MACMAHON'S DOUBLE VISION: PARTITION DIAMONDS REVISITED

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Abstract

Plane partition diamonds were introduced by Andrews, Paule, and Riese (2001) as part of their study of MacMahon's Ω-operator in search of integer partition identities. More recently, Dockery, Jameson, Sellers, and Wilson (2024) extended this concept to d-fold partition diamonds and found their generating function in a recursive form. We approach d-fold partition diamonds via Stanley's (1972) theory of P-partitions and give a closed formula for a bivariate generalization of the Dockery–Jameson–Sellers–Wilson generating function; its main ingredient is the Euler–Mahonian polynomial encoding descent statistics of permutations.

1. Introduction

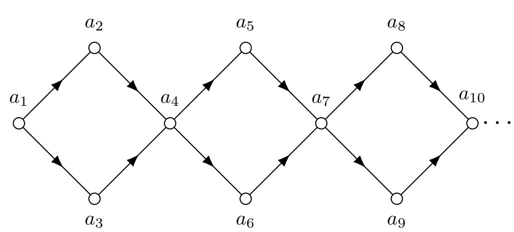

Figure 1: A plane partition diamond.

A *plane partition diamond* is an integer partition $a_1 + a_2 + \cdots + a_k$ whose parts satisfy the inequalities given by Figure 1, where each directed edge represents $\geq$.

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Plane partition diamonds were introduced by Andrews, Paule, and Riese [1], who found their generating function as

$$\prod_{n\geq 1} \frac{1+q^{3n-1}}{1-q^n} \,.$$

They proved this result as part of an impressive series of papers on partition identities via MacMahon's Ω-operator; indeed MacMahon himself used it for the case of a single diamond $\diamond$ [4, Volume 2, Section IX, Chapter II].

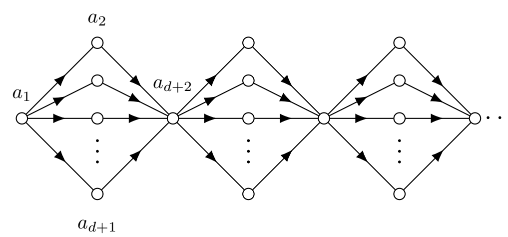

Figure 2: A d-fold partition diamond.

Dockery, Jameson, Sellers, and Wilson [3] recently generalized the above concept to a *d-fold partition diamond*, whose parts now follow the inequalities stipulated by Figure 2. They proved that their generating function equals

$$\prod_{n\geq 1} \frac{F_d\left(q^{(n-1)(d+1)+1},\, q\right)}{1-q^n}$$

where $F_d(q_0,w) \in \mathbb{Z}[q_0,w]$ is recursively defined via $F_1(q_0,w)=1$ and

$$F_d(q_0,w) \;=\; \frac{(1-q_0w^d)\,F_{d-1}(q_0,w) - w(1-q_0)\,F_{d-1}(q_0w,w)}{1-w} \,. \tag{1}$$

Once more the proof in [3] uses MacMahon's Ω-operator.

Our goal is to view the above results via Stanley's theory of P-partitions [5,6]. Our first result gives a closed formula for the generating function for d-fold partition diamonds. In a charming twist of fate, its main ingredient turns out to be the *Euler–Mahonian polynomial*

$$E_d(x,y) \;:=\; \sum_{\tau\in S_d} x^{\mathrm{des}(\tau)} y^{\mathrm{maj}(\tau)}$$

which first appeared in a completely separate part of MacMahon's vast body of work [4, Volume 2, Chapter IV, Section 462]; here $\mathrm{Des}(\tau) := \{j : \tau(j) > \tau(j+1)\}$ records the descent positions of a given permutation $\tau \in S_d$, with the statistics $\mathrm{des}(\tau) := |\,\mathrm{Des}(\tau)|$ and $\mathrm{maj}(\tau) := \sum_{j\in\mathrm{Des}(\tau)} j$.

Theorem 1. *The Dockery–Jameson–Sellers–Wilson polynomial $F_d(x, y)$ equals the Euler–Mahonian polynomial $E_d(x, y)$.*

This theorem consequently implies that Equation (1) defines the Euler–Mahonian polynomials recursively. We suspect that this recursion is known but could not find it in the literature.

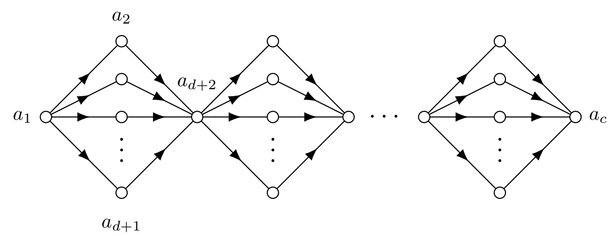

Figure 3: The d-fold partition diamond poset of length M, with $c = M(d + 1) + 1$.

Theorem 1 is actually a corollary of our main result, which gives a 2-variable refinement as follows. Let M be the number of $\diamond$s in a finite version of the d-fold partition diamond depicted in Figure 3; we call M the *length* of the diamond. We define $\sigma_{d,M}(a, b)$ to be the generating function of d-fold partition diamonds of length M, where a encodes the parts in the "folds" of the diamond, and b encodes the parts in the links connecting the $\diamond$s. That is,

$$\sigma_{d,M}(a, b) := \sum a^{a_2+\cdots+a_{d+1}+a_{d+3}+\cdots+a_{2d+2}+a_{2d+4}+\cdots+a_{M(d+1)}} \, b^{a_1+a_{d+2}+\cdots+a_{M(d+1)+1}} \tag{2}$$

where the sum is over all d-fold partition diamonds $a_1 + a_2 + \cdots + a_{M(d+1)+1}$.

Theorem 2. *For positive integers d and M,*

$$\sigma_{d,M}(a, b) = \frac{\prod_{n=1}^{M} E_d\left(a^{(n-1)d} b^n, \, a\right)}{\left(1 - a^{Md} b^{M+1}\right) \prod_{n=1}^{M} \prod_{j=0}^{d} \left(1 - a^{nd-j} b^n\right)} .$$

Naturally, Theorem 1 follows with $a = b = q$ and $M \to \infty$. Another special evaluation ($a = 1$) gives the generating function, already discovered by Dockery–Jameson–Sellers–Wilson [3], of *Schmidt type d-fold partition diamonds*, in which we sum only the links $a_1 + a_{d+2} + a_{2(d+1)+1} + \cdots$.

Corollary 1 ([3])**.** *The generating function for Schmidt type d-fold partition diamonds is given by*

$$\prod_{n \geq 1} \frac{E_d(q^n, 1)}{(1 - q^n)^{d+1}} .$$

The polynomial $E_d(x,1) = \sum_{\tau \in S_d} x^{\mathrm{des}(\tau)}$ is known as an *Eulerian polynomial*.

Section 2 contains our proof of Theorem 2. As we will see, it can be applied to more general situations, e.g., allowing folds within the diamond of different heights. We will outline this in Section 3.

2. Proofs

We now briefly review Stanley's theory of P-partitions [5,6]. Fix a finite partially ordered set $(P, \preceq)$. We may assume that $P = [c] := \{1, 2, \dots, c\}$ and that $j \preceq k$ implies $j \le k$; i.e., $(P, \preceq)$ is *naturally labelled.* A *linear extension* of $(P, \preceq)$ is an order-preserving bijection $(P, \preceq) \to ([c], \le)$. It is a short step to think about a linear extension as a permutation in S_c; accordingly we define the *Jordan–Hölder set* of $(P, \preceq)$ as

$$\mathrm{JH}(P, \preceq) \;:=\; \{\tau \in S_c : \preceq_\tau \text{ refines } \preceq\}$$

where $\preceq_\tau$ refers to the (total) order given by the linear extension corresponding to τ.

A *P-partition* is a composition $m_1 + m_2 + \cdots + m_c$ such that $m : P \to \mathbb{Z}_{\ge 0}$ is order preserving:[1]

$$j \preceq k \qquad \Longrightarrow \qquad m_j \le m_k \, .$$

It comes with the multivariate generating function

$$\sigma_P(z_1, z_2, \dots, z_c) \;:=\; \sum z_1^{m_1} z_2^{m_2} \cdots z_c^{m_c}$$

where the sum is over all P-partitions. The standard q-series for the P-partitions is, naturally, the specialization $\sigma_P(q, q, \dots, q)$.

One of Stanley's fundamental results, given here in the form of [2, Corollary 6.4.4], is that

$$\sigma_P(z_1, z_2, \dots, z_c) \;=\; \sum_{\tau \in \mathrm{JH}(P, \preceq)} \frac{\prod_{j \in \mathrm{Des}(\tau)} z_{\tau(j+1)} z_{\tau(j+2)} \cdots z_{\tau(c)}}{\prod_{j=0}^{c-1} \left(1 - z_{\tau(j+1)} z_{\tau(j+2)} \cdots z_{\tau(c)}\right)} \, . \tag{3}$$

We will apply Equation (3) to the poset depicted in Figure 3. This poset has a natural additive structure, and so we first review how to compute Equation (3) for

$$P \;=\; Q_0 \oplus Q_1 \oplus \cdots \oplus Q_M \, ,$$

where the linear sum $A \oplus B$ of two posets A and B is defined on the ground set $A \uplus B$, with the relations inherited among elements of A and those of B, together with $a \preceq b$ for any $a \in A$ and $b \in B$. Assuming that Q_j has ground set $[q_j]$, let

[1] Stanley defines P-partitions in an order-*reversing* fashion.

$s_0 := 0$ and $s_j := q_0 + q_1 + \cdots + q_{j-1}$. Each element $\tau \in \mathrm{JH}(P)$ is uniquely given via

$$\tau(j) = \begin{cases} \tau_0(j) & \text{if } j \in Q_0, \\ \tau_1(j - s_1) + s_1 & \text{if } j \in Q'_1, \\ \tau_2(j - s_2) + s_2 & \text{if } j \in Q'_2, \\ \quad \vdots & \\ \tau_M(j - s_M) + s_M & \text{if } j \in Q'_M, \end{cases}$$

for some $\tau_j \in \mathrm{JH}(Q_j)$, where $0 \le j \le M$. Here $Q'_k := \{s_k + 1, s_k + 2, \ldots, s_k + q_k\}$, with the relations induced by those in Q_k. Subsequently,

$$\mathrm{Des}(\tau) \;=\; \biguplus_{k=0}^{M} \{ j + s_k \;:\; j \in \mathrm{Des}(\tau_k) \}$$

and so the numerator in Equation (3) becomes

$$\begin{aligned} &\prod_{j \in \mathrm{Des}(\tau)} z_{\tau(j+1)} z_{\tau(j+2)} \cdots z_{\tau(c)} \\ &\quad = \prod_{k=0}^{M} \prod_{j \in \mathrm{Des}(\tau_k)} z_{\tau_k(j+1)+s_k} z_{\tau_k(j+2)+s_k} \cdots z_{\tau_k(q_k)+s_k} z_{\tau_{k+1}(1)+s_{k+1}} \cdots z_{\tau_M(q_M)+s_M} \,, \end{aligned} \tag{4}$$

with an analogous form for the denominator. We now apply these concepts to the partition diamond poset P in Figure 3.

Proof of Theorem 2. We have

$$P \;=\; \{1\} \oplus \underbrace{Q_d \oplus Q_d \oplus \cdots \oplus Q_d}_{M \text{ copies}} \,,$$

where Q_d is the poset in Figure 4, an antichain with d elements plus one more element that sits above the others.

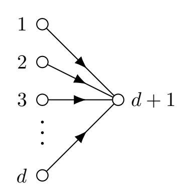

Figure 4: The poset Q_d.

The generating function $\sigma_{d,M}(a, b)$ defined in Equation (2) is the special evaluation

$$\sigma_{d,M}(a, b) \;=\; \sigma_P(z_1, z_2, \ldots, z_c) \qquad \text{where} \qquad z_j = \begin{cases} a & \text{if } j \not\equiv 1 \bmod d + 1, \\ b & \text{if } j \equiv 1 \bmod d + 1. \end{cases}$$

We now count how many a's and b's appear in Equation (4). Each $\tau_k(d+1) = d+1$, contributing a factor of b^{M-k+1} to Equation (4). The remaining variables contribute a^{d-j} for the index k and $a^{(M-k)d}$ for the indices greater than k. The denominator in Equation (3) is computed analogously; note that, unlike the numerator, it has a contribution stemming from the minimal element in P. In summary, this analysis yields

$$\begin{aligned}\sigma_{d,M}(a,b) &= \sum_{\tau \in \mathrm{JH}(P)} \frac{\prod_{k=1}^{M} \prod_{j \in \mathrm{Des}(\tau_k)} a^{(M-k+1)d-j} b^{M-k+1}}{(1-a^{Md}b^{M+1}) \prod_{k=1}^{M} \prod_{j=0}^{d} \left(1-a^{(M-k+1)d-j} b^{M-k+1}\right)} \\ &= \frac{\prod_{k=1}^{M} \sum_{\tau \in S_d} \prod_{j \in \mathrm{Des}(\tau)} a^{(M-k+1)d-j} b^{M-k+1}}{(1-a^{Md}b^{M+1}) \prod_{k=1}^{M} \prod_{j=0}^{d} \left(1-a^{(M-k+1)d-j} b^{M-k+1}\right)}, \qquad (5)\end{aligned}$$

where the second equation follows from the fact that each τ_k stemming from some $\tau \in \mathrm{JH}(P)$ fixes $d+1$, but can freely permute the remaining d elements.

By standard bijective arguments,

$$\begin{aligned}\sum_{\tau \in S_d} \prod_{j \in \mathrm{Des}(\tau)} a^{(M-k+1)d-j} b^{M-k+1} &= \sum_{\tau \in S_d} \prod_{j \in \mathrm{Asc}(\tau)} a^{(M-k+1)d-j} b^{M-k+1} \\ &= \sum_{\tau \in S_d} \prod_{d-j \in \mathrm{Asc}(\tau)} a^{(M-k)d+j} b^{M-k+1} \\ &= \sum_{\tau \in S_d} \prod_{j \in \mathrm{Des}(\tau)} a^{(M-k)d+j} b^{M-k+1},\end{aligned}$$

where $\mathrm{Asc}(\tau) := \{j : \tau(j) < \tau(j+1)\}$. Substituting this back into Equation (5) and making the change of parameters $n := M-k$ gives

$$\begin{aligned}\sigma_{d,M}(a,b) &= \frac{\prod_{n=0}^{M-1} \sum_{\tau \in S_d} \prod_{j \in \mathrm{Des}(\tau)} a^{nd+j} b^{n+1}}{(1-a^{Md}b^{M+1}) \prod_{n=0}^{M-1} \prod_{j=0}^{d} \left(1-a^{(n+1)d-j} b^{n+1}\right)} \\ &= \frac{\prod_{n=1}^{M} \sum_{\tau \in S_d} \prod_{j \in \mathrm{Des}(\tau)} a^{(n-1)d+j} b^{n}}{(1-a^{Md}b^{M+1}) \prod_{n=1}^{M} \prod_{j=0}^{d} (1-a^{nd-j} b^{n})} \\ &= \frac{\prod_{n=1}^{M} \sum_{\tau \in S_d} a^{\mathrm{maj}(\tau)} \left(a^{(n-1)d} b^{n}\right)^{\mathrm{des}(\tau)}}{(1-a^{Md}b^{M+1}) \prod_{n=1}^{M} \prod_{j=0}^{d} (1-a^{nd-j} b^{n})} \\ &= \frac{\prod_{n=1}^{M} E_d(a^{(n-1)d} b^{n},\, a)}{(1-a^{Md}b^{M+1}) \prod_{n=1}^{M} \prod_{j=0}^{d} (1-a^{nd-j} b^{n})}. \qquad \square\end{aligned}$$

3. Another Extension

The *ansatz* for our proof of Theorem 2 is, naturally, amenable to more general constructs. We give one sample here, the proof of which is analogous to that of Theorem 2.

Let $\{d_j\}_{j=1}^M$ be a finite sequence of positive integers. We define the *multifold partition diamond* corresponding to this sequence to be a partition whose parts follow a similar structure as those in Figure 3, but the jth diamond has d_j folds. The accompanying bivariate generating function is $\sigma_{d_1,\ldots,d_M}(a,b)$, where again a encodes the parts in the "folds" of the diamond, and b encodes the parts in the links connecting the $\diamond$s. Let $\omega_k := \sum_{j=k+1}^M d_j$.

Theorem 3. *For positive integers $d_1, \ldots, d_M$,*

$$\sigma_{d_1,\ldots,d_M}(a,b) \;=\; \frac{\prod_{k=1}^M E_{d_k}\left(a^{\omega_k} b^{M-k+1},\, a\right)}{\left(1 - a^{\omega_0} b^{M+1}\right) \prod_{k=1}^M \prod_{j=0}^{d_k} \left(1 - a^{\omega_k + d_k - j} b^{M-k+1}\right)}\,.$$

We conclude with an open question, namely, we are curious if the combinatorial perspective on partition diamonds via P-partitions sheds light on the partition congruences appearing in [1] and [3], and if so, if the congruences can be extended in light of Theorem 3.

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