(0)=1,\ m(1)=1,\ m(2)=2,\ m(3)=5.$$

This recursion of the coefficients

$$f_0(z)=\sum_{n\geq 0} m(n)z^n$$

is equivalent to the one given in [5] but derived in a (semi-)automatic fashion. The package gfun has several additional features of practical interest [3].

4. Switching the Roles of Even and Odd Levels

As the section title says, we switch to roles of "even" resp. "odd". The notation stays the same, and there are just a few modifications necessary (Figure 3). Then

$$F(z,u)=1+2zF(z,u)+\frac{z}{u}G(z,u)+zuG(z,u), \\ G(z,u)=zG(z,u)+\frac{z}{u}[F(z,u)-f_0]+zuF(z,u).$$

Solving the system,

$$F(u)=\frac{z^2u^2f_0+z^2f_0+zu^2-u^2}{z^2+z^2u^4-u^2+3zu^2} \quad \text{and} \quad G(u)=-\frac{zu(1-f_0+2zf_0+u^2)}{z^2+z^2u^4-u^2+3zu^2},$$

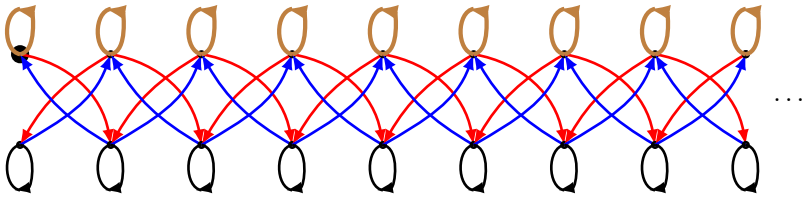

Figure 3: Brown loops refer to two variants of level steps, black loops just to one variant.

the denominators are the same as before. Consequently, dividing out the two bad factors $(u-s_3)(u-s_4)$,

$$\begin{aligned} f_0 &= \frac{1-3z+2z^2-W}{2(1-2z)z^2} \\ &= 1+2z+5z^2+13z^3+35z^4+97z^5+276z^6+804z^7+2391z^8+\ldots, \end{aligned}$$

a sequence that is not in the OEIS [4]. Furthermore,

$$F(u)=\frac{z^2f_0+z-1}{z^2(u-s_1)(u-s_2)} \quad \text{and} \quad G(u)=\frac{-u}{z(u-s_1)(u-s_2)}.$$

The function $G(u)$ has not changed from before (a little reflection shows us why this is the case). However,

$$F(u)=\frac{\frac{1-3z+2z^2-W}{2(1-2z)}+z-1}{z^2(u^2-s_1^2)}=-\frac{1+s_1^2}{(1-2z)(u^2-s_1^2)}=\frac{1+s_1^2}{(1-2z)s_1^2(1-u^2/s_1^2)}$$

and

$$[u^{2k}]F(u)=f_{2k}=\frac{1+s_1^2}{(1-2z)s_1^{2k+2}}.$$

Again, f_{2k} and g_{2k+1} can be expressed in terms of the function s_1^2.

References

[1] H. Prodinger, The kernel method: a collection of examples, *Sém. Lothar. Combin.* **50** (2003/04), #50f, 19pp.

[2] H. Prodinger, A walk in my lattice path garden, *Sém. Lothar. Combin.*, **87B** (2023), #1, 49pp.

[3] B. Salvy and P. Zimmermann, GFUN: a maple package for the manipulation of generating and holonomic functions in one variable, *Transactions on Mathematical Software* **20** (1994), 163–177.

[4] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `https://oeis.org`.

[5] Tong Niu, A short proof of Mathar's 2016 recurrence conjecture for OEIS A176677, preprint, `arXiv 2605.04369`.